Local Estimates for the Koornwinder Jacobi-Type Polynomials

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Abstract

In this paper we give some local estimates for the Koornwinder Jacobi-type polynomials by using asymptotic properties of Jacobi orthogonal polynomials.

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1. Introduction

Let

\[ \omega^{(\alpha,\beta)}(x) = (1-x)^{\alpha} \cdot (1+x)^{\beta}, x \in [-1,1] \]
be a Jacobi weight with $\alpha, \beta > -1$. Let also

$$p_n(x) = p_n^{(\alpha, \beta)}(x) = \gamma_n^{(\alpha, \beta)} x^n + \ldots, n \in \mathbb{N}_0$$

denote the unique Jacobi polynomials of precise degree $n$, with leading coefficients $\gamma_n^{(\alpha, \beta)} > 0$, fulfilling the orthogonal conditions

$$\int_{-1}^{1} p_n(x)p_m(x)\omega_n^{(\alpha, \beta)}(x) = \delta_{n,m}, n, m \in \mathbb{N}.$$

Felten (2007), introduced modified Jacobi weights as

$$\omega_n^{(\alpha, \beta)}(x) := \left( \sqrt{1-x} + \frac{1}{n} \right)^{2\alpha} \left( \sqrt{1+x} + \frac{1}{n} \right)^{2\beta}, x \in [-1,1], n \in \mathbb{N}. \quad (1)$$

He proved the following theorem [see Felten (2007)]:

**Theorem 1.1:**

Let $\alpha, \beta > -1$ and $n \in \mathbb{N}$. Then,

$$|p_n^{(\alpha, \beta)}(x)| \leq C \frac{1}{\omega_n^{\alpha, \beta, \frac{1}{2}, \frac{1}{2}}(x)}, \quad (2)$$

for all $x \in [-1,1]$ with a positive constant $C = C(\alpha, \beta)$ being independent of $n$ and $x$.

The above estimation first appeared in Lubinski and Totik (1994). Then for $\alpha, \beta \geq -\frac{1}{2}$, Felten (2004) extended the previous results as follows:

**Theorem 1.2:**

Let $\alpha, \beta \geq -\frac{1}{2}$ and $n \in \mathbb{N}$. Then,

$$|p_n^{(\alpha, \beta)}(t)| \leq C \frac{1}{\omega_n^{\alpha, \beta, \frac{1}{2}, \frac{1}{2}}(x)}, \quad (3)$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$, where
\[
U_n(x) := \left\{ t \in [-1,1] : t - x \leq \frac{\varphi_n(x)}{n} \right\} = \left[ x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n} \right],
\]

for \( n \in \mathbb{N} \) and \( x \in [-1,1] \) with \( \varphi_n(x) := \sqrt{1-x^2} + \frac{1}{n} \).

Koornwinder (1984), introduced the polynomials \( \left( P_n^{(\alpha,\beta,M,N)}(x) \right)_{n=0}^\infty \) defined as follows:

**Definition 1.3.**

Fix \( M, N \geq 0 \) and \( \alpha, \beta > -1 \). For \( n = 0,1,2, \cdots \) define

\[
P_n^{(\alpha,\beta,M,N)}(x) = \left( \frac{\alpha + \beta + 1}{n!} \right)^2 \left[ (\alpha + \beta + 1)^{-1} (B_n M (1-x) - A_n N (1+x) \frac{d}{dx} + A_n B_n) \right] P_n^{(\alpha,\beta)}(x),
\]

where

\[
A_n = \frac{(\alpha+1)_n n!}{(\beta+1)_n (\alpha+\beta+1)_n} + \frac{n(n+\alpha \beta +1)M}{(\beta+1)(\alpha+\beta+1)}, \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}
\]

and

\[
B_n = \frac{(\beta+1)_n n!}{(\alpha+1)_n (\alpha+\beta+1)_n} + \frac{n(n+\alpha \beta +1)N}{(\alpha+1)(\alpha+\beta+1)}.
\]

We call these polynomials the Koornwinder’s Jacobi-type polynomials.

The above defined polynomials are orthogonal on the interval \([-1,1]\) with respect to the measure \( \mu \) defined by

\[
\int_{-1}^{1} f(x) d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} f(x)(1-x)^\alpha (1+x)^\beta dx + Mf(-1) + Nf(1),
\]

where \( f \in C([-1,1]) \) and \( M, N \geq 0, \alpha, \beta > -1 \).

Clearly, for \( M = N = 0 \) one has

\[
P_n^{(\alpha,\beta,0,0)}(x) = P_n^{(\alpha,\beta)}(x).
\]
Also
\[
P_n^{(\alpha, \beta, M, N)}(-x) = (-1)^n \, P_n^{(\beta, \alpha, N, M)}(x).
\]  
\tag{9}

Some basic properties of \( P_n^{(\alpha, \beta, M, N)}(x) \) are given as below [Varona (1989), chapter IV]].
\[
P_n^{(\alpha, \beta, M, N)}(1) \sim \begin{cases} 
-\frac{\alpha}{2}, & \text{if } N > 0 \\
\frac{1}{2}, & \text{if } N = 0
\end{cases}
\]  
\tag{10}

and
\[
|P_n^{(\alpha, \beta, M, N)}(-1)| \sim \begin{cases} 
-\frac{\beta}{2}, & \text{if } M > 0 \\
\frac{1}{2}, & \text{if } M = 0.
\end{cases}
\]  
\tag{11}

**Theorem 1.4** [Varona (1989)]:

Let \( \alpha, \beta > -1, M, N > 0 \). For every \( x \in [-1,1] \), there exists a unique constant \( C \) such that the following relation holds for each \( n \in \mathbb{N} \):
\[
\left( h_n^{(\alpha, \beta, M, N)} \right)^{\frac{1}{2}} |P_n^{(\alpha, \beta, M, N)}(x)| \leq C \left( 1 - x + \frac{1}{n^2} \right)^{\frac{\alpha}{2} \frac{1}{4}} \left( 1 + x + \frac{1}{n^2} \right)^{\frac{\beta}{2} \frac{1}{4}},
\]
where
\[
h_n^{(\alpha, \beta, M, N)} = \int_{-1}^{1} (P_n^{(\alpha, \beta, M, N)}(x))^2 \, d\mu.
\]

Based on Theorem 1.4 and properties of Jacobi polynomials [see Lubinski and Totik (1994) and Szego (1975)], we get the following estimation for the Koornwinder Jacobi-type polynomials:
\[
|P_n^{(\alpha, \beta, M, N)}(\cos \theta)| = \begin{cases} 
0(\theta^{\frac{1}{2}}, & \text{if } \frac{c}{n} \leq \theta \leq \frac{\pi}{2} \\
0(n^{\frac{\alpha}{2}}), & \text{if } 0 \leq \theta \leq \frac{c}{n},
\end{cases}
\]  
\tag{12}

for
\[ \alpha \geq -1, \beta \geq -1 \text{ and } n \geq 1. \]

The aim of this paper is to prove similar results as those given in Theorem 1.1 and Theorem 1.2, for Koornwinder Jacobi-type polynomials, when \( \alpha, \beta \geq -1 \), respectively, for \( \alpha, \beta \geq -\frac{1}{2} \).

2. Results

The following Theorem is the main result of this note.

**Theorem 2.1:**

Let \( \alpha, \beta > -1 \) and \( n \in \mathbb{N} \). Then,

\[
\left| P_n^{(\alpha, \beta, M, N)}(x) \right| \leq D \frac{1}{\omega_n^{\alpha, \beta, M, N}}(x),
\]

for all \( x \in [-1,1] \) with a positive constant \( D = D(\alpha, \beta) \) being independent of \( n \) and \( x \).

**Proof:**

Proof of the Theorem is similar to Theorem 2.1 in Felten (2007). Let \( x \in [0,1] \), and let \( \theta \in \left[ 0, \frac{\pi}{2} \right] \) such that \( x = \cos \theta \). From (12), one has the following estimation

\[
\left| P_n^{(\alpha, \beta, M, N)}(\cos \theta) \right| \leq C \begin{cases} 
\theta^{-\frac{1}{2}}, & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\
\frac{\alpha + 1}{2}, & \text{if } 0 \leq \theta \leq \frac{c}{n} \\
\frac{\alpha + 1}{2}, & \text{if } 0 \leq \theta \leq \frac{c}{n} 
\end{cases}.
\]

If in the last relation, we substitute \( x = \cos \theta \), then we will have

\[
\left| P_n^{(\alpha, \beta, M, N)}(x) \right| \leq C \begin{cases} 
\frac{\alpha + 1}{2}, & \text{if } 0 \leq \arccos x \leq \frac{c}{n} \\
\left( \arccos x \right)^{\frac{\alpha + 1}{2}}, & \text{if } \frac{c}{n} \leq \arccos x \leq \frac{\pi}{2},
\end{cases}
\]
where $C$ is a fixed positive constant being independent of $n$ and $\theta$.

In what follows we will make use of the following estimates

$$
\frac{\pi}{2} \sqrt{1-x} = \frac{\pi}{\sqrt{2}} \sqrt{1-\frac{x}{2}} = \frac{\pi}{\sqrt{2}} \sin \frac{t}{2} \geq \frac{\pi}{\sqrt{2}} \left( \frac{2}{\pi} \frac{t}{\sqrt{2}} \right) = t = \arccos x
$$

(16)

and

$$
\sqrt{2} \sqrt{1-x} = 2 \sqrt{1-\frac{x}{2}} = 2 \sin \frac{t}{2} \leq 2 \frac{t}{2} = t = \arccos x.
$$

(17)

We differ two cases:

**Case 1.** $-1 < \alpha \leq -\frac{1}{2}$. In this case, $-\left( \alpha + \frac{1}{2} \right) \geq 0$.

If $0 \leq \arccos x \leq \frac{c}{n}$, then from (17) we obtain $\frac{c}{n} \geq \sqrt{2} \sqrt{1-x}$ and from (15) we get the following relation

$$
| P_n^{(0,0,M,N)} | \leq C n^{\alpha - \frac{1}{2}} = C \left( \frac{1}{n} \right)^{\alpha - \frac{1}{2}} \leq C_1 \left( \sqrt{1-x} \right)^{\alpha - \frac{1}{2}} \leq C_2 \left( \sqrt{1-x} + \frac{1}{n} \right)^{\alpha - \frac{1}{2}}.
$$

If $\frac{c}{n} \leq \arccos x \leq \frac{\pi}{2}$, then from relations (15) and (17) we get

$$
| P_n^{(0,0,M,N)} | \leq C_3 (\arccos x)^{\alpha - \frac{1}{2}} \leq C_4 \left( \sqrt{1-x} \right)^{\alpha - \frac{1}{2}} \leq C_5 \left( \sqrt{1-x} + \frac{1}{n} \right)^{\alpha - \frac{1}{2}}.
$$

**Case 2.** $\alpha > -\frac{1}{2}$. In this case, $-\left( \alpha + \frac{1}{2} \right) < 0$.

If $0 \leq \arccos x \leq \frac{c}{n}$, then from relations (15) and (17) we obtain

$$
| P_n^{(0,0,M,N)} | \leq C_6 n^{\alpha - \frac{1}{2}} = C_6 \left( \frac{c + \sqrt{2}}{n} \right)^{\alpha - \frac{1}{2}} \leq C_7 \left( \sqrt{1-x} + \frac{1}{n} \right)^{\alpha - \frac{1}{2}}.
$$

If $\frac{c}{n} \leq \arccos x \leq \frac{\pi}{2}$, again according to relations (15) and (17) we have
\[
| P_n^{(\alpha, \beta, M, N)} | \leq C_s(\arccos x)^{(-\frac{1}{2})} = C_s(\arccos x + \arccos x)^{(-\frac{1}{2})} \leq C_{10} \left( \sqrt{1-x + \frac{1}{n}} \right)^{(-\frac{1}{2})}.
\]

From previous cases we have proved that
\[
| P_n^{(\alpha, \beta, M, N)}(x) | \leq C_{11}(\alpha, \beta) \left( \sqrt{1-x + \frac{1}{n}} \right)^{(-\frac{1}{2})} \left( \sqrt{1+x + \frac{1}{n}} \right)^{(-\frac{1}{2})},
\]
for all \( x \in [0,1], n \in \mathbb{N} \) and \( \alpha, \beta \geq -1 \).

From (10) we obtain
\[
| P_n^{(\alpha, \beta, M, N)}(x) | \leq C_{12}(\beta, \alpha) \left( \sqrt{1+x + \frac{1}{n}} \right)^{(-\frac{1}{2})} \times \left( \sqrt{1-x + \frac{1}{n}} \right)^{(-\frac{1}{2})},
\]
for all \( x \in [-1,0], n \in \mathbb{N} \) and \( \alpha, \beta \geq -1 \).

The proof is completed.

Next, we will show that the local estimates of previous theorem can be further extended. We will prove that \( | P_n^{(\alpha, \beta, M, N)}(x) | \) in (14) can be replaced by \( | P_n^{(\alpha, \beta, M, N)}(t) | \), whenever \( t \) is in the interval \( U_n(x) = \left[ x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n} \right] \cap [-1,1] \). In order to do that we will make use of the following Lemma [see Felten (2007)].

**Lemma 2.2:**

Let \( a, b \leq 0, n \in \mathbb{N} \) and \( x \in [-1,1] \). Then,
\[
\omega_n^{(a,b)}(t) \leq 16^{-(a+b)} \omega_n^{(a,b)}(x),
\]
for all \( t \in U_n(x) \).

**Theorem 2.3:**

Let \( \alpha, \beta \geq -\frac{1}{2} \) and \( n \in \mathbb{N} \). Then,
\[ |P_n^{(\alpha, \beta, M, N)}(t)| \leq D \frac{1}{\omega_n^{\frac{1}{2} + \frac{1 \cdot 1}{2 + 1} + \frac{1}{4} - \frac{1}{4}}(x)}, \quad (19) \]

for all \( t \in U_n(x) \) and each \( x \in [-1,1] \), where \( D = D(\alpha, \beta) \) is a positive constant independent of \( n, t \) and \( x \).

**Proof:**

Since \( \alpha, \beta \geq -\frac{1}{2} \), it follows that \( \frac{\alpha}{2} + \frac{1}{2} + \frac{1}{4} \geq 0 \). Therefore, by Lemma 2.2 with \( a = -\frac{\alpha}{2} - \frac{1}{4} \) and \( \beta = -\frac{\alpha}{2} - \frac{1}{4} \), we obtain

\[
\frac{1}{\omega_n^{\frac{1}{2} + \frac{1 \cdot 1}{2 + 1} + \frac{1}{4} - \frac{1}{4}}(x)} = \omega_n^{\frac{1}{2} + \frac{1 \cdot 1}{2 + 1} + \frac{1}{4} - \frac{1}{4}}(x) \leq \frac{4^{\alpha + \beta + 1}}{\omega_n^{\frac{1}{2} + \frac{1 \cdot 1}{2 + 1} + \frac{1}{4} - \frac{1}{4}}(x)},
\]

for all \( t \in U_n(x) \). Applying Theorem 2.1 yields inequality (14) for all \( t \in U_n(x) \), as claimed.

**Corollary 2.4:**

Let \( n \in \mathbb{N} \) and \( \alpha, \beta \geq -\frac{1}{2}, x \in [-1,1] \). Then,

\[
\int_{U_n(x)} |P_n^{(\alpha, \beta, M, N)}(t)|^2 \omega_n^{(\alpha, \beta)}(t)dt \leq D(\alpha, \beta)^{\frac{1}{n}}.
\]

**Proof:**

Applying Theorem 2.3 we obtain

\[
\int_{U_n(x)} |P_n^{(\alpha, \beta, M, N)}(t)|^2 \omega_n^{(\alpha, \beta)}(t)dt \leq D \frac{1}{\omega_n^{\frac{1}{2} + \frac{1 \cdot 1}{2 + 1} + \frac{1}{4} - \frac{1}{4}}(x)} \int_{U_n(x)} \omega_n^{(\alpha, \beta)}(t)dt.
\]

Using the following result from Felten (2008), we obtain
\[ \int_{U_n(x)} \omega_n^{(\alpha, \beta)}(t) \, dt \leq \frac{D}{n} \omega_n^{(\alpha+\frac{1}{2}, \beta+\frac{1}{2})}(x) \]

and, thus, the proof is completed.

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