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A Hybrid Variational Iteration Method for Blasius Equation

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Abstract

The objective of this paper is to present the hybrid variational iteration method. The proposed algorithm is based on the combination of variational iteration and shooting methods. In the proposed algorithm the entire domain is divided into subintervals to establish the accuracy and convergence of the approximate solution. It is found that in each subinterval a three term approximate solution using variational iteration method is sufficient. The proposed hybrid variational iteration method offers not only numerical values, but also closed form analytic solutions in each subinterval. The method is implemented using an example of the Blasius equation. The results show that a hybrid variational iteration method is a powerful technique for solving nonlinear problems.

Keywords: Variational iteration method; shooting method; Blasius equation; subintervals

MSC 2010: 34G20, 76D10, 76M30

1. Introduction

In 1997, He (1997) proposed the variational iteration method (VIM) for solving nonlinear differential equations. Since then the VIM has been extensively used for solving this type of differential equations. In this paper, a hybrid variational iteration method is proposed to solve the well known Blasius equation (1979) which describes the flow over a flat plate. Blasius solved the equation using a series expansion method. The numerical solution which uses the Runge-Kutta method was provided by Toepfer in 1912. A more accurate numerical solution of the problem is given by Howarth (1938) and Ozisik (1977). Yu and Chen (1998) provided the solution of the Blasius equation using the differential transform method. Liao and Campo (2002) found the analytical solutions of the temperature distribution in Blasius viscous flow problems using the homotopy analysis method.

The solution of Blasius equations using variational iteration method was first given by He (1999). In a later study Wazwaz (2007) provided another solution using variational iteration method and more recently Aiyesimi and Niyi (2011) provided their own. In the present paper we have revisited the Blasius equation for the solution using the proposed hybrid variational iteration method where its accuracy and convergence are established by subdividing the domain into subintervals. The details of the proposed algorithm are given in the next section.

2. Hybrid variational iteration method

The dimensionless form of the Blasius equation is given by (1979)

$$f''' + \frac{1}{2} f f'' = 0, \tag{1}$$

subject to boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad \lim_{\eta \to \infty} f'(\eta) = 1, \tag{2}$$

in which f is a dimensionless velocity and η is a dimensionless independent variable. For the solution we use the shooting method in combination with a variational iteration method. The boundary value problem given in Equations (1) and (2) is converted into an initial value problem using the shooting method (1979) by assuming

$$f''(0) = s. \tag{3}$$

Here s is a missing condition which will be computed through the solution process. Differentiating Equations (1)-(3) with respect to s one gets

$$g''' + \frac{1}{2}(g f'' + f g'') = 0, \tag{4}$$

$$g(0) = 0, \quad g'(0) = 0, \quad g''(0) = 1. \tag{5}$$

The initial value problems given in Equations (1)-(5) can be transformed as follows

$$f' = F, \quad F' = G, \quad G' = -\frac{1}{2} fG, \tag{6}$$

$$g' = Y, \quad Y' = Z, \quad Z' = -\frac{1}{2}(gG + fZ), \tag{7}$$

$$f(0) = 0, \quad F(0) = 0, \quad G(0) = s, \tag{8}$$

$$g(0) = 0, \quad Y(0) = 0, \quad Z(0) = 1. \tag{9}$$

For the numerical computations we replace ∞ by a number η_∞ and divide the domain $0 \leq \eta \leq \eta_\infty$ into subintervals each represented by H_i such that

$$\sum H_i = \eta_\infty , \quad i = 1,2,3, ... \tag{10}$$

For fixed length subintervals each subinterval is represented by$[(i-1)H, iH]$, $i = 1,2,3, ...$ The initial value problem in each subinterval takes the form

$$\frac{df^i}{d\eta} = F^i, \quad \frac{dF^i}{d\eta} = G^i, \quad \frac{dG^i}{d\eta} = -\frac{1}{2} f^i G^i \tag{11}$$

$$\frac{dg^i}{d\eta} = Y^i, \quad \frac{dY^i}{d\eta} = Z^i, \quad \frac{dZ^i}{d\eta} = -\frac{1}{2}\left(g^i G^i + f^i Z^i\right), \tag{12}$$

and initial conditions in the first subinterval are

$$f^1(0) = 0, \quad F^1(0) = 0, \quad G^1(0) = s, \tag{13}$$

$$g^1(0) = 0, \quad Y^1(0) = 0, \quad Z^1(0) = 1. \tag{14}$$

The numerical values computed at the end point in the ith subinterval are the initial values in the $(i+1)st$ subinterval.

According to variational iteration method (1997) the correction functional of system (11) and (12) can be constructed as follows

$$f_{n+1}^i(\eta) = f_n^i(\eta) + \int_{(i-1)H}^{\eta} \lambda_1(\xi) \left\{\frac{df_n^i(\xi)}{d\eta} - \tilde{F}_n^i(\xi)\right\} d\xi, \tag{15}$$

$$F_{n+1}^i(\eta) = F_n^i(\eta) + \int_{(i-1)H}^{\eta} \lambda_2(\xi) \left\{\frac{dF_n^i(\xi)}{d\eta} - \tilde{G}_n^i(\xi)\right\} d\xi, \tag{16}$$

$$G_{n+1}^i(\eta) = G_n^i(\eta) + \int_{(i-1)H}^{\eta} \lambda_3(\xi) \left\{\frac{dG_n^i(\xi)}{d\eta} + \frac{1}{2}\tilde{f}_n^i(\xi)\tilde{G}_n^i(\xi)\right\} d\xi, \tag{17}$$

$$g_{n+1}^i(\eta) = g_n^i(\eta) + \int_{(i-1)H}^{\eta} \lambda_4(\xi) \left\{\frac{dg_n^i(\xi)}{d\eta} - \tilde{Y}_n^i(\xi)\right\} d\xi, \tag{18}$$

$$Y_{n+1}^i(\eta) = Y_n^i(\eta) + \int_{(i-1)H}^{\eta} \lambda_5(\xi) \left\{\frac{dY_n^i(\xi)}{d\eta} - \tilde{Z}_n^i(\xi)\right\} d\xi, \tag{19}$$

$$Z_{n+1}^i(\eta) = Z_n^i(\eta) + \int_{(i-1)H}^{\eta} \lambda_6(\xi) \left\{\frac{dZ_n^i(\xi)}{d\eta} + \frac{1}{2}\left[\tilde{f}_n^i(\xi)\tilde{Z}_n^i(\xi) + \tilde{g}_n^i(\xi)\tilde{G}_n^i(\xi)\right]\right\} d\xi, \tag{20}$$

where $\lambda_j: j = 1,2,3,\dots,6$ are general Lagrange multipliers and terms subscribed denote restricted variations, i.e.,

$$\delta\tilde{f}^i = \delta\tilde{F}^i = \delta\tilde{G}^i = \delta\tilde{g}^i = \delta\tilde{Y}^i = \delta\tilde{Z}^i = 0. \tag{21}$$

Making the correction functional (15)-(20) stationary, one can obtain the following stationary conditions

$$\frac{d\lambda_j(\xi)}{d\eta} = 0, \quad 1 + \lambda_j(\xi)\Big|_{\xi=\eta} = 0, \quad j = 1,2,3,\dots,6. \tag{22}$$

The Lagrange multipliers take the values

$$\lambda_j = -1, \quad j = 1,2,3,\dots,6. \tag{23}$$

Using the above values in the correction functional given in Equations (15)-(20), one yields

$$f_{n+1}^i(\eta) = f_n^i\big((i-1)H\big) + \int_{(i-1)H}^{\eta} F_n^i(\xi)\, d\xi, \tag{24}$$

$$F_{n+1}^i(\eta) = F_n^i\big((i-1)H\big) + \int_{(i-1)H}^{\eta} G_n^i(\xi)\, d\xi, \tag{25}$$

$$G_{n+1}^i(\eta) = G_n^i\big((i-1)H\big) - \frac{1}{2}\int_{(i-1)H}^{\eta} f_n^i(\xi) G_n^i(\xi)\, d\xi, \tag{26}$$

$$g_{n+1}^i(\eta) = g_n^i\big((i-1)H\big) + \int_{(i-1)H}^{\eta} Y_n^i(\xi)\, d\xi, \tag{27}$$

$$Y_{n+1}^i(\eta) = Y_n^i\big((i-1)H\big) + \int_{(i-1)H}^{\eta} Z_n^i(\xi)\, d\xi, \tag{28}$$

$$Z_{n+1}^i(\eta) = Z_n^i\big((i-1)H\big) - \frac{1}{2}\int_{(i-1)H}^{\eta} \big\{f_n^i(\xi) Z_n^i(\xi) + g_n^i(\xi) G_n^i(\xi)\big\}\, d\xi. \tag{29}$$

With the starting initial approximations

$$f_0^1(0) = F_0^1(0) = g_0^1(0) = Y_0^1(0) = 0, \quad G_0^1(0) = s, \quad Z_0^1(0) = 1. \tag{30}$$

Now the solution proceeds as follows. First an approximate value of s is chosen and the iteration formulas (24)-(29) are evaluated for $i = 1$ with three terms i.e. $n = 1,2,3$. Then the values of the functions are evaluated at the final point of the first subinterval i.e. $\eta = H$. These values are the initial conditions for the second subinterval. The process is repeated and an analytic solution is evaluated at each subinterval. A zero finding algorithm is chosen to evaluate the correct value of s, which leads to $F_0^N(\eta_\infty) = 1$.

3. Numerical results and discussion

The procedure proposed in the previous section is implemented in MATHEMATICA for finding the numerical values of the velocity field and the missing condition. In each subinterval a three term solution using variational iteration method is computed. The hybrid variational iteration method not only provides the numerical values but also the analytic solution of the Blasius

equation in each subinterval. The obtained numerical values for the functions f, f' and f'' are given in Tables 1-3, respectively.

The results obtained using hybrid variational iteration method agrees well with the Howarth (1938) solution. However, the results presented with the standard variational iternation method by He (1999) have very high percentage error. Hence the presented hybrid variational iteration method is more accurate than the standard variational iteration method. In a recent study Yun (2010) reported the missing value of $f''(0) = 0.33205733621519630$.

However, in the present study by implementing proposed hybrid variational iteration method we have found $f''(0) = 0.332057337331755$. The presented results prove that the hybrid variational iteration method is an effective method for solving nonlinear boundary value problems. It is hoped that the application of this proposed algorithm will lead to many interesting results for future studies.

Table 1. Comparison of the present numerical values of f with Howarth (1938) exact solution and He (1999) standard VIM solution

η	Present	Howarth (1938)	He (1999)
0	0	0	0
1	0.16557	0.16557	0.19319
2	0.65003	0.65003	0.67940
3	1.39682	1.39682	1.39106
4	2.30576	2.30576	2.24573
4.8	3.08534	3.08534	2.98719
5	3.28329	3.28329	3.17448
6	4.27964	4.27964	4.14688
7	5.27926	5.27926	5.13359
8	6.27923	6.27923	6.12796
8.8	7.07923	7.07923	6.92593

Table 2. Comparison of the present numerical values of f' with Howarth (1938) exact solution and He (1999) standard VIM solution

η	Present	Howarth (1938)	He (1999)
0	0	0	0
1	0.32979	0.32979	0.35064
2	0.62977	0.62977	0.61218
3	0.84605	0.84605	0.79640
4	0.95552	0.95552	0.90185
4.8	0.98779	0.98779	0.94744
5	0.99115	0.99115	0.95523
6	0.99868	0.99868	0.988032
7	0.99992	0.99992	0.99158
8	1	1	0.99618
8.8	1	1	0.99813

Table 3. Comparison of the present numerical values of f'' with Howarth (1938) exact solution and He (1999) standard VIM solution

η	Present	Howarth (1938)	He (1999)
0	0.33206	0.33206	0.54360
1	0.32301	0.32301	0.27141
2	0.26675	0.26675	0.22784
3	0.16136	0.16136	0.14117
4	0.06424	0.06424	0.07469
4.8	0.02187	0.02187	0.04184
5	0.01591	0.01591	0.03600
6	0.00240	0.00240	0.01645
7	0.00022	0.00022	0.00724
8	0.00001	0.00001	0.00310
8.8	0	0	0.00155

4. Conclusion

The hybrid variational iteration method is proposed in this paper. For validation purpose the Blasius flow problem is considered. The developed algorithm combines the features of shooting and variational iteration methods. Its comparison with existing methods is very favorable. The present approach also provides more accurate results when compared to the standard variational iteration method.

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