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Analytical Solution of Time-Fractional Advection Dispersion Equation

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Abstract

In this paper, we get exact solution of the time-fractional advection-dispersion equation with reaction term, where the Caputo fractional derivative is considered of order $\alpha \in (0,2]$. The solution is achieved by using a function transform, Fourier and Laplace transforms to get the formulas of the fundamental solution, which are expressed explicitly in terms of Fox's *H*-function by making use of the relationship between Fourier and Mellin transforms. As special cases the exact solutions of time-fractional diffusion and wave equations are also obtained, and the solutions of the integer order equations are mentioned.

Keywords: Fractional Derivatives; Laplace Transform; Fourier Transform; Mellin Transform; Fox's H-function

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1. Introduction

Time fractional partial differential equations, obtained by replacing the first order time derivative by a fractional derivative (of order $0 < \alpha \le 2$, in Caputo sense) have been treated in different contexts by a number of researchers. Mainardi and Pagnini (2003) studied the time fractional diffusion equation and the fundamental solutions (Green functions) by using Fourier-Laplace transforms. Liu, Anh, Turner and Zhang (2003) considered the time fractional advection dispersion equation by replacing the first order derivative in time by a fractional derivative of order α $(0<\alpha\le 1)$ and they have used variable transformation, Mellin and Laplace transforms to achieve a complete solution. Saxena, Mathai and Haubold (2002, 2004a, 2004b, 2006a, 2006b, 2006c) and Haubold, Mathai and Saxena (2007) used integral transform methods to obtain exact solutions for the fractional kinetic, diffusion and reaction diffusion equations .Other research articles handling time fractional partial differential equations by using integral transforms can be found in the literature by a number of authors, see, e.g. the reviews in Mainardi (1996), Mainardi, Luchko and Pagnini (2001), Mainardi, Pagnini and Saxena (2005), Mainardi and Pagnini (2007), Mainardi, Pagnini and Gorenflo (2007), Momani and Odibat (2007) and Wang, Xu and Li (2007) and the references therein.

In this paper we study the time fractional advection dispersion equation with reaction

$$\frac{\partial^{\alpha}C(x,t)}{\partial t^{\alpha}}=D\frac{\partial^{2}C(x,t)}{\partial x^{2}}-b\frac{\partial C(x,t)}{\partial x}+\lambda C(x,t), \tag{1.1}$$

which describes the transient transport of solutes through a homogeneous soil, where

C: is the solute concentration (ML^{-3}),
t: is the time (T),
x: is the soil depth (L),
$(b>0)$: is the pore water velocity (LT^{-1}),
$(D>0)$: is the dispersion coefficient (L^2T^{-1}),
$(\lambda\ge 0)$: is the first order reaction rate coefficient (T^{-1}), and
α $(0<\alpha\le 2)$ is the order of the time fractional derivative which is intended in the Caputo sense.

For a detailed discussion on this fractional derivative, we refer to Podlubny (1999). When α is not integer ($\alpha\ne 1,2$), the Caputo fractional derivative is written as

$$\frac{\partial^{\alpha}C(x,t)}{\partial t^{\alpha}}=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t\left[\frac{\partial}{\partial\tau}C(x,\tau)\right]\frac{\partial\tau}{(t-\tau)^{\alpha}}, & if \quad 0<\alpha<1\\[2ex] \dfrac{1}{\Gamma(2-\alpha)}\displaystyle\int_0^t\left[\frac{\partial^2}{\partial\tau^2}C(x,\tau)\right]\frac{\partial\tau}{(t-\tau)^{\alpha-1}}, & if \quad 1<\alpha<2\end{cases} \tag{1.2}$$

and if α is an integer ($\alpha=1,2$), the Caputo fractional derivative is identical to the corresponding partial derivative of integer order.

Now, using the relation (see Gorenflo and Mainardi (1997))

$$J^{\alpha}\left(\frac{\partial^{\alpha}C(x,t)}{\partial t^{\alpha}}\right)=\begin{cases}C(x,t)-C(x,0^{+}), & if \quad 0<\alpha\leq 1\\ \\ C(x,t)-C(x,0^{+})-tC_{t}(x,0^{+}), & if \quad 1<\alpha\leq 2,\end{cases} \tag{1.3}$$

where $J^{\alpha} f(t)$ is the Riemann-Liouville fractional integral operator defined by

$$I^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1} f(\tau)d\tau\ , \tag{1.4}$$

then we can eliminate the time fractional derivative in (1.1) and obtain the integro-differential equation

$$C(x,t)=C\left(x,0^{+}\right)+\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}\left[D\frac{\partial^{2}C(x,\tau)}{\partial x^{2}}-b\frac{\partial C(x,\tau)}{\partial x}+\lambda C(x,\tau)\right]d\tau\ , \tag{1.5}$$

if $0<\alpha\leq 1$, and

$$\begin{aligned}C(x,t)=C\left(x,0^{+}\right)+tC_{t}\left(x,0^{+}\right)\\ +\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}\left[D\frac{\partial^{2}C(x,\tau)}{\partial x^{2}}-b\frac{\partial C(x,\tau)}{\partial x}+\lambda C(x,\tau)\right]d\tau,\end{aligned} \tag{1.6}$$

if $1<\alpha\leq 2$.

In order to correctly formulate and solve the Cauchy problem for (1.1), we have to select explicit initial conditions concerning $C\left(x,0^{+}\right)$, if $0<\alpha\leq 1$ and $C\left(x,0^{+}\right), C_{t}\left(x,0^{+}\right)$, if $1<\alpha\leq 2$. If $\phi(x)$ and $\psi(x)$ denote sufficiently well-behaved real functions defined on $\Re$, the Cauchy problem consists in finding the solution of (1.1) subjected to the initial conditions

$$C\left(x,0^{+}\right)=\phi(x) \quad ,x\in\Re,\quad if \quad 0<\alpha\leq 1, \tag{1.7a}$$

$$C\left(x,0^{+}\right)=\varphi(x),\ C_{t}\left(x,0^{+}\right)=\psi(x),\ x\in\Re, if \quad 1<\alpha\leq 2. \tag{1.7b}$$

Now, we give some basic definitions of the Laplace transform and the Fourier transform and some required formulas. The Laplace transform of a function $f(t)$ on $\Re^{+}$ is defined by Asmar (1999) as

$$\ell(f(t),p)=\int_{0}^{\infty} f(t)e^{-pt}dt=\tilde{f}(p),\quad \mathrm{Re}(p)>0 \tag{1.8}$$

and the Laplace transform for the Caputo derivative $\dfrac{\partial^{\alpha} f(t)}{\partial t^{\alpha}}$, $(0 < \alpha \le 2)$ is intended to be

$$\ell\left\{\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}}; p\right\} = \begin{cases} p^{\alpha}\tilde{f}(p) - p^{\alpha-1} f(0^{+}), & if \quad 0 < \alpha \le \\ 1, & \\ p^{\alpha}\tilde{f}(p) - p^{\alpha-1} f(0^{+}) - p^{\alpha-2} f_t(0^{+}), & if \quad 1 < \alpha \le 2, \end{cases} \tag{1.9}$$

and the inverse Laplace transform is written as

$$f(t) = \ell^{-1}\left\{\tilde{f}(p), t\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\tilde{f}(p)\,dp, \quad \gamma = \mathrm{Re}(p). \tag{1.10}$$

The Fourier transform of a function $f(x)$ on $\Re$ is defined by Mainardi and Pagnini (2003) as

$$\Im\{f(t);\kappa\} = \hat{f}(\kappa) = \int_{-\infty}^{\infty} e^{i\kappa t} f(t)\,dt \tag{1.11}$$

and its inverse is written as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\kappa)e^{i\kappa t}\,d\kappa. \tag{1.12}$$

In application of Fourier transforms to physical problems, it is useful to have the formula:

$$\Im\left\{f^{(n)}(x);\kappa\right\} = (-i\kappa)^{n}\tilde{f}(\kappa), \tag{1.13}$$

where $f^{(n)}(x) = \dfrac{d^{n} f(x)}{dx^{n}}, \quad and \quad i = \sqrt{-1}$.

2. The Green Function

To reduce (1.5) and (1.6) to a more familiar form, we use the following function transform. Let

$$C(x,t) = u(\zeta,t)\exp(\mu\zeta),\ \zeta = \frac{x}{\sqrt{D}},\ \mu = \frac{b}{2\sqrt{D}}. \tag{2.1}$$

Then, (1.5) and (1.6) yield the integro-differential equations

$$u(\zeta,t)=\Phi(\zeta)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\frac{\partial^2 u(\zeta,\tau)}{\partial\zeta^2}-\theta^2 u(\zeta,\tau)\right]d\tau\,, \tag{2.2}$$

for $0<\alpha\leq 1$, and

$$u(\zeta,t)=\Phi(\zeta)+t\Psi(\zeta)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\frac{\partial^2 u(\zeta,\tau)}{\partial\zeta^2}-\theta^2 u(\zeta,\tau)\right]d\tau\,, \tag{2.3}$$

for $1<\alpha\leq 2$, where $\Phi(\zeta)=\phi\left(\zeta\sqrt{D}\right)\exp(-\mu\zeta)$, $\Psi(\zeta)=\psi\left(\zeta\sqrt{D}\right)\exp(-\mu\zeta)$

and $\theta^2=\mu^2+\lambda$.

Now, applying the Laplace transform with respect to t and the Fourier transform with respect to ζ, then equations (2.2) and (2.3) yield

$$\hat{\tilde{u}}(\kappa,p)=\frac{p^{\alpha-1}}{p^\alpha+\kappa^2+\theta^2}\hat{\Phi}(\kappa), \quad \text{for } 0<\alpha\leq 1 \tag{2.4}$$

and

$$\hat{\tilde{u}}(\kappa,p)=\frac{p^{\alpha-1}}{p^\alpha+\kappa^2+\theta^2}\hat{\Phi}(\kappa)+\frac{p^{\alpha-2}}{p^\alpha+\kappa^2+\theta^2}\hat{\Psi}(\kappa), \text{ for } 1<\alpha\leq 2\,. \tag{2.5}$$

By fundamental solutions (or Green functions) of the above Cauchy problems, we mean the solutions corresponding to initial conditions

$$G_\alpha^{(1)}\left(\zeta,0^+\right)=\delta(\zeta), \quad \text{for} \quad 0<\alpha\leq 1 \tag{2.6a}$$

$$\begin{cases} G_\alpha^{(1)}\left(\zeta,0^+\right)=\delta(\zeta) \\ \dfrac{\partial}{\partial t}G_\alpha^{(1)}\left(\zeta,0^+\right)=0 \end{cases} \quad \begin{cases} G_\alpha^{(2)}\left(\zeta,0^+\right)=0 \\ \\ \dfrac{\partial}{\partial t}G_\alpha^{(2)}\left(\zeta,0^+\right)=\delta(\zeta) \end{cases}, \quad \text{for} \quad 1<\alpha\leq 2\,, \tag{2.6b}$$

where $\delta(\zeta)$ is the Dirac-delta generalized function whose Fourier transform is known to be one ($\hat{\delta}(\zeta)=1$). Thus, the Fourier-Laplace transforms of these Green functions are

$$\hat{\tilde{G}}_\alpha^{(j)}(\kappa,p)=\frac{p^{\alpha-j}}{p^\alpha+\kappa^2+\theta^2}, \quad 0<\alpha\leq 2, \quad j=1,2\,. \tag{2.7}$$

Note that by making use of the formula [see Asmar (1999)]

$$\Im^{-1}\left\{\frac{a_j}{b+\kappa^2};\zeta\right\}=\frac{a_j}{2\sqrt{b}}e^{-|\zeta|\sqrt{b}}, \quad b>0 \tag{2.8}$$

and setting $a_j = p^{\alpha-j}, \quad b = p^{\alpha}+\theta^2$, we get

$$\tilde{G}_{\alpha}^{(j)}(\zeta,p)=\frac{p^{\alpha-j}}{2\sqrt{p^{\alpha}+\theta^2}}e^{-|\zeta|\sqrt{p^{\alpha}+\theta^2}}, \quad j=1,2. \tag{2.9}$$

It is obvious from (2.9) that $\tilde{G}_{\alpha}^{(2)}(\zeta,p)=\tilde{G}_{\alpha}^{(1)}(\zeta,p)/p$. So,

$$G_{\alpha}^{(2)}(\zeta,t)=\int_0^t G_{\alpha}^{(1)}(\zeta,\tau)d\tau. \tag{2.10}$$

This means that it is enough to obtain $G_{\alpha}^{(1)}(\zeta,t)$ since the other Green function $G_{\alpha}^{(2)}(\zeta,t)$ can be obtained by using (2.10) if $G_{\alpha}^{(1)}(\zeta,t)$ is known. Unfortunately, inverting the Laplace transform from equation (2.9) is problematic.

However, we can invert the Laplace transform in equation (2.7) by first rewriting $\hat{\tilde{G}}_{\alpha}^{(1)}(\kappa,p)$ in the form

$$\hat{\tilde{G}}_{\alpha}^{(1)}(\kappa,p)=\frac{p^{\alpha-1}}{p^{\alpha}+\kappa^2}\cdot\frac{1}{1+\dfrac{\theta^2}{p^{\alpha}+\kappa^2}}. \tag{2.11}$$

Now, expanding the second fraction and simplifying, we get

$$\hat{\tilde{G}}_{\alpha}^{(1)}(\kappa,p)=\sum_{m=0}^{\infty}\frac{(-\theta^2)^m p^{\alpha-1}}{(p^{\alpha}+\kappa^2)^{m+1}}. \tag{2.12}$$

Then, by making use of the inverting Laplace transform formula (see Podlubny (1999))

$$\ell^{-1}\left\{\frac{p^{\alpha-\beta}}{(p^{\alpha}+a)^{m+1}};t\right\}=\frac{t^{\alpha m+\beta-1}}{m!}E_{\alpha,\beta}^{(m)}(-at^{\alpha}), \tag{2.13}$$

where

$$E_{\alpha,\beta}^{(m)}(t) = \frac{d^m E_{\alpha,\beta}(t)}{dt^m} = \sum_{j=0}^{\infty} \frac{(j+m)!\,t^j}{j!\,\Gamma(\alpha(j+m)+\beta)} \tag{2.14}$$

is the derivative of the Mittag-Leffler function $E_{\alpha,\beta}(t)$, we obtain

$$\hat{G}_{\alpha}^{(1)}(\kappa,t) = \sum_{m=0}^{\infty} \frac{\left(-\theta^2\right)^m t^{\alpha m}}{m!} E_{\alpha}^{(m)}\left(-\kappa^2 t^{\alpha}\right). \tag{2.15}$$

Langlands (2006) has shown that the Fourier inverse of the derivative of the Mittag-Leffler functions in equation (2.15) can be obtained by first rewriting the derivative in terms of *H*-function

$$E_{\alpha}^{(m)}\left(-\kappa^2 t^{\alpha}\right) = H_{1,2}^{1,1}\left[\kappa^2 t^{\alpha} \,\middle|\, \begin{matrix} (-m,1) \\ (0,1),(-\alpha m,\alpha) \end{matrix}\right]. \tag{2.16}$$

In order to evaluate the inverse Fourier transform of the *H*-function, we need the following relationship between Fourier transform and Mellin transform (see Langlands (2006))

$$M\left\{\hat{f}(\kappa);s\right\} = 2\Gamma(s)\cos\left(\frac{\pi s}{2}\right) M\left\{f(x);1-s\right\}, \tag{2.17}$$

where the Mellin transform of *f(x)* is written as

$$M\left\{f(x);s\right\} = \int_0^{\infty} x^{s-1} f(x)\,dx. \tag{2.18}$$

To find the Mellin transform of equation (2.16), we note that the Mellin transform of Fox's *H*-function is given by Kilbas and Saigo (2004), (see also Srivastava, Gupta and Goyal (1982) and Mathai and Saxena (1978)).

$$M\left\{H_{p,q}^{m,n}\left[ax \,\middle|\, \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix}\right];s\right\} = a^{-s} \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}, \tag{2.19}$$

when the following conditions are satisfied

$$\delta = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j > 0, \quad A = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j+1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j > 0,$$

$$\left|\arg(a)\right| < \frac{A\pi}{2} \quad and \quad -\min_{1\le j\le m}\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right] < \mathrm{Re}(s) < \min_{1\le j\le n}\left[\mathrm{Re}\left(\frac{1-a_j}{\alpha_j}\right)\right].$$

Another useful identity is [see Oberhettinger (1974)]

$$M\left\{x^{\nu} f\left(ax^{\gamma}\right); s\right\} = \frac{1}{\gamma} a^{-(s+\nu)/\gamma} M\left\{ f(x); \frac{s+\nu}{\gamma}\right\}, \tag{2.20}$$

for $\gamma > 0$, $a > 0$. We now invert equation (2.15) by using equations (2.19) and (2.20) along with equation (2.7) to get

$$M\left\{G_{\alpha}^{(1)}(\zeta,t); s\right\} = \frac{t^{-\frac{\alpha}{2}}}{4\pi} \sum_{m=0}^{\infty} \frac{\left(-\theta^2\right)^m}{m!} \cdot t^{\frac{\alpha s}{2}} \frac{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+m+\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}{\Gamma(1-s)\Gamma\left(1+\alpha m-\frac{\alpha}{2}+\frac{\alpha s}{2}\right)}. \tag{2.21}$$

Comparing with equation (19), we fined by inverting the Mellin transform that

$$G_{\alpha}^{(1)}(\zeta,t) = \frac{t^{-\alpha/2}}{4\pi} \sum_{m=0}^{\infty} \frac{\left(-\theta^2 t^{\alpha}\right)^m}{m!} \times H_{3,3}^{2,2}\left[\zeta\, t^{-\alpha/2} \left| \begin{matrix} (1/2,1/2),(0,1/2),\left(1+\alpha m-\frac{\alpha}{2},\alpha/2\right) \\ \left(\frac{1}{2}+m,1/2\right),(0,1/2),(0,1) \end{matrix} \right.\right]. \tag{2.22}$$

Now, applying the relation [see Srivastava, Gupta and Goyal (1982)]

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} H_{p,q}^{m,n}\left[zx^{\mu}(t-x)^{\nu} \left| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right.\right] dx$$

$$= t^{\rho+\sigma-1} H_{p+2,q+2}^{m,n+2}\left[zt^{\mu+\nu} \left| \begin{matrix} (1-\rho,\mu),(1-\sigma,\nu),(a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q},(1-\rho-\sigma,\mu+\nu) \end{matrix} \right.\right] \tag{2.23}$$

to equation (2.10), and eliminating we get

$$G_{\alpha}^{(2)}(\zeta,t)=\frac{t^{-\frac{\alpha}{2}+1}}{4\pi}\sum_{m=0}^{\infty}\frac{\left(-\theta^{2}t^{\alpha}\right)^{m}}{m!}\times H_{3,3}^{2,2}\left[\zeta\, t^{-\alpha/2}\middle|\begin{matrix}(1/2,1/2),(0,1/2),\left(2+\alpha m-\frac{\alpha}{2},\alpha/2\right)\\ \left(\frac{1}{2}+m,1/2\right),(0,1/2),(0,1)\end{matrix}\right]. \quad (2.24)$$

3. The Complete Solution and Special Cases

The Green function allows us to represent the solutions of the Cauchy problems through the relevant integral formulas

$$u(\zeta,t)=\int_{-\infty}^{\infty}G_{\alpha}^{(1)}(y,t)\Phi(\zeta-y)dy,\quad 0<\alpha\leq 1, \quad (3.1)$$

$$u(\zeta,t)=\int_{-\infty}^{\infty}\left[G_{\alpha}^{(1)}(y,t)\Phi(\zeta-y)+G_{\alpha}^{(2)}(y,t)\Psi(\zeta-y)\right]d\zeta,\quad 1<\alpha\leq 2. \quad (3.2)$$

So, substituting equations (2.22) and (2.24) in (3.1) and (3.2), respectively, we get

$$u(\zeta,t)=\frac{t^{-\alpha/2}}{4\pi}\sum_{m=0}^{\infty}\frac{\left(-\theta^{2}t^{\alpha}\right)^{m}}{m!} \quad (3.3)$$
$$\times\int_{-\infty}^{\infty}H_{3,3}^{2,2}\left[y\, t^{-\alpha/2}\middle|\begin{matrix}(1/2,1/2),(0,1/2),\left(1+\alpha m-\frac{\alpha}{2},\alpha/2\right)\\ \left(\frac{1}{2}+m,1/2\right),(0,1/2),(0,1)\end{matrix}\right]\cdot\Phi(\zeta-y)\,dy,$$

for $0<\alpha\leq 1$, and

$$u(\zeta,t)=\frac{t^{-\alpha/2}}{4\pi}\sum_{m=0}^{\infty}\frac{\left(-\theta^{2}t^{\alpha}\right)^{m}}{m!}\int_{-\infty}^{\infty}\left\{\begin{matrix}H_{3,3}^{2,2}\left[y\, t^{-\alpha/2}\middle|\begin{matrix}(1/2,1/2),(0,1/2),\left(1+\alpha m-\frac{\alpha}{2},\alpha/2\right)\\ \left(\frac{1}{2}+m,1/2\right),(0,1/2),(0,1)\end{matrix}\right]\cdot\Phi(\zeta-y)\\ +tH_{3,3}^{2,2}\left[y\, t^{-\alpha/2}\middle|\begin{matrix}(1/2,1/2),(0,1/2),\left(2+\alpha m-\frac{\alpha}{2},\alpha/2\right)\\ \left(\frac{1}{2}+m,1/2\right),(0,1/2),(0,1)\end{matrix}\right]\cdot\Psi(\zeta-y)\end{matrix}\right\}dy, \quad (3.4)$$

for $1<\alpha\leq 2$.

Now, applying the substitutions in (2.1) allows us to get explicit forms for $C(x,t)$ as

$$C(x,t)=\frac{e^{-\frac{xb}{D}}t^{-\alpha/2}}{4\pi}\sum_{m=0}^{\infty}\frac{\left(-\theta^2t^{\alpha}\right)^m}{m!}$$
$$\times\int_{-\infty}^{\infty}H_{3,3}^{2,2}\left[y\,t^{-\alpha/2}\left|\begin{matrix}(1/2,1/2),(0,1/2),\left(1+\alpha m-\frac{\alpha}{2},\alpha/2\right)\\ \left(\frac{1}{2}+m,1/2\right),(0,1/2),(0,1)\end{matrix}\right.\right]\cdot\varphi\left(x-\sqrt{D}y\right)e^{\frac{b}{2\sqrt{D}}y}dy, \tag{3.5}$$

for $0<\alpha\leq 1$, and

$$C(x,t)=\frac{e^{-\frac{xb}{D}}t^{-\alpha/2}}{4\pi}\sum_{m=0}^{\infty}\frac{\left(-\theta^2t^{\alpha}\right)^m}{m!}$$
$$\times\int_{-\infty}^{\infty}\left\{\begin{matrix}H_{3,3}^{2,2}\left[y\,t^{-\alpha/2}\left|\begin{matrix}(1/2,1/2),(0,1/2),\left(1+\alpha m-\frac{\alpha}{2},\alpha/2\right)\\ \left(\frac{1}{2}+m,1/2\right),(0,1/2),(0,1)\end{matrix}\right.\right]\cdot\phi\left(x-\sqrt{D}y\right)\\ +tH_{3,3}^{2,2}\left[y\,t^{-\alpha/2}\left|\begin{matrix}(1/2,1/2),(0,1/2),\left(2+\alpha m-\frac{\alpha}{2},\alpha/2\right)\\ \left(\frac{1}{2}+m,1/2\right),(0,1/2),(0,1)\end{matrix}\right.\right]\cdot\psi\left(x-\sqrt{D}y\right)\end{matrix}\right\}e^{\frac{b}{2\sqrt{D}}y}dy, \tag{3.6}$$

for $1<\alpha\leq 2$ and $\theta^2=\dfrac{b^2}{4D}+\lambda$.

Now, if we consider the limiting case $\theta\to 0$ (i.e., $b\to 0$ and $\lambda\to 0$), then we obtain the solution of the time-fractional diffusion equation

$$\frac{\partial^{\alpha}C(x,t)}{\partial t^{\alpha}}=D\frac{\partial^2C(x,t)}{\partial x^2},\quad 0<\alpha\leq 1. \tag{3.7}$$

Subject to initial conditions $C(x,0^+)=\phi(x)$, directly from equation (3.5) as

$$C(x,t)=\frac{t^{-\alpha/2}}{4\pi}\int_{-\infty}^{\infty}H_{3,3}^{2,2}\left[y\,t^{-\alpha/2}\left|\begin{matrix}(1/2,1/2),(0,1/2),\left(1-\frac{\alpha}{2},\alpha/2\right)\\(1/2,1/2),(0,1/2),(0,1)\end{matrix}\right.\right]\cdot\varphi\left(x-\sqrt{D}y\right)dy. \tag{3.8}$$

Making use of the known property of the Gamma function $\Gamma(1-z)\Gamma(z)=\dfrac{\pi}{\sin\pi z}$ and simplify, then equation (3.8) can be written as

$$C(x,t)=\frac{t^{-\alpha/2}}{2}\int_{-\infty}^{\infty}H_{1,1}^{1,0}\left[y\,t^{-\alpha/2}\left|\begin{matrix}\left(1-\frac{\alpha}{2},\alpha/2\right)\\(0,1)\end{matrix}\right.\right]\cdot\phi\left(x-\sqrt{D}y\right)dy. \tag{3.9}$$

Similarly, the solution of the time-fractional wave equation

$$\frac{\partial^{\alpha}C(x,t)}{\partial t^{\alpha}}=D\frac{\partial^{2}C(x,t)}{\partial x^{2}},\quad 1<\alpha\le 2, \tag{3.10}$$

with initial conditions $C(x,0^{+})=\phi(x)$ and $C_t(x,0^{+})=\psi(x)$ can be achieved directly from equation (3.6), and after simplifying it can be written as

$$C(x,t)=\frac{t^{-\alpha/2}}{2}\int_{-\infty}^{\infty}H_{1,1}^{1,0}\left[y\,t^{-\alpha/2}\left|\begin{matrix}\left(1-\frac{\alpha}{2},\alpha/2\right)\\(0,1)\end{matrix}\right.\right]\cdot\phi\left(x-\sqrt{D}y\right)dy$$

$$+\frac{t^{1-\alpha/2}}{2}\int_{-\infty}^{\infty}H_{1,1}^{1,0}\left[y\,t^{-\alpha/2}\left|\begin{matrix}\left(2-\frac{\alpha}{2},\alpha/2\right)\\(0,1)\end{matrix}\right.\right]\cdot\psi\left(x-\sqrt{D}y\right)dy. \tag{3.11}$$

The results of equation (3.9) and equation (3.10) are in full agreement with the results of equation (3.10) and equation (4.9) recently obtained by Mainardi and Pagnini (2003) and Mainardi, Pagnini and Saxena (2005), respectively.

The integer order advection-dispersion equation is obtained by setting $\alpha=1$ in equation (1.1), and its solution yields directly from equation (3.5) as

$$C(x,t)=\frac{e^{-\frac{xb}{D}}t^{-1/2}}{4\pi}\sum_{m=0}^{\infty}\frac{\left(-\theta^{2}t\right)^{m}}{m!}\times\int_{-\infty}^{\infty}H_{3,3}^{2,1}\left[y\,t^{-1/2}\left|\begin{matrix}(1/2,1/2),(0,1/2)\\(0,1/2),(0,1)\end{matrix}\right.\right]\cdot\phi\left(x-\sqrt{D}y\right)e^{\frac{b}{2\sqrt{D}}y}dy, \tag{3.12}$$

and the solutions of the integer order diffusion and wave equations can be obtained by setting $\alpha = 1$ in equation (3.9) and $\alpha = 2$ in equation (3.11).

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