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Integrability and Exact Solutions for a (2+1)-dimensional Variable-Coefficient KdV Equation

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Abstract

By using the WTC method and symbolic computation, we apply the Painlevé test for a (2+1)-dimensional variable-coefficient Kortweg-de Vries (KdV) equation, and the considered equation is found to possess the Painlevé property without any parametric constraints. The auto-Bǎcklund transformation and several types of exact solutions are obtained by using the Painlevé truncated expansion method. Finally, the Hirota's bilinear form is presented and multi-soliton solutions are also constructed.

Keywords: (2+1)-dimensional variable-coefficient KdV equation; Painlevé property; Hirota's bilinear form; soliton solution; symbolic computation

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1. Introduction

Due to the potential applications of soliton theory in mathematics, physics, biology, communications and astrophysics, the studies on the nonlinear evolution equations (NLEEs) have attracted researchers' attention. Many scientists have concentrated their researches on sorts of exact solutions and remarkable properties of NLEEs, which attributes to have a better understanding for those nonlinear mechanism [Ablowitz and Satsuma (1978), Yan (1996), Peng (2005)]. Up to now, various powerful methods have been presented to obtain the exact solutions for the NLEEs, such as the inverse scattering transformation [Ablowitz and Clarkson (1991)], the Darboux transformation [Matveev and Salle (1991)], the Bǎcklund transformations [Rogers and Shadwick (1982)], the Hirota's bilinear method [Hirota (2004).], the Painlevé analysis [Weiss, Tabor and Carnevale (1983)], homotopy perturbation method[Ganji and Rafei (2006)], tanh function method [Ma and Fuchssteiner (1996), Wazwaz (2005)], sine-cosine method [Abdou (2007)], the homogenous balance method [Fan and Zhang (1998)], variational iteration method [Tang and Liang (2006), Peng (2006)], exp-function method [He and Wu (2006)], Jacobi elliptic function expansion method [Chen et al. (2005)], F-expansion method [Abdou (2007)], the sub-ODE method [Li and Wang (2007)], and so on.

Considering that coefficient functions are able to reflect the slowly-varying inhomogeneities, nonuniformities of boundaries, and external forces, recently nonlinear equations with variable coefficients have attracted considerable attention in the literature. Although the variable coefficients increase the difficulty of our investigation, many researchers have investigated the integrable property and exact solutions of the variable-coefficient nonlinear evolution equations [Deng (2006), Xu (2009), Lü et al. (2010), Kraenkel et al. (2011)]. It is easily found that solitons can be compressed and their dynamics effectively controlled through these variable parameters.

The Kortweg-de Vries equation is one of the most important systems in mathematical physics and many physical situations are governed by variable-coefficient KdV equation. To our knowledge, the study of the (1+1)-dimensional variable-coefficient KdV equation with different form has been paid attention by some authors, and the authors have obtained the integrable property and exact solutions of these equations [Fan (2011), Yan (2008), Zhu et al. (2010)]. Furthermore, there are some researches about the applications and numerous interesting properties of the (2+1)-dimensional variable-coefficient KdV equation. For example, Emmanuel Yomba et al. constructed the exact solutions of the following equation:

$$(u_t + 6uu_x + u_{xxx})_x + e(t)u_x + n(t)u_{yy} = 0, \tag{1}$$

by means of the tanh function method, the homogenous balance method and other methods [Yomba (2004), Moussa and M. El-Shiekh (2011)]. Very recently, Peng et al. [Peng (2010)] derived a new (2+1)-dimensional KdV equation with variable coefficients via the Lax pair generating technique, the equation is given as follows:

$$4u_t - a(y,t)(4uu_y + 2\int u_y dx u_x + u_{xxy}) - b(y,t)(6uu_x + u_{xxx}) = 0, \tag{2}$$

where a(y, t) and b(y, t) are two arbitrary functions of indicated variables. To our knowledge, equation (2) was considered for special choices with $a(y,t) = b(y,t) = -4$. Therefore, in this paper, we will study the Painlevé property and analytic solutions of (2) in the general form. The rest of this paper is arranged as follows. In Section 2, we perform the Painlevé analysis of (2). It is proven that equation (2) possesses the Painlevé property without any parametric constraints. In

Section 3, several types of exact solutions will be constructed in terms of the Auto-Bǎcklund transformation. In Section 4, the Hirota's bilinear form of (2) will be presented, and the multi-soliton solutions will be derived for equation (2) with a special choice of parameters. Finally, our conclusions and discussions will be given in Section 5.

2. Painlevé analysis

The integrable classes of KdV equations intrigue researchers for the past few decades due to their rich variety of solutions. The Painlevé analysis is one of the powerful methods for identifying the integrable properties of nonlinear partial differential equations. In this section we apply the Painlevé test for integrability to (2), using the well-known WTC method and symbolic computation [Xu and Li (2003), (2004), Qu et al. (2010)]. In order to apply the Painlevé analysis, we introduce the transformation $u_y = v_x$, and equation (2) reduces to the following coupled system:

$$\begin{aligned} &4u_t - a(y,t)(4uu_y + 2vu_x + u_{xxy}) - b(y,t)(6uu_x + u_{xxx}) = 0, \\ &u_y = v_x. \end{aligned} \tag{3}$$

Equations (3) is said to possess the Painlevé property if its solutions are "single-valued" about arbitrary non-characteristic, movable singularity manifolds. In other words, all solutions of (3) can be expressed as Laurent series,

$$u(x,y,t) = \sum_{j=0}^{\infty} u_j \phi^{j+\alpha}, \qquad v(x,y,t) = \sum_{j=0}^{\infty} v_j \phi^{j+\beta}, \tag{4}$$

with sufficient number of arbitrary functions among u_j, v_j in addition to ϕ, and ϕ, u_j, v_j are analytic functions of x, y, t. Moreover, the leading orders α, β should be negative integers.

First, find the leading order and coefficients. To reach this aim, we insert

$$u(x,y,t) = u_0\phi^{\alpha}, \qquad v(x,y,t) = v_0\phi^{\beta} \tag{5}$$

into equations (3). Upon balancing the dominant terms, we obtain,

$$\alpha = \beta = -2, \quad u_0 = -2\phi_x^{\ 2}, \quad v_0 = -2\phi_x\phi_y. \tag{6}$$

Next, in order to find the resonances that are powers at which the arbitrary coefficients enter into the Laurent series (4), we substitute,

$$u(x,y,t) = u_0\phi^{-2} + u_j\phi^{-2+j}, \qquad v(x,y,t) = v_0\phi^{-2} + u_j\phi^{-2+j} \tag{7}$$

into equations (3). Inserting equations (6) and vanishing the coefficients of (ϕ^{j-5}, ϕ^{j-3}) yields,

$$\begin{bmatrix} a(y,t)\phi_y(j^3-9j^2+14j+16)+b(y,t)\phi_x(j+1)(j-4)(j-6) & 8a(y,t)\phi_x \\ (2-j)\phi_y & (j-2)\phi_x \end{bmatrix}\begin{bmatrix} u_j \\ v_j \end{bmatrix}=0. \tag{8}$$

The eigenvalues of the above matrix gives the following resonance equation for the exponent *j*:

$$(j-2)(j-4)(j-6)(j+1)\phi_x(a(y,t)\phi_y+b(y,t)\phi_x)=0. \tag{9}$$

Thus the resonances occur at $j=-1,2,4,6$. As usual, the resonance $j=-1$ corresponds to the arbitrariness of the singular manifold $\phi(x,y,t)$.

In order to check the existence of a sufficient number of arbitrary functions at the resonance values, the truncated Laurent expansions

$$u(x,y,t)\sim -2\phi^{-2}+\phi^{-2}\sum_{j=1}^{6}u_j\phi^j, \qquad v(x,y,t)\sim -2\phi^{-2}+\phi^{-2}\sum_{j=1}^{6}v_j\phi^j \tag{10}$$

are substituted into equations (3). To simplify the calculations, we make use of the Kruskal ansatz $\phi(x,y,t)=x+\psi(y,t)$, and ψ is an arbitrary function of *y* and *t*. Then the coefficients functions u_j and v_j in equations (10) will be functions of *y*, and *t*, too.

Equating the coefficients of (ϕ^{-4},ϕ^{-2}) to zero, we obtain a linear system with respect to u_1 and v_1,

$$\begin{aligned} &v_1-u_1\psi_y=0,\\ &4a(y,t)v_1+15b(y,t)u_1+11a(y,t)\psi_y u_1=0, \end{aligned} \tag{11}$$

from which we have,

$$u_1=v_1=0. \tag{12}$$

Setting the coefficients of (ϕ^{-3},ϕ^{-1}) to zero, we obtain a linear system with respect to u_2 and v_2,

$$\begin{aligned} &a(y,t)(8u_2\psi_y-v_1u_1-2u_1^2\psi_y-3u_{1y}+4v_2)+12b(y,t)u_2-3b(y,t)u_1^2-8\psi_t=0,\\ &u_{1y}=0. \end{aligned} \tag{13}$$

Solving it, we get,

$$v_2=\frac{2\psi_t-3b(y,t)u_2-2a(y,t)u_2\psi_y}{a(y,t)}, \tag{14}$$

where u_2 is an arbitrary function which corresponds to the resonance $j=2$.

Collecting the coefficients of (ϕ^{-2},ϕ^{0}) and using the values of u_0,v_0,u_1,v_1 and v_2, we have,

$$
\begin{aligned}
&u_{2y}+u_3\psi_y-v_3=0,\\
&3b(y,t)u_3-2a(y,t)u_{2y}+a(y,t)u_3\psi_y+2a(y,t)v_3=0.
\end{aligned}
\tag{15}
$$

Solving the above equations with respect to u_3,v_3 yields,

$$
u_3=0,\qquad v_3=u_{2y}. \tag{16}
$$

In this way by proceeding further and collecting the coefficients of $(\phi^{-1},\phi\ )$, (ϕ^{0},ϕ^{2}) and $(\phi\ ,\phi^{3})$, one can obtain other coefficients of Laurent series (4),

$$
\begin{aligned}
v_4&=u_4\psi_y,\\
u_5&=-\frac{6u_{2t}-6au_2u_{2y}+5au_{4y}}{9(b+a\psi_y)},\\
v_5&=-\frac{-3bu_{4y}+2au_{4y}\psi_y+6u_{2t}\psi_y-6au_2u_{2y}\psi_y}{9(b+a\psi_y)},\\
v_6&=[6a^2\psi_y u_{2y}^2+6abu_{2y}^2-6au_{2yt}\psi_y-5a^2\psi_y u_{4yy}-5ba_y u_{4y}\\
&\quad+36a^2u_6\psi_y^3+36b^2u_6\psi_y-6a^2u_2u_{2y}\psi_{yy}-6au_2u_{2y}b_y\\
&\quad+6a_yu_{2t}\psi_y-6bu_{2yt}+6au_{2t}\psi_{yy}+6b_yu_{2t}+5a^2\psi_{yy}u_{4y}\\
&\quad+5ab_yu_{4y}+6a^2u_2u_{2yy}\psi_y+6abu_2u_{2yy}+6ba_yu_2u_{2y}\\
&\quad+72abu_6\psi_y^2]/[36(a^2\psi_y^2+b^2+2ab\psi_y)],
\end{aligned}
\tag{17}
$$

where $a\equiv a(y,t), b\equiv b(y,t)$, and u_4, u_6 are arbitrary functions which correspond to the resonances $j=4,6$.

Up to now, we establish the required number of arbitrary functions corresponding to $j=2$, $j=4$ and $j=6$ without any additional restrictions on the parameters. Thus we can conclude that equations (3) has Painlevé property and hence is expected to be integrable.

3. Auto-Bǎcklund transformation and exact solutions

In order to get auto-Bǎcklund transformation, we may truncate the Laurent series (4) at the constant terms, namely,

$$u=\phi^{-2}u_0+\phi^{-1}u_1+u_2, \qquad v=\phi^{-2}v_0+\phi^{-1}v_1+v_2, \tag{18}$$

Where $\phi, u_j, v_j\,(j=0,1,2)$ are analytic functions of x, y, t. Then substituting (18) into (3) and setting the coefficients of $\phi(x,y,t)$ with different powers to zero yields,

$$
\begin{aligned}
&v_0\varphi_x-u_0\varphi_y=0,\\
&u_{0y}-v_{0x}+v_1\varphi_x-u_1\varphi_y=0,\\
&u_{1y}-v_{1x}=0,\\
&u_0(3bu_0\varphi_x+2au_0\varphi_y+av_0\varphi_x+6b\varphi_x^3+6a\varphi_x^2\varphi_y=0,\\[6pt]
&9bu_0u_1\varphi_x-6au_{0x}\varphi_x\varphi_y-9bu_{0x}\varphi_x^2+6au_0u_1\varphi_y+2au_0v_1\varphi_x-6au_0\varphi_x\varphi_{xy}\\
&\quad -av_0u_{0x}+av_0u_1\varphi_x+3bu_1\varphi_x^3-3au_0\varphi_{xx}\varphi_y+3au_1\varphi_x^2\varphi_y\\
&\quad -3au_{0y}\varphi_x^2-9bu_0\varphi_{xx}\varphi_x-3bu_0u_{0x}-2au_0u_{0y}=0,\\[6pt]
&2au_{0xy}\varphi_x-4u_0\varphi_t+3bu_1^2\varphi_x-3bu_{1x}\varphi_x^2-3bu_1u_{0x}-av_1u_{0x}-2au_1u_{0y}-2au_0u_{1y}\\
&\quad -av_0u_{1x}+2av_2u_0\varphi_x+av_1u_1\varphi_x+4au_2u_0\varphi_y-2au_{1x}\varphi_x\varphi_y+6bu_2u_0\varphi_x\\
&\quad +2au_1^2\varphi_y-au_{1y}\varphi_x^2-3bu_0u_{1x}+2au_{0x}\varphi_{xy}-2au_1\varphi_x\varphi_{xy}-au_1\varphi_{xx}\varphi_y\\
&\quad -3b\varphi_{xx}\varphi_x+3bu_{0x}\varphi_{xx}+3bu_{0xx}\varphi_x+au_0\varphi_{xxy}+bu_0\varphi_{xxx}+au_{0y}\varphi_{xx}+au_{0xx}\varphi_y=0,\\[6pt]
&6bu_1u_2\varphi_x-6bu_2u_{0x}+3bu_{1x}\varphi_{xx}+3bu_{1xx}\varphi_x-au_{0xxy}+au_1\varphi_{xxy}\\
&\quad -6bu_1u_{1x}+4u_{0t}+2au_{1xy}\varphi_x+au_{1y}\varphi_{xx}+2au_1v_2\varphi_x\\
&\quad +2au_{1x}\varphi_{xy}+bu_1\varphi_{xxx}-2av_2u_{0x}-4au_0u_{2y}-4au_2u_{0y}\\
&\quad -6bu_0u_{2x}+4au_1u_2\varphi_y-2av_0u_{2x}-2av_1u_{1x}-4au_1u_{1y}\\
&\quad -bu_{0xxx}-4u_1\varphi_t+au_{1xx}\varphi_y=0,\\[6pt]
&4u_{1t}-au_{1xxy}-2av_1u_{2x}-6bu_1u_{2x}-2av_2u_{1x}-bu_{1xxx}\\
&\qquad -4au_1u_{2y}-4au_2u_{1y}-6bu_{1x}u_2=0,\\[6pt]
&4u_{2t}-6bu_2u_{2x}-4au_2u_{2y}-bu_{2xxx}-au_{2xxy}-2au_{2x}v_2=0,\\
&u_{2y}-v_{2x}=0.
\end{aligned}
\tag{19}$$

From (19), one can easily obtain,

$$u_0=-2\phi_x, \quad v_0=-2\phi_x\phi_y, \quad u_1=2\phi_x^2, \quad v_1=2\phi_{xy}. \tag{20}$$

From (18) and (20), we obtain an auto-Bǎcklund transformation as follows,

$$u = 2(\ln\phi)_{xx} + u_2, \qquad v = 2(\ln\phi)_{xy} + v_2. \tag{21}$$

From the last two equations of (19), it is easily seen that (u_2, v_2) is a set of solutions of (3), so we may take the trivial vacuum solution $u_2 = v_2 = 0$ as the seed solution, then (21) are reduced to,

$$u = 2(\ln\phi)_{xx}, \qquad v = 2(\ln\phi)_{xy}, \tag{22}$$

where ϕ satisfies the following constraint conditions,

$$3b\phi_{xx}^2\phi_x + 4\phi_x^2\phi_t - 4b\phi_x^2\phi_{xxx} + a\phi_{xx}^2\phi_y - 2a\phi_x\phi_y\phi_{xxx} + 2a\phi_x\phi_{xy}\phi_{xx} - 2a\phi_x^2\phi_{xxy} = 0, \tag{23}$$

$$4a\phi_x\phi_{xxxy} - 8\phi_x\phi_{xt} - 2a\phi_{xxy}\phi_{xx} - 2b\phi_{xx}\phi_{xxx} + 5b\phi_x\phi_{xxxx} + a\phi_y\phi_{xxxx} - 4\phi_{xx}\phi_t = 0, \tag{24}$$

$$a\phi_{xxxxy} + b\phi_{xxxxx} - 4\phi_{xxt} = 0. \tag{25}$$

Once Equations (23)-(25) are solved, we can get exact solutions of (3) by means of the transformation (22). Equations (23)-(25) are homogeneous differential equations, we may suppose that the solutions of (23)-(25) are in the form,

$$\phi(x,y,t) = 1 + f(\xi)\exp(\eta) = 1 + f(kx + l(y,t))\exp(nx + m(y,t)), \tag{26}$$

where k, n are constants, l(y, t), m(y, t) are the arbitrary analytic functions, whereas functions $f(\xi)$ may be sine, cosine, hyperbolic sine, hyperbolic cosine and so on. In this section, we only consider some special cases.

Case 1. $f(\xi) = 1$

In this case, the solutions of (23)-(25) read,

$$\phi_1 = 1 + \exp(nx + m(y,t)), \tag{27}$$

where the parameters satisfy the condition,

$$b = \frac{4m_t - an^2 m_y}{n^3} \tag{28}$$

Case 2. $f(\xi) = \cos(\xi)$ or $f(\xi) = \sin(\xi)$

In this case, we obtain the solutions of (23)-(25) as follows,

$$\phi_2 = 1 + \cos(l(y,t))\exp(nx + m(y,t)), \tag{29}$$

$$\phi_3 = 1 + \sin(l(y,t))\exp(nx + m(y,t)), \tag{30}$$

where the parameters satisfy the conditions,

$$a = \frac{4l_t}{n^2 l_y}, \qquad b = \frac{4m_t l_y - 4m_y l_t}{n^3 l_y}. \tag{31}$$

Case 3. $f(\xi) = \cosh(\xi)$ or $f(\xi) = \sinh(\xi)$

Similarly, we can get the solutions of (23)-(25),

$$\phi_4 = 1 + \cosh(l(y,t))\exp(nx + m(y,t)), \tag{32}$$
$$\phi_5 = 1 + \sinh(l(y,t))\exp(nx + m(y,t)), \tag{33}$$

where the parametric conditions are given by (31).

Inserting equation (27), equation (29), equation (30), equation (32) and equation (33) into equation (22), one can obtain five types of exact solutions of (3). For sake of simplicity, we only list one solution of (3). Substituting the expression (27) into equation (22), we obtain the following solution

$$u = \frac{n^2}{2}\sec h^2(nx + m(y,t)), \qquad v = \frac{nm(y,t)_y}{2}\sec h^2(nx + m(y,t)), \tag{34}$$

where b satisfies the constraint condition

$$b = \frac{4m_t - an^2 m_y}{n^3} \tag{35}$$

4. Hirota's Bilinear form and Soliton Solutions

As the first step, one should transform (3) into the bilinear forms with the help of dependent variable transformation. To this purpose, we consider the standard Painlevé truncated expansion,

$$u = 2(\ln\ f)_{xx} + u_0, \qquad v = 2(\ln\ f)_{xy} + v_0 \tag{36}$$

for simplicity we may take the seed solutions $u_0 = v_0 = 0$. With the transformation (36), we get the following bilinear form of (3),

$$\begin{aligned} &(D_x^4 - D_x D_s) f \cdot f = 0, \\ &(4D_x D_t - \frac{2a}{3} D_x^3 D_y - \frac{a}{3} D_y D_s - bD_x^4) f \cdot f = 0, \end{aligned} \tag{37}$$

where a and b are the functions with respect to x, y and t, s is the auxiliary variable, the well-known Hirota bilinear operators D_x, D_y, D_t are defined by [Hirota (2004)]

$$D_x^m D_y^n D_t^k a \cdot b = (\partial x - \partial x^{'})^m (\partial y - \partial y^{'})^n (\partial t - \partial t^{'})^k a(x,y,t) b(x^{'},y^{'},t^{'})\Big|_{x^{'}=x, y^{'}=y, t^{'}=t}. \tag{38}$$

For the simplicity of computation, we will consider equation (2) with a special choice of parameters with *a*(*y*, *t*) = 3, *b*(*y*, *t*) = 1. In this case, equation (2) becomes,

$$4u_t - 3(4uu_y + 2\int u_y dx u_x + u_{xy}) - (6uu_x + u_{xxx}) = 0. \tag{39}$$

Accordingly, its bilinear form reads,

$$\begin{aligned} &(D_x^4 - D_x D_s) f \cdot f = 0, \\ &(4D_x D_t - 2D_x^3 D_y - D_y D_s - D_x^4) f \cdot f = 0. \end{aligned} \tag{40}$$

The next step is the usual one. Let us represent f by a formal series

$$f = 1 + f^{(1)}\varepsilon + f^{(2)}\varepsilon^2 + f^{(3)}\varepsilon^3 + \cdots, \tag{41}$$

where ε serves as a parameter. Proceeding as in the Hirota method, we substitute (41) into (40) and equate to zero the different powers of ε,

$$\begin{aligned} o(\varepsilon): &(D_x^4 - D_x D_s) f^{(1)} \cdot 1 = 0 \\ &(4D_x D_t - 2D_x^3 D_y - D_y D_s - D_x^4) f^{(1)} \cdot 1 = 0 \end{aligned} \tag{42}$$

$$\begin{aligned} &o(\varepsilon^2): 2(D_x^4 - D_x D_s) f^{(2)} \cdot 1 = -(D_x^4 - D_x D_s) f^{(1)} \cdot f^{(2)} \\ &2(4D_x D_t - 2D_x^3 D_y - D_y D_s - D_x^4) f^{(2)} \cdot 1 \\ &\qquad = -(4D_x D_t - 2D_x^3 D_y - D_y D_s - D_x^4) f^{(1)} \cdot f^{(1)} \end{aligned} \tag{43}$$

$$\begin{aligned} &o(\varepsilon^3): (D_x^4 - D_x D_s) f^{(3)} \cdot 1 = -(D_x^4 - D_x D_s) f^{(1)} \cdot f^{(2)} \\ &(4D_x D_t - 2D_x^3 D_y - D_y D_s - D_x^4) f^{(3)} \cdot 1 \\ &\qquad = -(4D_x D_t - 2D_x^3 D_y - D_y D_s - D_x^4) f^{(1)} \cdot f^{(2)} \\ &\qquad \vdots \end{aligned} \tag{44}$$

First we consider one-soliton solution, suppose that

$$f^{(1)} = e^{\eta_1}, \quad \eta_1 = k_1 x + l_1 y + \omega_1 t + \xi_0^{(1)}, \tag{45}$$

where k_1, l_1, ω_1 are constants to be determined. Inserting it to (42) leads to the dispersion relation,

$$\omega_1 = \frac{3k_1^2 l_1 + k_1^3}{4}. \tag{46}$$

We find, in fact, that the right-hand side of (43) is equal to zero and so we can set $f_n = 0$ for $n \le 2$. Therefore, the series (41) truncates, then the exact solution to (40) reads,

$$f = 1 + \varepsilon e^{\eta_1}, \quad \eta_1 = k_1 x + l_1 y + \frac{3k_1^2 l_1 + k_1^3}{4} t + \xi_0^{(1)}. \tag{47}$$

Applying the transformation (36) yields the one-soliton solution of (39) (for simplicity, we take $\varepsilon = 1$)

$$u = 2(\ln f)_{xx} = 2\ln(1+e^{\eta_1})_{xx} = \frac{k_1^2}{2} \sec h^2 (\frac{k_1 x + l_1 y + \omega_1 t + \xi_0^{(1)}}{2}), \tag{48}$$

where $\omega_1 = \dfrac{3k_1^2 l_1 + k_1^3}{4}$.

We now proceed to search for two-soliton solution. Suppose that,

$$f^{(1)} = e^{\eta_1} + e^{\eta_2}, \quad \eta_i = k_i x + l_i y + \omega_{i1} t + \xi_0^{(i)}, \qquad (i = 1, 2), \tag{49}$$

from (42) one gets,

$$\omega_i = \frac{3k_i^2 l_i + k_i^3}{4}, \qquad i = 1, 2. \tag{50}$$

Together the values of f_1, solving (43) with respect to f_2 yields,

$$f^{(2)} = e^{\eta_1 + \eta_2 + a_{12}}, \quad e^{a_{12}} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \tag{51}$$

With the help of Maple, the right-hand side of (44) is reduced to zero. Consequently we can set $f_i = 0$ for $i \le 3$. Therefore the series (41) truncates, the exact solution for (39) is obtained,

$$f = 1 + \varepsilon e^{\eta_1} + \varepsilon e^{\eta_2} + \varepsilon^2 \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{\eta_1 + \eta_2}, \tag{52}$$

where

$$\eta_i = k_i x + l_i y + \frac{3k_i^2 l_i + k_i^3}{4} t + \xi_0^i, \quad i = 1, 2.$$

For simplicity one may let $\varepsilon = 1$, using the transformation (36), the explicit two-soliton solution of (39) reads,

$$u = 2\ln(1+\varepsilon e^{\eta_1} + \varepsilon e^{\eta_2} + \varepsilon^2 \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{\eta_1+\eta_2})_{xx}. \tag{53}$$

Generally, the N-soliton solution of (39) can be expressed as,

$$\begin{aligned} &u = 2\ln(\sum_{\mu=0,1} \exp(\sum_{j=1}^{n} \mu_j \eta_j + \sum_{1\le j<i}^{n} \mu_j \mu_i a_{ji}))_{xx}, \\ &\eta_j = k_j x + l_j y + \frac{3k_j^2 l_j + k_j^3}{4} t + \xi_0^{(j)}, \quad e^{a_{ji}} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, \end{aligned} \tag{54}$$

where the first $\sum_{\mu=0,1}$ means a summation over all possible combinations of $\mu_1 = 0,1,\ \mu_2 = 0,1,\ \ldots,\ \mu_n = 0,1,$ and $\sum_{1\le j<i}^{n}$ means a summation over all possible pairs ($j < i$).

5. Conclusions

In this work we have considered a (2+1)-dimensional variable-coefficient *KdV* equation. Through the Painlevé analysis, the considered equation is found to possess the Painlevé property without any parametric constraints. Using the Painlevé truncated expansion method, the auto-Băcklund transformation and five types of exact solutions are obtained. Moreover, the Hirota’s bilinear form of the (2+1)-dimensional variable-coefficient *KdV* equation is constructed. The multi-soliton solutions are constructed for the special choice of parameters. The obtained exact solutions may be useful for describing the correspond physical phenomena. It is deserved to make considerations on obtaining other integrable properties of this equation, such as the Lax pair, Băcklund transformation, conservation laws, and so on.

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