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On the L^p-spaces techniques in the existence and uniqueness of the fuzzy fractional Korteweg-de Vries equation's solution

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Abstract

In this paper, is proposed the existence and uniqueness of the solution of all fuzzy fractional differential equations, which are equivalent to the fuzzy integral equation. The techniques on L^p-spaces are used, defining the $L^p_F([0,1])$ for $1 \leq p \leq \infty$, its properties, and using the functional analysis methods. Also the convergence of the method of successive approximations used to approximate the solution of fuzzy integral equation be proved and an iterative procedure to solve such equations is presented.

Keywords: Convergence; Existence; Fractional calculus; Korteweg-de Vries equation; The method of successive approximations; Uniqueness

MSC 2010 No.: 34A12, 26A33, 34K36, 37L65, 35Q53

1. Introduction

Fuzzy differential and integral equations are one of the important parts of the fuzzy analysis theory, that play major role in numerical analysis. Zadeh (1956) published his pioneering study in fuzzy theory, the nature of uncertainty in the behavior of a given system processes in fuzzy, then stochastic, nature. The idea of fuzzy derivatives was first established by Chang et al. (1972). Recently the study of the existence of solutions for nonlinear integral equations has been done by Mishra et al. (2016) and the study of the existence of a unique solution for fuzzy integral equations using fixed point theorems is carried out by Mordeson et al. (1995), Friedman et al. (1999), Park et al. (1999), and Balachandran et al. (2005). Fuzzy fractional differential equations on L^p-spaces have not been done.

The numerical methods for solving fuzzy integral equations involve various techniques. The method of successive approximations and its iterative methods are applied by Friedman et al. (1999), Bede et al. (2004), and Bica et al. (2008). We want to solve the fuzzy fractional Korteweg-de Vries equation, which is denoted by FFKdVE in this paper.

We will show that the FFKdVE is equivalent to fuzzy integral equation. Since most of the methods to solve integral equations lead to solving linear systems and the singularity of these systems may be causing problems, using iterative methods based on successive approximations are very useful. So we propose an efficient iterative method to solve FFKdVE. Also the existence and uniqueness of the solution and convergence of the proposed method are proved in details on L^p-spaces.

The paper is organized as follows: in Section 2 we present some concepts and results about the fuzzy number and some properties for fuzzy number valued functions and we give some definitions about the method of successive approximations and L^p-spaces. The equivalency the FFKdVE to the fuzzy integral equation with the conformable fuzzy fractional derivative is proved in Section 3. Then the convergence of the method on L^p-spaces is discussed, where we define the $L^p_F([0,1])$ for $1 \leq p \leq \infty$ and its properties and use the functional analysis methods. In Section 4, we derive the proposed method to get numerical solutions of FFKdVE based on an iterative procedure. Finally, we give some numerical examples.

2. Preliminaries

We now recall some definitions and symbols needed through the paper. We follow Zadeh et al. (1974) in definitions and notations.

Definition 2.1.

A fuzzy number is a function $u : \mathbb{R} \to [0,1]$ satisfying the following properties:

a. u is upper semicontinuous on $\mathbb{R}$,
b. $u(x) = 0$ outside of some interval $[c,d]$,
c. there are the real numbers a and b with $c \leq a \leq b \leq d$, such that u is increasing on $[c,a]$, decreasing on $[b,d]$ and $u(x) = 1$ for each $x \in [a,b]$,
d. u is fuzzy convex set (that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$, $\forall x, y \in \mathbb{R}, \lambda \in [0,1]$).

The set of all fuzzy numbers is denoted by $\mathbb{R}_F$.

Definition 2.2.

For any $u \in \mathbb{R}_F$ the $\alpha-$cut set of u is denoted dy $[u]^\alpha$ and defined by $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$, where $0 \leq \alpha \leq 1$. The notation,

$$[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]; \qquad \alpha \in [0,1],$$

refers to the lower and upper branches on u, in other words,

$$\underline{u}^\alpha = \min\{x \mid x \in u^\alpha\}, \qquad \overline{u}^\alpha = \max\{x \mid x \in u^\alpha\}.$$

It is obvious that $[u]^0 = \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ Yue et al. (1998).

An arbitrary fuzzy number u is represented, in parametric form, by an ordered pair of functions $u = (\underline{\mathrm{u}}, \overline{u})$, which define the end points of the $\alpha-$cuts, satisfying the three conditions:

a. $\underline{\mathrm{u}}$ is a bounded non-decreasing left continuous function on $(0,1]$, and right continuous at 0,
b. $\overline{u}$ is a bounded non-increasing left continuous function on $(0,1]$, and right continuous at 0,
c. $\underline{\mathrm{u}}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

For arbitrary $u = (\underline{\mathrm{u}}, \overline{u})$, $v = (\underline{v}, \bar{v})$ and $k \geq 0$, addition $(u+v)$ and multiplication by k as $(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r)$, $(\overline{u+v})(r) = \overline{u}(r) + \bar{v}(r)$, $\underline{ku}(r) = k\underline{u}(r)$, $\overline{ku}(r) = k\overline{u}(r)$, $\;k \geq 0$ and $\underline{ku}(r) = k\overline{u}(r)$, $\overline{ku}(r) = k\underline{u}(r)$, $\;k < 0$ are defined.

It is well-known that the addition and multiplication operations of real numbers can be extended to $\mathbb{R}_F$. In other words, for any $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^\alpha = [u]^\alpha \oplus [v]^\alpha, \quad [\lambda \odot u]^\alpha = \lambda [u]^\alpha, \qquad \forall \alpha \in [0,1].$$

$\ominus$ is the Hukuhara difference (H-difference). It means that $w \ominus v = u$ if and only if $u \oplus v = w$ for all $u, v, w \in \mathbb{R}_F$.

Definition 2.3.

For arbitrary fuzzy number $u = (\underline{\mathrm{u}}(r), \overline{u}(r)),\ v = (\underline{v}(r), \overline{v}(r))$, the Hausdorff distance between these fuzzy numbers, given by $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$,

$$D(u,v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \bar{v}(r)|\},$$

where D is a metric on $\mathbb{R}_F$ and has the following properties (Zadeh et al., 1965).

a. $D(u \oplus w, \nu \oplus w) = D(u, \nu), \forall u, \nu, w \in \mathbb{R}_F$,
b. $D(k \odot u, k \odot \nu) = |k| D(u, \nu), \forall k \in \mathbb{R}, u, \nu \in \mathbb{R}_F$,
c. $D(u \oplus \nu, w \oplus e) \leq D(u, w) + D(\nu, e), \forall u, \nu, w, e \in \mathbb{R}_F$,
d. $(D, \mathbb{R}_F)$ is a complete metric space.

Definition 2.4.

The function $f : T \subseteq \mathbb{R} \longrightarrow \mathbb{R}_F$ is called a fuzzy function.
So the $\alpha-$cut set of f is represented by

$$f(x;\alpha) = [\underline{f}(x;\alpha), \overline{f}(x;\alpha)]; \qquad \forall\alpha \in [0,1], t \in T.$$

A fuzzy function may have fuzzy domain and fuzzy range. So the function $f : \mathbb{R}_F \longrightarrow \mathbb{R}_F$ is also a fuzzy function.

Definition 2.5.

Let $f : \mathbb{R} \to \mathbb{R}_F$ be a fuzzy function. If for an arbitrary fixed number $t_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|t - t_0| < \delta \Longrightarrow D(f(t), f(t_0)) < \varepsilon,$$

then f is continuous at t_0.

In the following, we consider the concept of integral of a fuzzy

Definition 2.6.

Let $f : [a,b] \longrightarrow \mathbb{R}_F$. For each partition $p = \{x_1, x_2, ..., x_m\}$ of $[a,b]$ and for arbitrary

$x_{i-1} \leq \xi_i \leq x_i, \quad 2 \leq i \leq m$, let $R_p = \Sigma_{i=2}^{m} f(\xi_i)(x_i - x_{i-1})$.
The define integral of $f(x)$ over $[a,b]$ is,

$$\int_a^b f(x,y) = \lim R_p, \qquad \max|x_i - x_{i-1}| \to 0,$$

provided that this limit exists in metric D.

If the function f is continuous, its definite integral exists Goetschel et al.(1986).

Furthermore:

$$\underline{\int_a^b f(x;\alpha)}dx = \int_a^b \underline{f}(x;\alpha)dx, \qquad \overline{\int_a^b f(x;\alpha)}dx = \int_a^b \overline{f}(x;\alpha)dx.$$

More details about the properties of the fuzzy integral are given by (Goetschel et al., 1986).

Definition 2.7.

Let $f : [a,b] \longrightarrow \mathbb{R}_F$. The fuzzy $\beta-$fractional integral of fuzzy-valued function f is defined as follows:

$$(I^\beta f)(x) = \int_a^x \frac{f(t)}{t^{1-\beta}}dt, \qquad x > a, \qquad 0 < \beta < 1.$$

Let us consider the α-cut representation of fuzzy-valued function f is $f(x;\alpha) = [\underline{f}(x;\alpha), \overline{f}(x;\alpha)]$, for $0 \leq \alpha \leq 1$,. Then we can show the fuzzy $\beta-$fractional integral of fuzzy-valued function f based on its lower and upper functions as follows.

Theorem 2.1.

Let $f : [a, b] \longrightarrow \mathbb{R}_F$. The fuzzy $\beta-$fractional integral of fuzzy-valued function f can be expressed as follows:

$$(I^\beta f)(x;\alpha) = [(I^\beta \underline{f})(x;\alpha), (I^\beta \overline{f})(x;\alpha)], \qquad 0 < \alpha < 1,$$

where

$$(I^\beta \underline{f})(x;\alpha) = \int_a^x \frac{\underline{f}(t;\alpha)}{t^{1-\beta}} dt, \qquad (I^\beta \overline{f})(x;\alpha) = \int_a^x \frac{\overline{f}(t;\alpha)}{t^{1-\beta}} dt.$$

Now, we introduce a new definition of fuzzy fractional derivative as follows (Khalil et al.2014).

Theorem 2.2.

For $\beta \in [0, 1)$, and $f : [a, b] \longrightarrow \mathbb{R}_F$,

$$D_t^\beta f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\beta}) \ominus f(t)}{\varepsilon}.$$

For $t > 0, \beta \in (0, 1).D_t^\beta f(t)$ is called the conformable fuzzy fractional derivative of f of order β (Abdeljawad et al., 2015).

Using this kind of fractional derivative and some useful formulas, it can be converted differential equations into integer-order differential equations.

Some properties for the suggested conformable fuzzy fractional derivative given by Khalil et al. (2014) are as follows,

$$D_t^\beta (t^\gamma) = \gamma\, t^{\gamma-\beta}, \qquad \gamma \in \mathbb{R}, \tag{1}$$

$$D_t^\beta (f(t)g(t)) = g(t) D_t^\beta f(t) + f(t) D_t^\beta g(t), \tag{2}$$

$$D_t^\beta f[g(t)] = f'_g[g(t)] D_t^\beta g(t). \tag{3}$$

Definition 2.8.

The fuzzy continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}_F$ is called to be fuzzy bounded if there exists $M > 0$ such that $\|f\|_{F.u} = \sup_{u \in \mathbb{R}} D(f(u), \hat{0}) \leqslant M$.

Theorem 2.3.

Let $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q), p, q > 0, y_0 \in \mathbb{R}_F$ and $f : R_0 \to \mathbb{R}_F$ be continuous such that $\|f(x, y)\|_f \leq M$ for all $(x, y) \in R_0$ and f satisfies the Lipschitz condition $D(f(x, y), f(x, z)) \leq L.$ $D(y, z), \forall (x, y), (x, z) \in R_0$ and $D(y, z) \leq q$. If there exists $d > 0$ such that for $x \in (x_0, x_0 + d)$ the sequence given by $\overline{y}_0(x) = y_0, \overline{y}_{n+1}(x) = y_0 - (-1) \odot \int_x^{x_0} f(t, \overline{y}_n)(t) dt$ is defined for any $n \in \mathbb{N}$, then the fuzzy initial value problem $y' = f(x, y), y(x_0) = y_0$ has two solutions $y, \overline{y} : [x_0, x_0+r] \to \overline{B}(y_0, q)$ where $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, d\}$ and the successive iterations in

$$y_0(x) = y_0,$$
$$y_{n+1}(x) = y_0 \oplus \int_x^{x_0} f(t, y_n)(t)dt,$$

and

$$\overline{y}_0(x) = y_0,$$
$$\overline{y}_{n+1}(x) = y_0 - (-1) \odot \int_x^{x_0} f(t, \overline{y}_n)(t)dt,$$

converge to these two solutions, respectively (Bede et al., 2005).

3. The fractional transfor

In this section, we reduce FFKdVE to an ordinary differential equation. Then we show that this equation is equivalent to fuzzy integral equation.

We consider the FFKdVE with homogeneous Dirichlet boundary condition as follows,

$$\frac{\partial^\beta \tilde{U}(t,x)}{\partial t^\beta} + \frac{\partial \tilde{U}(t,x)}{\partial x} + \frac{\partial^3 \tilde{U}(t,x)}{\partial x^3} = 0, \quad 0 < t < 1, \quad 0 \leqslant x < 1, \quad 0 < \beta \leqslant 1, \tag{4}$$

where β is a parameter describing the order of the fractional time derivative, and

$$\tilde{U}(t,0) = \tilde{k}(t), \quad \tilde{U}_x(t,0) = \tilde{h}(t), \qquad 0 < x < 1,$$

where $\tilde{U}(t,x) : (0,1) \times [0,1) \longrightarrow \mathbb{R}_F$ is fuzzy number-valued function and $\mathbb{R}_F$ is the set of all fuzzy numbers. The Korteweg-de Vries equation (KdVE) is involved in a range of physics phenomena as a model for the evolution and interaction of nonlinear waves. It was first derived as an evolution equation that governs a one-dimensional, small amplitude, long surface gravity waves propagating in a shallow channel of water (Korteweg et al., 1895).

Subsequently the KdVE has arisen in a number of other physical contexts as collision-free hydromagnetic waves, stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics, etc (Fung et al., 1997). Certain theoretical physics phenomena in the quantum mechanics domain are explained by a KdVE model. It is used in fluid dynamics, aerodynamics, and continuum mechanics as a model for shock wave formation, soliton, turbulence, boundary layer behavior, and mass transport.

All of the physical phenomena may be considered as nonconservative, so they can be described using fractal differential equations. Therefore, it is important to develop numerical methods for solving such equations.

Now, we introduce the following transformations,

$$\overline{U}(t,x) = \overline{U}(\xi), \qquad \xi = ax + \frac{bt^\beta}{\beta} \qquad a, b > 0 \;\; a + \frac{b}{\beta} < 1.$$

So, we can say that 0< ξ <1 by using (1) and (3) and, by substituting into equation (4), it is derived that

$$b\overline{U}' + a\overline{U}' + a^3\overline{U}''' = 0. \tag{5}$$

Integrating (5) with respect to ξ, then we have,

$$(a+b)\overline{U} + a^3\overline{U}'' = R, \tag{6}$$

where R is a constant of integration.

Now we show that this fuzzy equation is equivalent to fuzzy integral equation as the form

$$\overline{U}''(\xi) = \overline{f}(\xi) \Longrightarrow \overline{U}'(\xi) = \overline{U}'(0) + \int_0^\xi \overline{f}(z)dz,$$

$$\Longrightarrow \overline{U}(\xi) = \overline{U}(0) + \overline{U}'(0)\xi + \int_0^\xi \int_0^\xi \overline{f}(z)dzd\xi.$$

On the other hand,

$$\int_0^\xi \cdots \int_0^\xi f(\xi)(d\xi)^n = \frac{1}{(n-1)!}\int_0^\xi (\xi - z)^{n-1} f(z)dz.$$

Therefore,

$$\overline{U}(\xi) = \overline{U}(0) + \overline{U}'(0)\xi + \int_0^\xi \int_0^\xi (\xi - z)\overline{f}(z)dz.$$

By substituting into equation (6) we have,

$$\overline{f}(\xi) = \underbrace{\left[\frac{R - (a+b)[\overline{U}(0) + \overline{U}'(0)\xi]}{a^3}\right]}_{\overline{g}(\xi)} + \int_0^\xi \underbrace{\frac{-(a+b)(\xi - z)}{a^3}}_{K(\xi,z)} \overline{f}(z)dz = 0,$$

$$\overline{f}(\xi) = \overline{g}(\xi) + \int_0^\xi K(\xi,z)\overline{f}(z)dz.$$

Similarly,

$$\underline{f}(\xi) = \underline{g}(\xi) + \int_0^\xi K(\xi,z)\underline{f}(z)dz.$$

Existence and convergence analysis on L^p-spaces

Now, we prove the existence and uniqueness of the solution and convergence of the successive approximations method on the L^p-spaces by using the following assumptions. We consider fuzzy integral equations as follows,

$$f(\xi) = g(\xi) + \int_0^\xi K(\xi,z)f(z)dz,$$

where K is an arbitrary positive kernel on $[0,1] \times [0,\xi]$ and functions $f, g : [0,1] \longrightarrow \mathbb{R}_F$ are continuous fuzzy number-valued functions. We know that f, g can be represented as $f = (\underline{f}, \overline{f})$ and

$g = (\underline{g}, \overline{g})$, which define the end points of the α-cuts. We assume that K is continuous and therefore it is uniformly bounded, so there exists $M_1 > 0$ such that

$$|K(\xi, z)| \leqslant M_1 \quad 0 \leqslant \xi \leqslant 1, \;\; 0 \leqslant z \leqslant \xi.$$

Now for $1 \leq p \leq \infty$ consider the set

$$L_F^p([0,1]) = \{f = (\underline{f}, \overline{f}); f : [0,1] \longrightarrow \mathbb{R}_F, \underline{f}, \overline{f} \in L^p([0,1] \times [0,1])\},$$

It is easy to see that $L_F^p([0,1])$ for $1 \leq p \leq \infty$ is a vector space.

Now we put norm as follows,

$$\|f\|_{F.p} = (\|\underline{f}\;\|_p^p + \|\overline{f}\;\|_p^p)^{\frac{1}{p}},$$

and for $p = \infty$ the fuzzy norm with form

$$\|f\|_{F.\infty} = \max(\|\underline{f}\|_\infty, \|\overline{f}\|_\infty).$$

We can see that they are normed vector space of fuzzy function.

$\|.\|_{F.p}$ for $1 \leq p \leq \infty$ has the properties of a usual norm on $\mathbb{R}_F$, that is,

$$\|f\|_{F.p} \geq 0, \|f\|_{F.p} = 0 \quad \text{iff} \quad f = 0, \|\lambda.f\|_{F.p} = \mid \lambda \mid \|f\|_{F.p}$$

and

$$\|f + g\|_{F.p} \leq \|f\|_{F.p} + \|g\|_{F.p},$$

for any $f, g \in \mathbb{R}_F$.

Now, we define the set

$$C_F([0,1]) = \left\{ f = (\underline{f}, \overline{f}); f : [0,1] \longrightarrow \mathbb{R}_F;\; f \text{ is continuous} \right\},$$

which is the space of fuzzy continuous function.

We define the fuzzy uniform norm with form

$$\|f\|_{F.u} = \sup_{\xi \in [0,1]} D(f(\xi), \hat{0}).$$

It is obvious that for $1 \leq p \leq \infty, \;\; C_F([0,1]) \subseteq L_F^p([0,1])$.

In the next theorem we show that for $1 \leq p < \infty$, $C_F([0,1])$ with $\|.\|_{F.p}$ is dense in $L_F^p([0,1])$.

Theorem 3.1.

$$\overline{(C_F([0,1]), \|.\|_{F.u})} = L_F^p([0,1]) \quad \text{with} \quad \|.\|_{F.p}$$

Proof:

Let $f \in L^p_F([0,1])$; each f can be represented by $f = (\underline{f}, \overline{f})$ such that $\underline{f}, \overline{f} \in L^p([0,1] \times [0,1])$, so there exists a sequence $f_n = (\underline{f_n}, \overline{f_n}) \in C([0,1] \times [0,1])$ which $\underline{f_n} \to \underline{f}, \overline{f_n} \to \overline{f}$ with $\|.\|_p$, so for $\varepsilon > 0$, there exist $n \in \mathbb{N}$ such that for $n \geq \mathbb{N}$ we have

$$\|\underline{f_n} - \underline{f}\|_p < \frac{\varepsilon}{2^{\frac{1}{p}}}, \qquad \|\overline{f_n} - \overline{f}\|_p < \frac{\varepsilon}{2^{\frac{1}{p}}}.$$

So we have,

$$\begin{aligned} \|f_n - f\|_{F.p} &= (\|\underline{f_n} - \underline{f}\|_p^p + \|\overline{f_n} - \overline{f}\|_p^p)^{\frac{1}{p}} \\ &< (\frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2})^{\frac{1}{p}} \\ &< \varepsilon. \end{aligned}$$

Consequently,

$$(\underline{f_n}, \overline{f_n}) \to (\underline{f}, \overline{f}) \quad \text{with} \quad \|.\|_p.$$

So we infer that $L^p_F([0,1])$ is a complete normed space so $(L^p_F([0,1]), \|.\|_{F.p})$ is a Banach space. ∎

Now in the following theorem we show that $L^\infty_F([0,1])$ is a Banach space.

Theorem 3.2.

$(L^\infty_F([0,1]), \|.\|_{F.\infty})$ is a Banach space.

Proof:

Let f_n be a Cauchy sequence of $L^\infty_F([0,1])$. So there exist $f_n = (\underline{f_n}, \overline{f_n})$ such that $\underline{f_n}, \overline{f_n} \in L^\infty([0,1] \times [0,1])$. We claim that for some $f = (\underline{f}, \overline{f})$, $\lim \| f - f_n\|_{F.\infty} = 0$. As we know, $L^\infty([0,1] \times [0,1])$ is a Banach space(Royden, 1963). So there exists some $\underline{f}, \overline{f} \in L^\infty([0,1] \times [0,1])$ such that $\lim \|\underline{f} - \underline{f_n}\|_\infty = 0$ and $\lim \|\overline{f} - \overline{f_n}\|_\infty = 0$, since

$$\lim \|f - f_n\|_{F.\infty} = \max(\|\underline{f} - \underline{f_n}\|_\infty, \|\overline{f} - \overline{f_n}\|_\infty).$$

This show that $\lim \|f - f_n\|_{F.\infty} = 0$ and the proof of the theorem is complete .

We define the operator $T : L^p_F([0,1]) \longrightarrow L^p_F([0,1])$ for $1 \leq p \leq \infty$ by

$$T(f)(\xi) = g(\xi) + \int_0^\xi K(\xi, z) f(z)\, dz, \quad \forall \xi \in [0,1], \quad \forall f, g \in L^p_F([0,1]), \quad f, g : [0,1] \to \mathbb{R}_F.$$

$T(f)(\xi)$ can be represented as form $T(f)(\xi) = (\underline{T(f)}(\xi), \overline{T(f)}(\xi))$ which,

$$\underline{T(f)}(\xi) = \underline{g}(\xi) + \int_0^\xi K(\xi, z)\underline{f}(z)\, dz, \qquad \overline{T(f)}(\xi) = \overline{g}(\xi) + \int_0^\xi K(\xi, z)\overline{f}(z)\, dz.$$

Sufficient conditions for the existence of a unique solution for the above integral equation is given in the following theorem. ∎

Theorem 3.3.

let $K = K(\xi, z)$ be continuous and positive for $0 \leqslant \xi \leqslant 1, \quad 0 \leqslant z \leqslant \xi$ and $f, g : [0,1] \longrightarrow \mathbb{R}_F$ be fuzzy functions and belong to $L_F^p([0,1])$ for $1 \leq p \leq \infty$. If $B = M_1\xi < 1$, then the successive approximate method

$$f_0(\xi) = g(\xi),$$

$$f_m(\xi) = g(\xi) + \int_0^\xi K(\xi, z) f_{m-1}(z) dz, \qquad m \geqslant 1. \tag{7}$$

converges to the unique solution f.

Proof:

At first we want to prove that $T(L_F^p([0,1])) \subseteq L_F^p([0,1]))$ for $1 \leq p \leq \infty$, therefore, we show that for $1 \leq p < \infty$,

$$\|T(f)\|_{F.p} = (\|\underline{T}(f)\|_p^p + \|\overline{T}(f)\|_p^p)^{\frac{1}{p}} < \infty,$$

and for $p = \infty$,

$$\|T(f)\|_{F.\infty} = \max(\|\underline{T}(f)\|_\infty, \|\overline{T}(f)\|_\infty) < \infty.$$

So it is enough to show that for $1 \leq p < \infty, (\| \ \underline{T}(f)\|_p^p)^{\frac{1}{p}} < \infty$ and $(\|\overline{T}(f)\|_p^p)^{\frac{1}{p}} < \infty$ and for $p = \infty, \|\underline{T}(f)\|_\infty < \infty$ and $\|\overline{T}(f)\|_\infty < \infty$.

Consequently, for $1 \leq p < \infty$,

$$\begin{aligned}
\|\underline{T}(f)\|_p &= \|\underline{g}(\xi) + \int_0^\xi K(\xi, z)\underline{f}(z)\ dz\|_p \\
&\leq \| \ \underline{g}(\xi)\|_p + \| \int_0^\xi K(\xi, z)\underline{f}(z)\ dz\|_p \\
&= \| \ \underline{g}(\xi)\|_p + (\int_0^1 | \int_0^\xi K(\xi, z)\underline{f}(z)\ dz\ |^p \ \ d\xi)^{\frac{1}{p}} \\
&\leq \| \ \underline{g}(\xi)\|_p + (\int_0^1 \int_0^1 |\ K(\xi, z)\ |^p|\ \underline{f}(z)\ |^p \ \ dz\ d\xi)^{\frac{1}{p}} \\
&\leq \| \ \underline{g}(\xi)\|_p + M_1(\int_0^1 \int_0^1 |\ \underline{f}(z)\ |^p \ \ dz\ d\xi)^{\frac{1}{p}} \\
&= \| \ \underline{g}(\xi)\|_p + M_1(\int_0^1 \ d\xi)^{\frac{1}{p}}(\int_0^1 |\ \underline{f}(z)\ |^p \ \ dz)^{\frac{1}{p}} \\
&\leq \| \ \underline{g}(\xi)\|_p + M_1(1)^{\frac{1}{p}}\|\underline{f}\|_p \\
&= \| \ \underline{g}(\xi)\|_p + M_1\|\underline{f}\|_p \\
&< \infty.
\end{aligned}$$

By the fact that $f, g \in L_F^p([0,1])$, the last relation is obvious.

Similarly, $\|\overline{T}(f)\|_p < \infty$. Thus, $T(L_F^p([0,1])) \subseteq L_F^p([0,1])$.

For $p = \infty$, since,

$$\|\underline{T}(f)\|_\infty = \sup\{|\underline{T}(f)(\xi)|;\ \ \xi \in [0,1]\},$$

it is enough to show that $|\underline{T}(f)| \leq \infty$. Since $f, g \in L^\infty_F([0,1])$, we infer that $\exists M_2 > 0$ such that $\mid f \mid \leq M_2$ and $\exists M_3 > 0$ such that $\mid g \mid \leq M_3$,

$$\begin{aligned} \mid \underline{T}(f)(\xi) \mid &= |\underline{g}(\xi) + \int_0^\xi K(\xi,z)\underline{f}(z)\ dz| \\ &\leq |\underline{g}(\xi)| + |\int_0^\xi K(\xi,z)\underline{f}(z)\ dz| \\ &\leq |\underline{g}(\xi)| + \int_0^\xi \mid K(\xi,z) \mid\mid \underline{f}(z)|\ dz \\ &\leq M_3 + M_1\ M_2 \int_0^\xi\ dz \\ &= M_3 + M_1\ M_2\ \xi \\ &< \infty. \end{aligned}$$

Now, we show that the operator T is a contraction map. So for $1 \leq p \leq \infty$ there exist, $f, h \in L^p_F([0,1])$ and $\xi \in [0,1]$,

$$\begin{aligned} D(T(f)(\xi), T(h)(\xi)) &\leqslant D(g(\xi), g(\xi)) + D(\int_0^\xi K(\xi,z)f(z)dz, \int_0^\xi K(\xi,z)h(z)dz) \\ &= \int_0^\xi |K(\xi,z)| D(f(z), h(z))dz \\ &\leqslant M_1 \int_0^\xi D(f(z), h(z))dz \\ &= M_1 \xi D(f,h) \\ &= BD(f,h). \end{aligned}$$

Therefore,

$$D(T(f)(\xi), T(h)(\xi)) \leqslant B\ D(f,h).$$

Since $B < 1$, the operator T is a contraction on $L^p_F([0,1])$ for $1 \leq p \leq \infty$.

Consequently, the Banach's fixed point theorem implies that this integral equation has a unique solution f in $L^p_F([0,1])$. ■

4. Using the successive approximations method for FFKdVEn

In this section, we present an effective method for solving fuzzy linear Volterra integral equation by using the successive approximations method.

Consider the following equation as form:

$$f_m(\xi) = g(\xi) + \int_0^\xi K(\xi,z) f_{m-1}(z)dz, \quad m \geqslant 1,$$

where

$$g(\xi) = \frac{R - (a+b)[\overline{U}(0) + \overline{U}'(0)\xi]}{a^3}$$

and

$$K(\xi, z) = \frac{-(a+b)(\xi - z)}{a^3}.$$

We present this method in several steps:

- Step 1. Set $n \to 0$.
- Step 2. Calculate the recursive relations (7).
- Step 3. If $D(\widetilde{f}_{n+1}, \widetilde{f}_n) < \varepsilon$, then go to step 4, else $n \to n+1$ and go to step 2.
- Step 4. Print $\widetilde{f}_n(\xi)$ as the approximation of the exact solution.

Since $f_0(\xi) = g(\xi)$, then

$$\begin{aligned}
f_1(\xi) &= g(\xi) + \int_0^\xi K(\xi, z) f_0(z) dz = A + B\xi + \frac{AC}{2!}\xi^2 + \frac{BC}{3!}\xi^3, \\
f_2(\xi) &= A + B\xi + \frac{AC}{2!}\xi^2 + \frac{BC}{3!}\xi^3 + \frac{AC^2}{4!}\xi^4 + \frac{BC^2}{5!}\xi^5, \\
f_3(\xi) &= A + B\xi + \frac{AC}{2!}\xi^2 + \frac{BC}{3!}\xi^3 + \frac{AC^2}{4!}\xi^4 + \frac{BC^2}{5!}\xi^5 + \frac{AC^3}{6!}\xi^6 + \frac{BC^3}{7!}\xi^7, \\
&\vdots
\end{aligned}$$

where

$$A = \frac{R - (a+b)\overline{U}(0)}{a^3},$$

$$B = \frac{-(a+b)\overline{U}'(0)}{a^3},$$

$$C = \frac{-(a+b)}{a^3}.$$

Now we give some numerical examples for fuzzy linear Volterra integral equation (7) when

- $x = t = 0.5, \quad a = b = 0.14, \quad R = 0.$
- $\overline{u}(0) = 0.1\alpha, \quad \underline{u}(0) = -0.1\alpha, \quad \overline{u}'(0) = 0.2\alpha, \quad \underline{u}'(0) = -0.2\alpha.$

Table 1: Numerical results for $\alpha = 0.2, \quad \beta = \frac{1}{2}, \quad \varepsilon = 10^{-8}$.

f_n	$\overline{u}_n$	$\underline{u}_n$
f_1	5.522200000	- 5.522200000
f_2	0.615228100	- 0.615228100
f_3	1.774414801	- 1.774414801
f_4	1.625915144	- 1.625915144
f_5	1.637819824	- 1.637819824
f_6	1.637167094	- 1.637167094
f_7	1.637193101	- 1.637193101
f_8	1.637192313	- 1.637192313
f_9	1.637192332	- 1.637192332

Table 1 shows that the approximations solution of the fuzzy linear Volterra integral equation is convergent to the exact solution with 9 iterations by using the successive approximations method.

Table 2: Numerical results for $\alpha = 0.1, \quad \beta = \frac{1}{4}, \quad \varepsilon = 10^{-9}$.

f_n	$\overline{u}_n$	$\underline{u}_n$
f_1	18.57083333	- 18.57083333
f_2	- 27.94484375	27.94484375
f_3	16.54190011	- 16.54190011
f_4	- 6.763676737	6.763676737
f_5	0.9144724671	- 0.9144724671
f_6	- 0.8204939397	0.8204939397
f_7	- 0.5351268291	0.5351268291
f_8	- 0.5708058636	0.5708058636
f_9	- 0.5673013330	0.5673013330
f_{10}	- 0.5675788580	0.5675788580
f_{11}	- 0.5675607654	0.5675607654
f_{12}	- 0.5675617532	0.5675617532
f_{13}	- 0.5675617074	0.5675617074
f_{14}	- 0.5675617093	0.5675617093
f_{15}	- 0.5675617092	0.5675617092

Table 2 shows that the approximations solution of the fuzzy linear Volterra integral equation is convergent to the exact solution with 15 iterations by using the successive approximations method.

5. Conclusion

In this paper the FFKdVE is reduced to an ordinary differential equation, then the equivalency this equation to the fuzzy integral equation is proved. The fuzzy L^p-spaces for $1 \leq p \leq \infty$ is introduced. Then the existence, uniqueness and convergence of the solution of this integral equation on L^p-spaces are proved. So it is concluded that the FFKdVE has a unique solution. At the end the successive approximation method is presented to get numerical solutions of this equation with some numerical examples.

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