Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 6, Issue 1 (June 2011) pp. 162 – 170
(Previously, Vol. 6, Issue 11, pp. 1902 – 1910)

Applications and Applied
Mathematics:
An International Journal
(AAM)

Local Estimates for the Koornwinder Jacobi-Type Polynomials

Valmir Krasniqi, Naim L. Braha and
Armend Sh. Shabani

Department of Mathematics and Computer Sciences
University of Prishtin
Avenue "Mother Theresa"
Nr = 5, Prishtine, 10000
Republic o Kosova
vali.99@hotmail.com, nbraha@yahoo.com,
armend_shabani@hotmail.com

Received: July 23, 2010; Accepted: February 3, 2011

Abstract

In this paper we give some local estimates for the Koornwinder Jacobi-type polynomials by using asymptotic properties of Jacobi orthogonal polynomials.

Keywords: Koornwinder Jacobi-type polynomials, Jacobi orthogonal polynomials

AMS (2010) No.: 33C45, 42C05

1. Introduction

Let

$$\omega^{(\alpha,\beta)}(x) = (1-x)^{\alpha} \cdot (1+x)^{\beta}, x \in [-1,1]$$

be a Jacobi weight with $\alpha, \beta > -1$. Let also

$$p_n(x) = p_n^{(\alpha,\beta)}(x) = \gamma_n^{(\alpha,\beta)} x^n + \ldots, n \in \mathbb{N}_0$$

denote the unique Jacobi polynomials of precise degree n, with leading coefficients $\gamma_n^{(\alpha,\beta)} > 0$, fulfilling the orthogonal conditions

$$\int_{-1}^{1} p_n(x) p_m(x) \omega^{(\alpha,\beta)}(x) = \delta_{m,n}, n, m \in \mathbb{N}.$$

Felten (2007), introduced modified Jacobi weights as

$$\omega_n^{(\alpha,\beta)}(x) := \left(\sqrt{1-x} + \frac{1}{n}\right)^{2\alpha} \left(\sqrt{1+x} + \frac{1}{n}\right)^{2\beta}, x \in [-1,1], n \in \mathbb{N}. \tag{1}$$

He proved the following theorem [see Felten (2007)]:

Theorem 1.1:

Let $\alpha, \beta > -1$ and $n \in \mathbb{N}$. Then,

$$| p_n^{(\alpha,\beta)}(x) | \leq C \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4}\right)}(x)}, \tag{2}$$

for all $x \in [-1,1]$ with a positive constant $C = C(\alpha, \beta)$ being independent of n and x.

The above estimation first appeared in Lubinski and Totik (1994). Then for $\alpha, \beta \geq -\frac{1}{2}$, Felten (2004) extended the previous results as follows:

Theorem 1.2:

Let $\alpha, \beta \geq -\frac{1}{2}$ and $n \in \mathbb{N}$. Then,

$$| p_n^{(\alpha,\beta)}(t) | \leq C \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4}\right)}(x)}, \tag{3}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$, where

$$U_n(x) := \left\{ t \in [-1,1] : |t-x| \le \frac{\varphi_n(x)}{n} \right\} = \left[x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n} \right], \tag{4}$$

for $n \in \mathbb{N}$ and $x \in [-1,1]$ with $\varphi_n(x) := \sqrt{1-x^2} + \frac{1}{n}$.

Koornwinder (1984), introduced the polynomials $\left(P_n^{(\alpha,\beta,M,N)}(x)\right)_{n=0}^{\infty}$ defined as follows:

Definition 1.3.

Fix $M, N \ge 0$ and $\alpha, \beta > -1$. For $n = 0,1,2,\cdots$ define

$$P_n^{(\alpha,\beta,M,N)}(x) = \left(\frac{(\alpha+\beta+1)_n}{n!} \right)^2 \cdot \left[(\alpha+\beta+1)^{-1} (B_n M (1-x) - A_n N (1+x) \frac{d}{dx} + A_n B_n) \right] p_n^{(\alpha,\beta)}(x),$$

where

$$A_n = \frac{(\alpha+1)_n n!}{(\beta+1)_n (\alpha+\beta+1)_n} + \frac{n(n+\alpha\beta+1)M}{(\beta+1)(\alpha+\beta+1)}, \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{5}$$

and

$$B_n = \frac{(\beta+1)_n n!}{(\alpha+1)_n (\alpha+\beta+1)_n} + \frac{n(n+\alpha\beta+1)N}{(\alpha+1)(\alpha+\beta+1)}. \tag{6}$$

We call these polynomials the Koornwinder's Jacobi-type polynomials.

The above defined polynomials are orthogonal on the interval $[-1,1]$ with respect to the measure μ defined by

$$\int_{-1}^{1} f(x) d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta} dx + Mf(-1) + Nf(1), \tag{7}$$

where $f \in C([-1,1])$ and $M, N \ge 0, \alpha, \beta > -1$.

Clearly, for $M = N = 0$ one has

$$P_n^{(\alpha,\beta,0,0)}(x) = P_n^{(\alpha,\beta)}(x). \tag{8}$$

Also

$$P_n^{(\alpha,\beta,M,N)}(-x) = (-1)^n P_n^{(\beta,\alpha,N,M)}(x). \tag{9}$$

Some basic properties of $P_n^{(\alpha,\beta,M,N)}(x)$ are given as below [Varona (1989), chapter IV)].

$$P_n^{(\alpha,\beta,M,N)}(1) \sim \begin{cases} n^{-\alpha-\frac{3}{2}}, \ if \ \ N>0 \\ n^{\alpha+\frac{1}{2}}, if \ \ N=0 \end{cases} \tag{10}$$

and

$$\left|P_n^{(\alpha,\beta,M,N)}(-1)\right| \sim \begin{cases} n^{-\beta-\frac{3}{2}}, \ if \ \ M>0 \\ n^{\beta+\frac{1}{2}}, if \ \ M=0. \end{cases} \tag{11}$$

Theorem 1.4 [Varona (1989)]**:**

Let $\alpha,\beta>-1, M, N>0.$ For every $x\in[-1,1]$, there exists a unique constant C such that the following relation holds for each $n\in \mathbb{N}$:

$$\left(h_n^{\alpha,\beta,M,M}\right)^{-\frac{1}{2}} \left|P_n^{(\alpha,\beta,M,N)}(x)\right| \le C\left(1-x+\frac{1}{n^2}\right)^{-\frac{\alpha}{2}-\frac{1}{4}} \left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}},$$

where

$$h_n^{(\alpha,\beta,M,N)} = \int_{-1}^{1} (P^{(\alpha,\beta,M,N)}(x))^2 \, d\mu.$$

Based on Theorem 1.4 and properties of Jacobi polynomials [see Lubinski and Totik (1994) and Szego (1975)], we get the following estimation for the Koornwinder Jacobi-type polynomials:

$$\left|P_n^{(\alpha,\beta,M,N)}(\cos\theta)\right| = \begin{cases} 0(\theta^{-\alpha-\frac{1}{2}}), \ if \ \frac{c}{n} \le \theta \le \frac{\pi}{2} \\ 0(n^{\alpha+\frac{1}{2}}), \ if \ 0 \le \theta \le \frac{c}{n}, \end{cases} \tag{12}$$

for

$\alpha \geq -1, \beta \geq -1$ and $n \geq 1$.

The aim of this paper is to prove similar results as those given in Theorem 1.1 and Theorem 1.2, for Koornwinder Jacobi-type polynomials, when $\alpha, \beta \geq -1,$ respectively, for $\alpha, \beta \geq -\frac{1}{2}$.

2. Results

The following Theorem is the main result of this note.

Theorem 2.1:

Let $\alpha, \beta > -1$ and $n \in \mathbb{N}$. Then,

$$| P_n^{(\alpha,\beta,M,N)}(x) | \leq D \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)}, \tag{13}$$

for all $x \in [-1,1]$ with a positive constant $D = D(\alpha, \beta)$ being independent of n and x .

Proof:

Proof of the Theorem is similar to Theorem 2.1 in Felten (2007). Let $x \in [0,1],$ and let $\theta \in \left[0, \frac{\pi}{2}\right]$ such that $x = \cos\theta$. From (12), one has the following estimation

$$\left|P_n^{(\alpha,\beta,M,N)}(\cos\theta)\right| \leq C \begin{cases} \theta^{-\alpha-\frac{1}{2}}, \; if \; \frac{c}{n} \leq \theta \leq \frac{\pi}{2} \\ n^{\alpha+\frac{1}{2}}, \; if \; 0 \leq \theta \leq \frac{c}{n} \end{cases} . \tag{14}$$

If in the last relation, we substitute $x = \cos\theta$, then we will have

$$\left|P_n^{(\alpha,\beta,M,N)}(x)\right| \leq C \begin{cases} n^{\alpha+\frac{1}{2}}, \; if \; 0 \leq \arccos x \leq \frac{c}{n} \\ (\arccos x)^{-\left(\alpha+\frac{1}{2}\right)}, \; if \; \frac{c}{n} \leq \arccos x \leq \frac{\pi}{2}, \end{cases} \tag{15}$$

where C is fixed positive constant being independent of n and θ.

In what follows we will make use of the following estimates

$$\frac{\pi}{2}\sqrt{1-x}=\frac{\pi}{\sqrt{2}}\sqrt{\frac{1-x}{2}}=\frac{\pi}{\sqrt{2}}\sin\frac{t}{2}\geq\frac{\pi}{\sqrt{2}}\left(\frac{2}{\pi}\cdot\frac{t}{\sqrt{2}}\right)=t=\arccos x \tag{16}$$

and

$$\sqrt{2}\sqrt{1-x}=2\sqrt{\frac{1-x}{2}}=2\sin\frac{t}{2}\leq 2\cdot\frac{t}{2}=t=\arccos x\,. \tag{17}$$

We differ two cases:

Case 1. $-1<\alpha\leq-\frac{1}{2}$. In this case, $-\left(\alpha+\frac{1}{2}\right)\geq 0$.

If $0\leq\arccos x\leq\frac{c}{n}$, then from (17) we obtain $\frac{c}{n}\geq\sqrt{2}\sqrt{1-x}$ and from (15) we get the following relation

$$|P_n^{(\alpha,\beta,M,N)}|\leq Cn^{\alpha+\frac{1}{2}}=C\left(\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}\leq C_1\left(\sqrt{1-x}\right)^{-\left(\alpha+\frac{1}{2}\right)}\leq C_2\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

If $\frac{c}{n}\leq\arccos x\leq\frac{\pi}{2}$, then from relations (15) and (17) we get

$$|P_n^{(\alpha,\beta,M,N)}|\leq C_3(\arccos x)^{-\left(\alpha+\frac{1}{2}\right)}\leq C_4\left(\sqrt{1-x}\right)^{-\left(\alpha+\frac{1}{2}\right)}\leq C_5\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

Case 2. $\alpha>-\frac{1}{2}$. In this case $-\left(\alpha+\frac{1}{2}\right)<0.$

If $0\leq\arccos x\leq\frac{c}{n}$, then from relations (15) and (17) we obtain

$$|P_n^{(\alpha,\beta,M,N)}|\leq C_6 n^{\alpha+\frac{1}{2}}=C_6\left(\frac{c}{n}+\frac{\sqrt{2}}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}\leq C_7\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

If $\frac{c}{n}\leq\arccos x\leq\frac{\pi}{2}$, again according to relations (15) and (17) we have

$$|P_n^{(\alpha,\beta,M,N)}| \le C_8(\arccos x)^{-\left(\alpha+\frac{1}{2}\right)} = C_9(\arccos x + \arccos x)^{-\left(\alpha+\frac{1}{2}\right)} \le C_{10}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

From previous cases we have proved that

$$|P_n^{(\alpha,\beta,M,N)}(x)| \le C_{11}(\alpha,\beta)\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}\cdot\left(\sqrt{1+x}+\frac{1}{n}\right)^{-\left(\beta+\frac{1}{2}\right)},$$

for all $x\in[0,1], n\in\mathbb{N}$ and $\alpha,\beta\ge -1$.

From (10) we obtain

$$|P_n^{(\alpha,\beta,M,N)}(x)| \le C_{12}(\beta,\alpha)\left(\sqrt{1+x}+\frac{1}{n}\right)^{-\left(\beta+\frac{1}{2}\right)}\times\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)},$$

for all $x\in[-1,0), n\in\mathbb{N}$ and $\alpha,\beta\ge -1$.

The proof is completed.

Next, we will show that the local estimates of previous theorem can be further extended. We will prove that $|P_n^{(\alpha,\beta,M,N)}(x)|$ in (14) can be replaced by $|P_n^{(\alpha,\beta,M,N)}(t)|$, whenever t is in the interval $U_n(x)=\left[x-\frac{\varphi_n(x)}{n}, x+\frac{\varphi_n(x)}{n}\right]\cap[-1,1]$. In order to do that we will make use of the following Lemma [see Felten (2007)].

Lemma 2.2:

Let $a,b\le 0, n\in\mathbb{N}$ and $x\in[-1,1]$. Then,

$$\omega_n^{(a,b)}(t) \le 16^{-(a+b)}\omega_n^{(a,b)}(x), \qquad (18)$$

for all $t\in U_n(x)$.

Theorem 2.3:

Let $\alpha,\beta\ge -\frac{1}{2}$ and $n\in\mathbb{N}$. Then,

$$|P_n^{(\alpha,\beta,M,N)}(t)| \le D \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)}, \tag{19}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$, where $D = D(\alpha,\beta)$ is a positive constant independent of n, t and x.

Proof:

Since $\alpha, \beta \ge -\frac{1}{2}$, it follows that $\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4} \ge 0$. Therefore, by Lemma 2.2 with $a = -\frac{\alpha}{2} - \frac{1}{4}$ and $\beta = -\frac{\alpha}{2} - \frac{1}{4}$, we obtain

$$\frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} = \omega_n^{\left(-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}\right)}(x) \le \frac{4^{\alpha+\beta+1}}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)},$$

for all $t \in U_n(x)$. Applying Theorem 2.1 yields inequality (14) for all $t \in U_n(x)$, as claimed.

Corollary 2.4:

Let $n \in$ □ and $\alpha, \beta \ge -\frac{1}{2}, x \in [-1,1]$. Then,

$$\int_{U_n(x)} |P_n^{(\alpha,\beta,M,N)}(t)|^2 \, \omega_n^{(\alpha,\beta)}(t)dt \le D(\alpha,\beta) \cdot \frac{1}{n}.$$

Proof:

Applying Theorem 2.3 we obtain

$$\int_{U_n(x)} |P_n^{(\alpha,\beta,M,N)}(t)|^2 \, \omega_n^{(\alpha,\beta)}(t)dt \le D \cdot \frac{1}{\omega_n^{\left(\alpha+\frac{1}{2},\beta+\frac{1}{2}\right)}(x)} \cdot \int_{U_n(x)} \omega_n^{(\alpha,\beta)}(t)dt.$$

Using the following result from Felten (2008), we obtain

$$\int_{U_n(x)} \omega_n^{(\alpha,\beta)}(t)dt \le \frac{D}{n} \cdot \omega_n^{\left(\alpha+\frac{1}{2},\beta+\frac{1}{2}\right)}(x)$$

and, thus, the proof is completed.

Acknowledgment

The authors would like to thank anonymous referees for their suggestions, which contributed to the quality of the note.

REFERENCES

Felten, M. (2007). Local estimates for Jacobi polynomials, J. Inequal. Pure Appl. Math., Vol. 8, pp. 1-7.

Felten, M. (2008). Uniform boundedness of $(C,1)$ means of Jacobi expansions in weighted sup norms, Acta Math. Hung., Vol.118, pp. 227-263.

Koornwinder, T. H. (1984). Orthogonal polynomials with weight function $(1-x)^{\alpha} \cdot (1+x)^{\beta} + M\delta(x+1) + N\delta(x-1),$ Canad. Math. Bull., Vol. 27, pp. 205-214.

Lubinski, D. S. and Totik, V. (1994). Best weighted polynomial approximation via Jacobi expansion, SIAM J. Math. Ana. Vol. 25, pp.555-570.

Szego, G. (1975). Orthogonal Polynomials, 4^{th} ed. American Mathematical Society, Providence, R.I., American Mathematical Society, Colloquium Publications, Vol. XXIII.

Varona, J. L. (1989). Convergencia en L^p con pesos de la serie de Fourier respecto de algunos sistemas ortogonales, Ph. D. Thesis, Sem. Mat. Garci de Galdeano, sec. 2, no. 22, Zaragoza.