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Relation between Hilbert Algebras and *BE*–Algebras

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Abstract

Hilbert algebras are introduced for investigations in intuitionistic and other non - classical logics and *BE*-algebra is a generalization of dual *BCK*-algebra. In this paper, we investigate the relationship between Hilbert algebras and *BE*-algebras. In fact, we show that a commutative implicative *BE*-algebra is equivalent to the commutative self distributive *BE*-algebra, therefore Hilbert algebras and commutative self distributive *BE*-algebras are equivalent.

Keywords: (Self-distributive, commutative, implicative) *BE*-algebras; dual *BCK*–algebra; (commutative, implicative) Hilbert algebra; implication algebra

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1. Introduction

Hilbert algebras are important tools for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value “true”. The concept of Hilbert algebra was introduced by L. Henkin and T. Skolem for investigations in intuitionistic and other non-classical logics. Diego (1966) proved that Hilbert algebras form a variety which is locally finite. The prepositions of Hilbert algebras in algebraic logic were displayed by Chajda et al. (2002), Jun (1994) and Abott (1967). Kim et al. (2007) introduced the notion of a BE–algebra as a generalization of dual BCK–algebra. Using the notion of upper sets they gave an equivalent condition of upper sets in BE–algebras and discussed some of their properties. Moreover, Ahn et al. (2008) introduced the concept of ideals to generalize the notion of upper sets in BE–algebras. Rezaei and Borumand Saeid (2012) introduced the idea of commutative ideals in BE–algebras and proved several characterizations of such ideals. Walendziak (2008) investigated the relationship between BE–algebras, implication algebras and J–algebras. Moreover, he defined commutative BE–algebras and proved that these algebras are equivalent to the commutative dual BCK–algebras. It is known that many properties of dual BCK–algebras, implication algebras and Hilbert algebras are similar to ones of BE–algebras, which motivates us to explore the interrelations. The concepts and methods of their respective algebras can therefore be applied to study deeply BE–algebras.

In this paper, we show that a commutative implicative BE–algebra is equivalent to the commutative self distributive BE–algebra. Also, we prove that every Hilbert algebra is a self distributive BE–algebra and commutative self distributive BE–algebra is a Hilbert algebra and that one cannot remove the conditions of commutativity and self distributivity.

2. Preliminaries

Definition 2.1. By a BE-algebra we shall mean an algebra $(X;*,1)$ of type $(2,0)$ satisfying the following axioms:

(BE1) $x*x=1$,
(BE2) $x*1=1$,
(BE3) $1*x=x$,
(BE4) $x*(y*z)=y*(x*z)$, for all $x,y,z\in X$.

A relation "$\leq$" on X is defined by $x\leq y$ if and only if $x*y=1$. In what follows, let X be a BE–algebra unless otherwise specified.

***Proposition* 2.1.**

Let X be a BE–algebra and $x,y\in X$. Then,

(i) $x*(y*x)=1$,

(ii) $y*((y*x)*x)=1$.

Definition 2.2.

A BE-algebra X is said to be self distributive if

$$x*(y*z)=(x*y)*(x*z), \text{ for all } x,y,z\in X.$$

***Proposition* 2.2.**

Let X be a self distributive BE–algebra. If $x\le y$, then

(i) $z*x\le z*y$ and $y*z\le x*z$,

(ii) $y*z\le(z*x)*(y*x)$, for all $x,y,z\in X$.

Definition 2.3.

Let X be a BE–algebra. We say that X is commutative if

$$(x*y)*y=(y*x)*x, \text{ for all } x,y\in X.$$

***Proposition* 2.3.**

If X is a commutative BE–algebra and $x*y=y*x=1$, then $x=y$, for all $x,y\in X$.

We note that "$\le$" is reflexive by $(BE1)$. If X is self distributive, then relation "$\le$" is a transitive order on X, because if $x\le y$ and $y\le z$, then

$$x*z=1*(x*z)=(x*y)*(x*z)=x*(y*z)=x*1=1.$$

Hence, $x\le z$. If X is commutative then by Proposition 2.3, relation "$\le$" is antisymmetric. Hence if X is a commutative self distributive BE–algebra, then relation "$\le$" is a partially ordered set on X.

Example 2.1.

Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Define the binary operation $*$ on $\mathbb{N}_0$ by:

$$x*y=\begin{cases}0 & if \quad y\le x\\ y-x & if \quad x<y\end{cases}.$$

Then, $(\mathbb{N}_0;*,0)$ is a commutative BE–algebra.

Definition 2.4.

An implication algebra is a set X with a binary operation "$*$" which satisfies the following axioms:

(I1) $(x*y)*x = x$,

(I2) $(x*y)*y = (y*x)*x$,

(I3) $x*(y*z) = y*(x*z)$, for all $x, y, z \in X$.

Lemma 2.1.

In any implication algebra $(X,*)$ the following identities hold:

(i) $x*(x*y) = x*y$,

(ii) $x*x = y*y$,

(iii) There exists a unique element $1 \in X$ such that,

(a) $x*x = 1$, $1*x = x$ and $x*1 = 1$,

(b) if $x*y = 1$ and $y*x = 1$, then $x = y$, for all $x, y \in X$.

***Proposition* 2.4.**

Any implication algebra is a BE–algebra.

3. Hilbert Algebras are Equivalent to Commutative Self Distributive BE–Algebras

Since there exist various modifications of the definition of Hilbert algebra, we use that of Busneag (1985).

Definition 3.1.

A Hilbert algebra is a triple $(H;*,1)$, where H is a non-empty set, $*$ is a binary operation on H and 1 is a fixed element of H (i.e., a nullary operation) such that the following three axioms are satisfied, for all $x, y, z \in H$:

(H1) $x*(y*x)=1$,

(H2) $(x*(y*z))*((x*y)*(x*z))=1$,

(H3) if $x*y=1$ and $y*x=1$, then $x=y$.

We introduce a relation "$\leq$" on X by $x\leq y$ if and only if $x*y=1$. It is easy to see that the relation "$\leq$" is a partial order on H which is called the natural ordering on H. We say that H is commutative if $(x*y)*y=(y*x)*x$, for all $x,y\in H$.

Example 3.1. Let $H=\{1,a,b,c,d\}$ be a set with the following table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	c	1	c	d
c	1	b	b	1	d
d	1	a	b	c	1

Then, $(H;*,1)$ is a Hilbert algebra.

***Proposition* 3.1. (**Digo (1966))

Let $(H;*,1)$ be a Hilbert algebra and $x,y,z\in X$. Then,

(i) $x*x=1$,

(ii) $1*x=x$,

(iii) $x*1=1$,

(iv) $x*(y*z)=y*(x*z)$,

(v) $x*(y*z)=(x*y)*(x*z)$,

(vi) if $x\leq y$, then $z*x\leq z*y$ and $y*z\leq x*z$.

Theorem 3.1.

Every Hilbert algebra is a self distributive BE–algebra.

Proof:

By Proposition 3.1, the proof is complete.

In the following example, we show that the converse of Theorem 3.1, is not correct in general.

Example 3.2. (i) Let $H=\{1,a,b\}$. The following table shows the self distributive BE-algebra structure on X.

*	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

But $(H;*,1)$ is not a Hilbert algebra, because $a*b=b*a=1$, while $a\neq b$.

(ii) Let $\mathbb{N}$ be the set of all natural numbers and $*$ be the binary operation on $\mathbb{N}$ defined by:

$$x*y=\begin{cases} y & if \quad x=1 \\ 1 & if \quad x\neq 1 \end{cases}.$$

Then, $(\mathbb{N};*,1)$ is a non-commutative BE–algebra. Also, it is not a Hilbert algebra, because $3*4=4*3=1$, but $3\neq 4$.

(iii) Example 2.1, is not a Hilbert algebra, because

$$(5*(6*7))*((5*6)*(5*7))=1\neq 0.$$

(iv) Let $H=\{1,a,b,c\}$ be a set with the following table:

*	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then, $(H;*,1)$ is a commutative BE–algebra which is not a Hilbert algebra, because

$$(b*(a*c))*((b*a)*(b*c))=(b*a)*(1*a)=1*(1*a)=1*a=a\neq 1.$$

(v) Let $X=\{1,a,b,c,d\}$ be a set with the following table:

*	1	a	b	c	d	o
1	1	a	b	c	d	o
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
o	1	1	1	1	1	1

Then, $(X;*,1)$ is a BE–algebra, but is not self distributive. Also, it is not a Hilbert algebra, since

$$(d*(a*o))*((d*a)*(d*o)) = (d*d)*(1*a) = 1*(1*a) = 1*a = a \neq 1.$$

Theorem 3.2.

Every commutative self distributive BE–algebra is a Hilbert algebra.

Proof:

By Propositions 2.2 and 2.3, the proof is clear.

Corollary 3.1.

$(X;*,1)$ is a commutative Hilbert algebra if and only if it is a commutative self distributive BE–algebra.

Proof:

By Theorems 3.1 and 3.2, the proof is clear.

Corollary 3.2.

$(X;*)$ is an implication algebra if and only if it is a commutative self distributive BE–algebra.

Proof:

Halas (2002) showed that commutative Hilbert algebras are implication algebras and Diego (1966) proved that implication algebras are Hilbert algebras. Now, from these results and Corollary 3.1, we have:

commutative self distributive BE–algebra $\leftrightarrow$ commutative Hilbert algebra
$\leftrightarrow$ implication algebra

Definition 3.2.

A BE–algebra X is said to be implicative BE-algebra if it satisfies the implicative condition $x = (x * y) * x$, for all $x, y \in X$.

Example 3.3.

(i) Let $X = \{1, a, b, c\}$ be a set with the following table:

*	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

Then, $(X;*,1)$ is an implicative BE–algebra.

(ii) Let $X = \{1, a, b, c, d\}$ be a set with the following table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	d
c	1	1	1	1	d
d	1	1	b	c	1

Then, $(X;*,1)$ is a BE–algebra which is not implicative, since $(a * c) * a = c * a = 1 \neq a$.

Example 3.4.

In Example 3.2(i), H is an implicative BE–algebra which is not an implication algebra, because

$$(a * b) * b = 1 * b = b \neq (b * a) * a = 1 * a = a.$$

Corollary 3.3.

Every commutative implicative BE–algebra is an implication algebra.

Theorem 3.3.

A commutative BE–algebra X is self distributive if and only if it is an implicative BE–algebra.

Proof:

$(\Rightarrow)$ Let $x, y \in X$. Then, by $(BE4)$, $(BE1)$ and $(BE2)$ we have $x \le (x*y)*x$. Now, by commutativity, $(BE4)$ and self distributivity, we have

$$((x*y)*x)*x = (x*(x*y))*(x*y) = x*((x*y)*y) = (x*y)*(x*y) = 1,$$

and so $(x*y)*x \le x.$ Hence $(x*y)*x = x.$

$(\Leftarrow)$. By Corollary 3.3, X is an implication algebra. Now, by Corollary 3.2, the proof is complete.

In the following example we show that the condition of commutativity in Theorem 3.3, is necessary.

Example 3.5.

Self distributive BE–algebra from Example 3.2(i), is an implicative but not a commutative BE–algebra because

$$(a*b)*b = 1*b = b \ne (b*a)*a = 1*a = a.$$

The following example shows that the condition self distributivity of Theorem 3.3, is necessary.

Example 3.6.

A non self distributive BE–algebra from Example 2.1, is a commutative but not an implicative BE–algebra because

$$(2*3)*2 = 1*2 = 1 \ne 2.$$

Definition 3.3.

A Hilbert algebra H is said to be implicative Hilbert algebra if it satisfies the implicative condition $x = (x*y)*x$, for all $x, y \in H$.

Example 3.7.

(i) In Example 3.1, $(H;*,1)$ is an implicative BE–algebra.

(ii) Let $X = \{1, a, b, c\}$ be a set with the following table:

*	1	a	b	c
1	1	a	b	c
a	1	1	1	1

b	1	a	1	1
c	1	a	b	1

Then, $(X;*,1)$ is a Hilbert algebra but it is not implicative because

$$(c*b)*c = b*c = 1 \neq c.$$

Also, since

$$(c*b)*b = b*b = 1 \neq (b*c)*c = 1*c = c,$$

we see that it is not an implication algebra.

Theorem 3.4.

Let $(H;*,1)$ be a Hilbert algebra. Then the following conditions are equivalent:

(i) H is an implicative Hilbert algebra,

(ii) H is a commutative Hilbertalgebra.

Proof:

(i $\Rightarrow$ ii). Let H be an implicative Hilbert algebra, that is H satisfying in condition $x = (x*y)*x$, for all $x, y \in H$. Then $y = (y*x)*y$, for any $x, y \in H$, too. Since $x \leq (x*y)*y$, then

$$(y*x)*x \leq (y*x)*((x*y)*y) = (x*y)*((y*x)*y) = (x*y)*y$$

Now, by replacing the roles of x and y, we have $(x*y)*y \leq (y*x)*x$. Hence, $(x*y)*y = (y*x)*x$ and so H is commutative.

(ii $\Rightarrow$ i). By Theorems 3.1 and 3.3, the proof is complete.

Corollary 3.4.

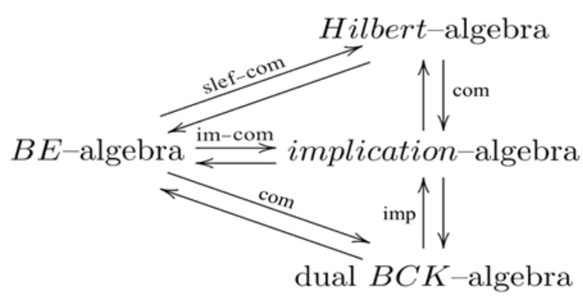
Hilbert algebra H is an implicative if and only if it is an implication algebra.

Corollary 3.5.

Hilbert algebra H is an implicative if and only if it is a commutative self distributive $BE-$ algebra.

4. Conclusion and Future Research

Meng (1996) proved that implication algebras are a dual to implicative BCK–algebras. Also, Halas (2002) showed that commutative Hilbert algebras are implication algebras and Digo (1966) proved that implication algebras are Hilbert algebras. Recently, Walendziak (2008) showed that an implication algebra is a BE–algebra and commutative BE–algebras are dual BCK–algebras. Furthermore, in this note we show that every Hilbert algebra is a self distributive BE–algebra and commutative self distributive BE–algebra is a Hilbert algebra. Now, in the following diagram we summarize the results of this paper and the previous results in this field and we give the relations among BE-algebras, dual BCK-algebras, Hilbert algebras and implication algebras. The mark $A \to B$ ($A \xrightarrow{ex} B$ or $A \xrightarrow[ex]{} B$), means that A conclude B (respectively, A conclude B with the condition "*example*" briefly "*ex*").



For the future research, we investigate new structures on BE-algebras.

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