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A New Approach to the Numerical Solution of Fractional Order Optimal Control Problems

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Abstract

In this article, a new numerical method is proposed for solving a class of fractional order optimal control problems. The fractional derivative is considered in the Caputo sense. This approach is based on a combination of the perturbation homotopy and parameterization methods. The control function u(t) is approximated by polynomial functions with unknown coefficients. This method converts the fractional order optimal control problem to an optimization problem. Numerical results are included to demonstrate the validity and applicability of the method.

Keywords: Fractional order optimal control, Homotopy perturbation method, Caputo fractional derivative.

AMS-MSC 2010 No.: 49J21, 26A33

1. Introduction

Fractional optimal control problems (FOCPs) are optimal control problems associated with fractional dynamic systems. The fractional optimal control theory is a very new topic in mathematics. FOCPs may be defined in terms of different types of fractional derivatives. But the most important types of fractional derivatives are the Riemann-Liouville and the Caputo fractional derivatives. In Agrawal (2004), Agrawal and Baleanu (2007) the authors obtained necessary conditions for FOCPs with the Riemann-Liouville derivative and were able to solve the problem numerically. Agrawal (2008a) presented a quadratic numerical scheme for a class of fractional optimal control problems (FOCPs). In Agrawal (2008c), the FOCPs are formulated for a class of distributed systems where the fractional derivative is defined in the Caputo sense, and a numerical technique for FOCPs presented. Baleanu et al. (2009) used a direct numerical scheme to find a solution of the FOCPs. In Biswas and Sen (2011a), FOCPs with fixed final time are considered and a transversely condition is obtained. FOCPs with dynamic constraint involving integer and fractional derivatives are also considered Biswas and Sen (2011b). Based on the expansion formula for fractional derivatives, a new solution scheme was proposed in Jelicic and Petrovacki (2009). Lotfi et al. (2011) used Legendre orthonormal polynomial basis to solve the FOCPs.

A direct method using Eigen functions to solve the FOCPs of a 2-dimensional system was presented in Özdemir et al. (2009), where the Grünwald-Letnikov approximation was used to approximate the fractional derivatives. Similar attempts have been made by several researchers for solving the FOCP of distributed systems (Hasan et al. (2011), Rapaic and Jelicic (2010)). Tricaud and Chen (2010a) presented a numerical scheme for FOCPs based on integer order optimal controls problem.

In Youse et al. (2011) the usage of Legendre multiwavelet basis and collocation method was proposed for obtaining the approximate solution of FOCPs. Tricaud and Chen (2010b) proposed a rational approximation based on the Hankel data matrix of the impulse response to obtain a solution for the general time-optimal problem. The interested reader is referred to Evirgen and Özdemir (2012), Wang and Zhou (2011), Jarad (2010), Özdemir (2009), Agrawal (2008b), and Frederico et al. (2008) for further information.

The homotopy perturbation method (HPM) was applied to solve ODE and PDE equations in He (2003), He (2005), Biazar and Ghazvini (2009). The numerical method for FOCPs presented in this paper follows the approach presented in Keyanpour and Azizsefat (2011), Keyanpour and Akbarian (2011), Borzabadi et al. (2010), Borzabadi and Azizsefat (2010). Of course in this paper we develop a hybrid of parametrization and modified homotopy perturbation method to solve FOCPs.

This paper is organized as follows: In Section 2 we present some basic definitions. In section 3 we describe our method. In section 4 we report our numerical results. Finally section 5 we conclude the paper.

2. Fractional Optimal Control Problem Statement

2.1. Basic Definitions

We present some basic definitions related to fractional derivatives. The Left Riemann-Liouville derivative of fractional order $m-1<\nu\leq m$ for a function $f(t)$ is defined by:

$$ {}_aD_t^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \left(\frac{d}{dx}\right)^m \int_a^x (x-t)^{m-\nu-1} f(t)dt, \tag{1} $$

while the Right Riemann-Liouville fractional derivative is given as:

$$ {}_tD_b^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \left(-\frac{d}{dx}\right)^m \int_x^b (t-x)^{m-\nu-1} f(t)dt. \tag{2} $$

Another fractional derivative, the left Caputo fractional derivative, is defined as:

$$ {}_aD_{*t}^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \int_a^x (t-x)^{m-\nu-1} \left(\frac{d^m f(t)}{dt^m}\right)dt, \tag{3} $$

while the right Caputo fractional derivative given by:

$$ {}_tD_{*b}^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \int_x^b (t-x)^{m-\nu-1} (-1)^m \left(\frac{d^m f(t)}{dt^m}\right)dt, \tag{4} $$

where $m \in \mathrm{N}$.

2.2. Fractional Optimal Control Problem Formulation

The FOCPs in the sense of Caputo are formulated as follows:

$$ Minimize \quad J(x,u) = \int_a^b f(t, x(t), u(t))dt, \tag{5} $$

subject to:

$$ {}_aD_{*t}^{\nu} x(t) = G(x(t), u(t), t), \tag{6} $$

the initial conditions of the problems are ($k = m$-1).

$$ \begin{aligned} x^k(a) &= c_k, \\ x^k(b) &= s_k, \end{aligned} \tag{7} $$

where

$$G(t)=(g_1(t),\dots,g_l(t))^t,\quad u(t)=(u_1(t),\dots,u_m(t))^t,$$
$$x(t)=(x_1(t),\dots,x_l(t))^t,$$
$$f\in C([a,b]\times R^l\times R^m),$$

in which x and u are the state and control variables respectively, $t\in[a,b]$ stands for the time and f and G are given nonlinear functions. Here, we assume that FOCPs have a unique solution. The basic existence and uniqueness follow from the Lipschitz condition by using contraction mapping theorem and a weighted norm with Mittag-Leffler in Yakar et al. (2012), Lakshmikantham and Mohapatra (2001), Podlubny (1999), Samko et al. (1993), Shaw and Yakar (1999).

3. Description of Method

In this section the proposed method is described and an associated algorithm is presented. The continuous control function $u(t)$ is approximated with a finite combination of elements of a basis Rudin (1976) as follow:

$$u(t)=\sum_{j=0}^{r}w_j q_j. \tag{8}$$

Since the FOCPs are solved by homotopy perturbation method, we construct a convex homotopy as follows:

$$\frac{dx^m}{dt^m}=p\left(\frac{dx^m}{dt^m}+G(x(t),u(t),t)-{}_aD_{*t}^{\nu}x(t)\right),\quad p\in[0,1] \tag{9}$$

and suppose the solution of Equation(6) has the following form:

$$x=x_0+px_1+p^2x_2+\dots, \tag{10}$$

where $x_j(t,w_0,w_1,\dots,w_r),\ j=0,1,\dots,r$ are unknown functions. Substituting Equation (10) into Equation (9) for $x(t)$, and equating the coefficients of the terms with identical powers of p, we derive:

$$\begin{cases} p^0:\dfrac{dx_0^m}{dt^m}=0 \qquad x_0^k(a)=c_k, \\ p^1:\dfrac{dx_1^m}{dt^m}=\dfrac{dx_0^m}{dt^m}+G(x_0(t),u(t),t)-{}_aD_{*t}^{\nu}x_0(t), \qquad x_1^k(a)=0, \\ p^2:\dfrac{dx_2^m}{dt^m}=\dfrac{dx_1^m}{dt^m}+G(x_1(t),u(t),t)-{}_aD_{*t}^{\nu}x_1(t), \qquad x_2^k(a)=0, \\ \vdots \end{cases} \tag{11}$$

As $p \to 1$, Equation (9) tends to Equation (6) and Equation (10), in most cases converges to an approximate solution of Equation (6), i.e.,

$$x = x_0 + x_1 + x_2 + \ldots. \tag{12}$$

By substitution of Equation (8) and Equation (12) into Equation (5) and Equation (7) we obtain an approximate solution of FOCPs as follows:

$$\begin{aligned} &\min_{(w_0,\ldots,w_r)} && J_r(w_0,\ldots,w_r) = \int_a^b f(t, x(t, w_0,\ldots,w_r), \sum_{j=0}^{r} w_j q_j(t))dt, \\ &s.t: && x^k(a, w_0, w_1,\ldots,w_r) = c_k, \\ & && x^k(b, w_0, w_1,\ldots,w_r) = s_k. \end{aligned} \tag{13}$$

Let J_r^* be the optimal value of Equation (13). A stopping criterion is proposed as follows:

$$\left| J_r^* - J_{r-1}^* \right| < \varepsilon, \tag{14}$$

where the small positive number ε is chosen according to the accuracy desired. We propose the following algorithm, which is presented in two stages.

Algorithm:

Initialization step:
Choose ε for the accuracy desired and set $r = 1$, and go to the main step.

Main step
Step 1. Set $u(t)$ by Equation (8) and go to Step 2.

Step 2. Compute $x(t)$ by Equation (12) and go to Step 3.

Step 3. Then compute $J_r^* = \inf\ J_r$ by Equation (13). If $r = 1$ go to step (5). Otherwise, go to Step 4.

Step 4. If the stopping criterion Equation (14) holds, then stop; else , go to Step 5.

Step 5. $r = r + 1$ and go step 1.

3. Numerical Results

In this section we apply the method presented in Section 3 to solve the following two test examples. All computations carried out by the package MAPLE 13.

Example 1. Consider the following time invariant problem

$$J = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right]dt, \tag{15}$$

subject to:

$${}_0D_{*t}^{\nu} x(t) = -x(t) + u(t),$$

with the initial condition

$$x(0) = 1.$$

The exact solution for $\nu = 1$ is

$$x(t) = \cosh\left(\sqrt{2}t\right) + \beta \sinh\left(\sqrt{2}t\right),$$

$$u(t) = \left(1 + \sqrt{2}\beta\right)\cosh\left(\sqrt{2}t\right) + \left(\sqrt{2} + \beta\right)\sinh\left(\sqrt{2}t\right),$$

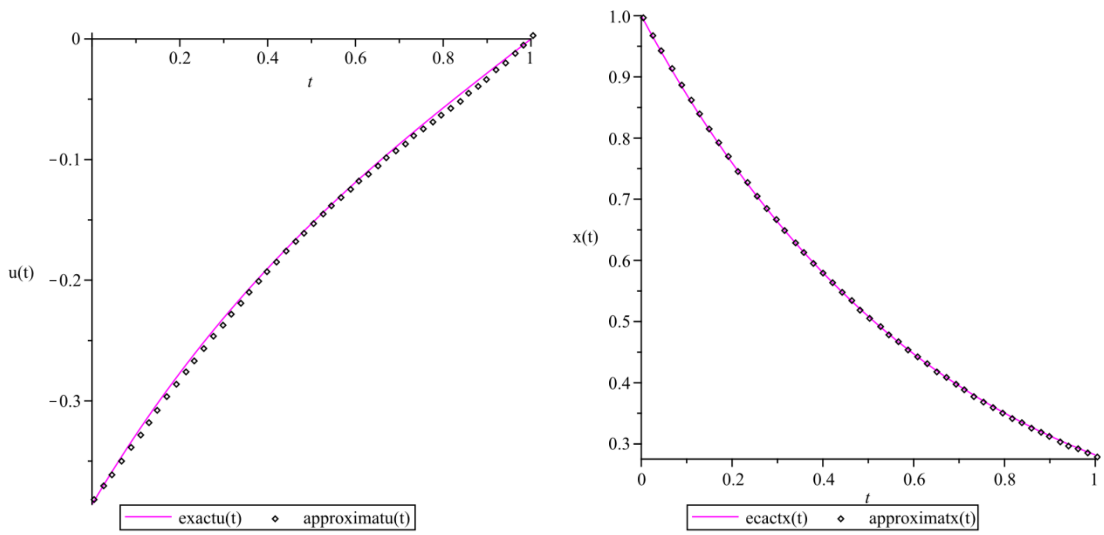
where

$$\beta = -\frac{\cosh\left(\sqrt{2}\right) + \sqrt{2}\sinh\left(\sqrt{2}\right)}{\sqrt{2}\cosh\left(\sqrt{2}\right) + \sinh\left(\sqrt{2}\right)} \approx -0.98,$$

with objective value $J^*(x,u) = 0.192909$. In Figure 1, the state variable $x(t)$ and the control variable $u(t)$ are plotted for $\nu = 1$. It is obvious that by applying the algorithm presented in section 3, the approximate values of $x(t)$ and $u(t)$ converge to the exact solutions. Figure 2 shows the state $x(t)$ and the control input $u(t)$ as functions of time t for different values of ν. Choosing $\varepsilon = 10^{-5}$, the results of the applying the given algorithm are presented in Table 1.

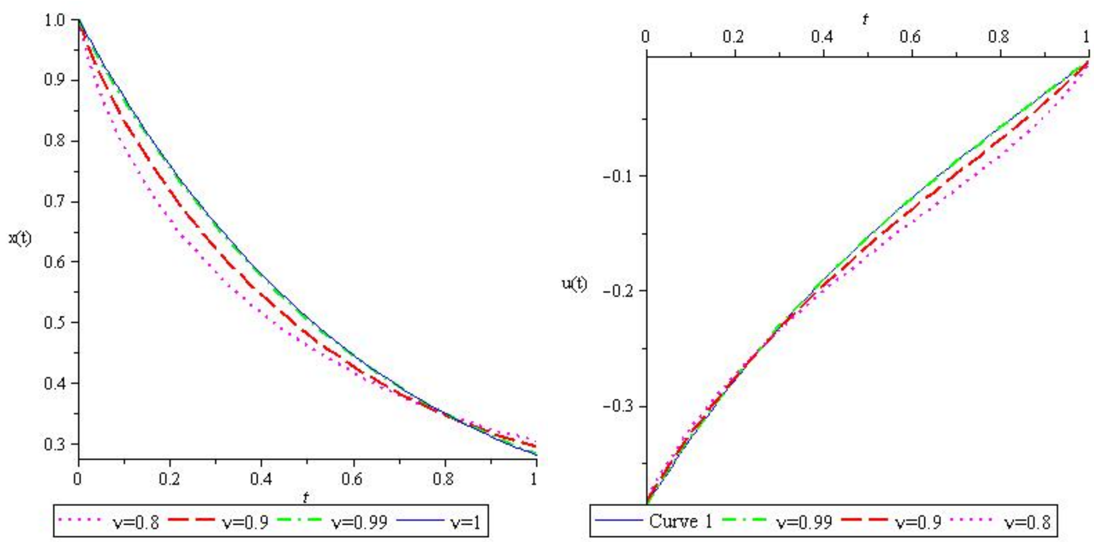
Table 1. Numerical results in Example 1

ν	r	n	J^*
1	4	7	0.192909
0.99	6	6	0.19153
0.9	7	6	0.17952
0.8	6	5	0.16729



(a) (b)

Figure 1. (a) Approximate solutions and exact solution of $u(t)$ for $(n = 7, \nu = 1)$
(b) Approximate solutions and exact solution of $x(t)$ for $(n = 7, \nu = 1)$ for Example 1



(a) (b)

Figure.2: (a) State $x(t)$ as a function of t for different values of ν
(b) Control $u(t)$ as a function of t for different values of ν for Example 1

Example 2. In this example, a time varying FOCP is considered to find the control $u(t)$, which minimizes the performance index

$$J = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right]dt, \tag{16}$$

subject to:

$${}_0D_{*t}^{\nu} x(t) = tx(t) + u(t),$$

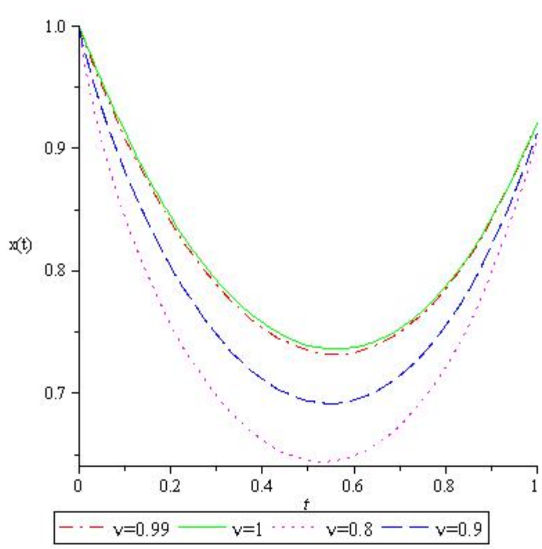
with free terminal condition and the initial condition

$$x(0) = 1.$$

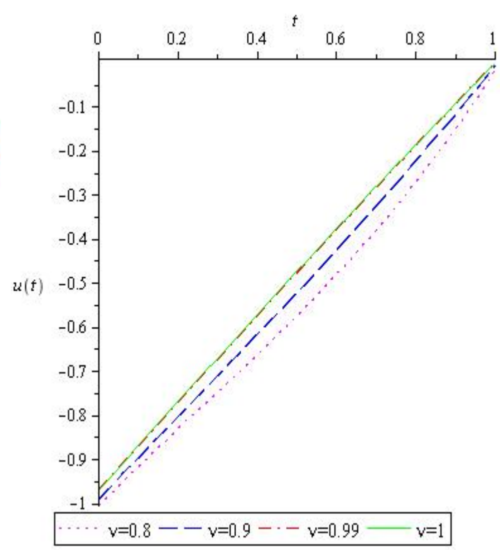
Figure 3 demonstrates the approximation of $x(t)$ and $u(t)$ for different values of ν. The results of applying the algorithm are presented in Table 2. Figure 4 shows the state and the control variables, respectively, as a function of time for $\nu = 0.8$ for different values of n. It is obvious that the approximate values $x(t)$ and $u(t)$ converge to the exact solutions by increasing the values of n.

Table 2. Numerical results in Example 2

ν	r	n	J^*
1	3	5	0.48427
0.99	3	5	0.48347
0.9	3	5	0.47605
0.8	3	5	0.46722

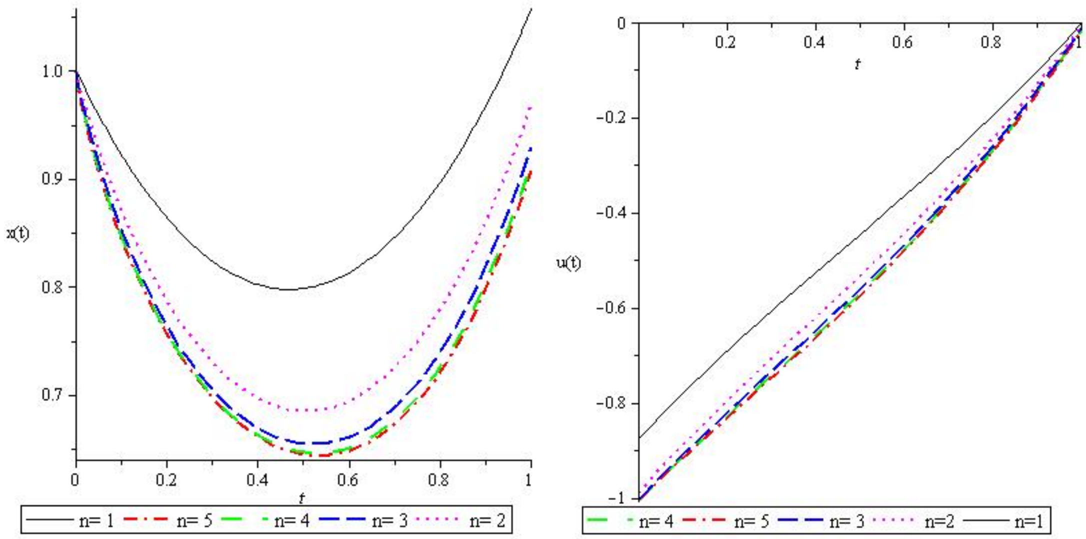


(a)



(b)

Figure 3. (a) State $x(t)$ as a function of t for different values of ν
(b) Control $u(t)$ as a function of t for different values of ν for Example 2



(a) (b)

Figure 2: (a) Convergence of the state variable for the time-varying system for $\nu = 0.8$
(b) Convergence of the control variable for the time- varying system for $\nu = 0.8$.

Test problems 1 and 2 were solved in Agrawal (2008a) in a different way. Our results, shown in Figures 1-4 are in good agreement with the results demonstrated in Agrawal (2008a). But, we achieved satisfactory numerical results in only 5 iterations while in Agrawal (2008a), the number of approximations starts in 10 and increases up to 320. So it is significant that we achieved our numerical results with a very small order of approximations. Also we find the approximate optimal value of the objective function for each ν.

4. Conclusion

In this paper, we have developed the homotopy perturbation and parameterization methods for solving a class of fractional optimal control problems. By the proposed method we are able to reduce the main problem to an optimization problem. The numerical results have demonstrated the high accuracy of the proposed method.

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