Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 5, Issue 1 (June 2010) pp. 128 - 147
(Previously, Vol. 5, No. 1)

Applications and Applied Mathematics:
An International Journal
(AAM)

Approximations of Sturm-Liouville Eigenvalues Using Sinc-Galerkin and Differential Transform Methods

Marwan Taiseer Alquran and Kamel Al-Khaled
Mathematics and Statistics
Faculty of Science
Jordan University of Science and Technology
Irbid, 22110, Jordan
marwan04@just.edu.jo; kamel@just.edu.jo

Received: September 23, 2009; Accepted: March 7, 2010

Abstract

In this paper, we present a comparative study of Sinc-Galerkin method and differential transform method to solve Sturm-Liouville eigenvalue problem. As an application, a comparison between the two methods for various celebrated Sturm-Liouville problems are analyzed for their eigenvalues and solutions. The study outlines the significant features of the two methods. The results show that these methods are very efficient, and can be applied to a large class of problems. The comparison of the methods shows that although the numerical results of these methods are the same, differential transform method is much easier, and more efficient than the Sinc-Galerkin method.

Keywords: Sinc-Galerkin Method, Differential Transform Method, Sturm-Liouville Problem, Approximate Methods, Ordinary Differential Equations

MSC (2000) No.: 34B 05, 34A 45, 65L15, 65L 20

1. Introduction

The concept of an eigenvalue problem is rather important, both in pure and applied mathematics, in physical systems such as pendulums and vibrating and rotating shafts. The Sturm-Liouville systems arise from vibration problems in continuum mechanics. In physics, they describe boundary value problems corresponding to simply harmonic standing waves. A general Sturm-Liouville problem (SLP) can be written as the following differential equation

$$L(y) = \frac{d}{dx}\left[p(x)\frac{dy(x)}{dx} \right] + \left[\lambda r(x) - q(x) \right] y(x) = f(x), \;\; x \in (0,1), \tag{1.0.1}$$

subject to the boundary conditions

$$y(0) = 0, y(1) = 0, \tag{1.0.2}$$

where $p(x) \geq 0, r(x) \geq 0$ and $q(x) \geq 0$, also, $p(x)$, $r(x)$, $q(x)$, and $f(x)$ all are continuous on the closed interval [0,1]. The values of the parameter λ for which equation (1.0.1), together with appropriate boundary conditions, like (1.0.2), gives rise to non-trivial solution *y*, are called eigenvalues, while the corresponding non-trivial solutions y are called eigenfunctions of the problem. It is well known that there exists an infinite number of eigenvalues for equation (1.0.1) together with the associated conditions (1.0.2). They are real, simple, countable, and isolated see Chanane (2007).

In recent years there has been a considerable renewal of interest in the SLP, from the point of view of both mathematics and its applications to physics and engineering. For many important applications in science and engineering, it is required to determine the eigenvalues as well as the corresponding eigenfunctions. In an application involving vibration and stability of deformable bodies, for example, the viral piece of information required is the smallest eigenvalue Bujurke, et al. (2009). Engineers are often interested in the location of the smallest eigenvalue since this gives potentially the most visual structure of dynamical systems. The eigenvalues are also crucial in finding the stability region of solutions of SLP Bender and Orszag (1987). For the solution of SLP, some studies have been carried out. Bujurke et al. (2009) used truncated Haar wavelet series for the computation of eigenvalues and solutions of SLPs. The collocation method of the weight residual methods is investigated for the approximate computation of higher SLP Ibrahim (2005). Chanane (2007) has used Shannon sampling theory to compute the eigenvalues of regular SLP. Asymptotic formulas for eigenvalues associated with Hill's equation have been studied by Guseinov and Karaca (2004). The Sinc-Galerkin method was used to approximate solutions of nonlinear problems involving nonlinear second, fourth, and sixth-order differential equation El-Gamel and Zayed (2004). The Sinc-Galerkin method was also used by Ramos (2005) to solve two-point boundary value problem with applications to chemical reactor theory. Adel and El-Gamel (2009) compared the performance of the collocation and Galerkin methods using Sinc bases for solving linear and nonlinear second-order two-point boundary value problem.

Differential transform method was used by Siraj-Ul et al. (2009) to find numerical solution of special 12th order boundary value problems with two point boundary conditions. In order to obtain more efficient numerical results, several ways have been devised in the last years, for example, see Farlow (1982), Kadakal and Mukhtarov (2007), Trim (1990) and Ugour (2006).

In this paper we introduce two methods for solving (1.0.1), (1.0.2), namely, Sinc-Galerkin Method, and Differential Transform Method (DTM for short). DTM, which is based on Taylor series expansion, has been introduced by Zhou (1986) in a study about electrical circuits. It gives exact values of the nth derivative of an analytical function at a point in terms of known and unknown boundary conditions in a fast manner for more detail, see Vedat and Shaher (2007). Stenger (1993) originally proposed the numerical solution of ordinary differential equations with the Sinc-Galerkin method. Excellent expositions of the use of Sinc function to approximate differential equations are found in Norman et al. (1987), McArthur and Kelly (1989), and Stenger (1993). A basis element may be transformed to any connected subset of the real line via a composition with a suitable conformal map in conjunction with the Galerkin method for differential equations.

There are several reasons to approximate by Sinc functions. First, they are easily implemented and give good accuracy for problems with singularities. Secondly, the most distinctive feature of the basis is its resulting exponential convergence rate of the error $\mathrm{O}\left(e^{-c\sqrt{N}}\right)$, where $c > 0$ and the $2N+1$ basis functions is used to build the approximation. We should mention that the effect of any such singularities will appear in some form for any scheme of numerical solution rather that the Sinc function.

Moreover, the convergence rate maintains when the solution of the differential equation has boundary singularities. Of equal practical significance is that the technique's implementation requires no modification in the presence of singularities. Specifically, the statement of the quadrature, the mesh definition and the resulting matrix structure depend only on the parameters of the differential equation whether it is singular or nonsingular.

This paper is organized as follows. The Sinc solution together with the Galerkin method and the development of the scheme is treated in Section 2. Also we formulate an iterative procedure to solve (1.0.1), (1.0.2) using Differential Transform method in Section 3. Section 4 provides numerical examples which demonstrate the exponential convergence of the Sinc method and compare its performance with DTM.

2. Sinc Function Approximation

A thorough review of Sinc function properties and the general Sinc-Galerkin method can be found in Norman et al. (1987) and Stenger (1993). In this section, an overview of the basic formulation of the Sinc function required for our subsequent development is presented.

2.1 Sinc Function Properties

The Sinc-Galerkin procedure for solving problem (1.0.1) subject to (1.0.2) begins by selecting composite Sinc functions appropriate to the interval (0,1) so that their translates form a basis functions for the expansion of the approximate solution $y(x)$. In what follows, overviews of the properties of the Sinc function that will be used in this paper are given. The Sinc function (known in engineering as the band-limited function) is defined on the whole real line by

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For $h > 0$, the translated Sinc functions with evenly spaced nodes are given as

$$S(k,h)(x) = \operatorname{sinc}\left(\frac{x-kh}{h}\right) = \begin{cases} \dfrac{\sin\left(\dfrac{\pi}{h}(x-kh)\right)}{\dfrac{\pi}{h}(x-kh)}, & x \neq kh,\ k = 0, \pm 1, \pm 2, ..., \\ 1, & x = kh,\ k = 0, \pm 1, \pm 2, ... \end{cases}$$

If f is defined on the real line, then for $h > 0$ the series

$$f(x) = C(f,h)(x) = \sum_{k=-\infty}^{k=\infty} f(kh) S(k,h)(x)$$

is called the Whittaker cardinal series of f whenever this series converges.

Definition 2.1:

Let $d > 0$ and D_d denote the open strip

$$D_d = \{ z \in C : |\operatorname{Im}(z)| < d \}. \tag{2.1.1}$$

Theorem 2.1:

Stenger (1993): Let D be a simply connected domain, and $a, b \in \partial D$ be such that $a \neq b$. Then, there exists a conformal map $\phi : D \to D_d$ satisfying $\psi(R) = \phi^{-1}(R) = \Gamma$, and such that for $z \in \Gamma$, we have $\lim_{z \to a^+} \phi(z) = -\infty$ and $\lim_{z \to b^-} \phi(z) = \infty$.

The class of functions such that the known exponential error estimates exist for Sinc interpolation is denoted by $B(D)$ and defined as follows:

Definition 2.2:

Let $B(D)$ be the class of functions such that f is holomorphic on the simply connected domain D;

$$\int_{\psi(t+L)} |f(w)dw| = \mathrm{O}(|t|^a),\ t \to \infty,$$

where $a \in [0,1)$, and $L = \{iy : |y| < d\}$; and

$$N(f;D) = \lim_{C \subset D} \inf_{C \to \partial D} \int_C |f(w)dw| < \infty,$$

where C is a simple closed curve in D.

Definition 2.3:

Let $f \in B(D)$, ϕ be a conformal one-to-one mapping of D onto D_d with inverse ψ and $t \in \Gamma = \psi(R) = \phi^{-1}(R)$. Then, f/ϕ' is said to decay exponentially with respect to ϕ, if there exist positive constants M and α such that

$$\left|\frac{f(t)}{\phi'(t)}\right| \le M \exp(-\alpha|\phi(t)|),\ t \in \Gamma. \tag{2.1.2}$$

The importance of the class $B(D)$ with regard to numerical integration is summarized in the following theorem, whose proof can be found in Stenger (1993).

Theorem 2.2:

If f satisfies Definition 2.3 and $z_k = \psi(kh)$, then for h sufficiently small we have

$$\left|\int_\Gamma f(z)dz - h\sum_{k=-\infty}^{\infty} \frac{f(z_k)}{\phi'(z_k)}\right| \le \frac{\mathrm{N}(f;D)}{1-e^{-d}} e^{-2\pi d/h}.$$

Further, if f/ϕ' decays exponentially with respect to ϕ, and taking $h = \sqrt{2\pi d/(\alpha N)}$ gives

$$\left| \int_{\Gamma} f(z)dz - h \sum_{k=-N}^{N} \frac{f(z_k)}{\phi'(z_k)} \right| \le M\, e^{-2\pi d\alpha/h}. \tag{2.1.3}$$

2.2. The General Sinc-Galerkin Method

The orthogonalization of the residual in the Sinc-Galerkin method for a differential equation of the form

$$\boldsymbol{L}\, u = F \tag{2.2.1}$$

can be treated as follows:

Define the approximate solution by

$$y_T(x) = \sum_{j=-N}^{N} y_j S_j(x), \tag{2.2.2}$$

where $S_j(x) = S(j,h) \circ \phi(x)$. The unknown coefficients $\{y_j\}$ in (2.2.2) are determined by orthogonalizing the residual $\boldsymbol{L}u_T - F$ with respect to the functions $\{S_j\}_{j=-N}^{N}$, which yields the discrete system

$$\left\langle \boldsymbol{L}u_T - F, S_j \right\rangle = 0,\ -N \le j \le N. \tag{2.2.3}$$

The most direct development of the discrete system for (2.2.1) is obtained by substituting (2.2.2) into (2.2.3). This approach, however, obscures the analysis that is necessary for applying Sinc quadrature formulas (2.1.3). An alternative approach is to analyze instead

$$\left\langle \boldsymbol{L}y - F, S_j \right\rangle = 0,\ -N \le j \le N, \tag{2.2.4}$$

where, for now, a general weight function w is used in the inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x) w(x) dx.$$

The discrete system resulting from (2.2.4) is the same as that arising from (2.2.3) within the accuracy of the method. The accurate approximation of the integrals arising from (2.2.3) is accomplished in the next subsection.

2.3. The Sinc Methodology

Approximations using the Sinc functions on (0,1) are obtained from corresponding approximations on R via a conformal map. For a function $f \in C^{\infty}(R)$ to be approximated on *R*, f must obey certain analyticity and boundedness conditions in a strip in the complex plane *C*, which contains *R*.

Through the conformal map, we obtain a corresponding "eye-shaped" region see Figure 1.7, pp. 68, of Stenger (1993) containing the interval (0,1). Our integrand must obey certain analyticity and boundedness conditions. To construct approximations on the interval (0,1), which is used in this paper, the eye-shaped domain in the z-plane

$$D_E = \left\{ z = x + iy \in C : \left| \arg(z / (1-z)) \right| < d \le \pi / 2 \right\} \tag{2.3.1}$$

is mapped conformally onto the infinite strip D_d via $w = \phi(z) = \ln\left(\frac{z}{1-z}\right)$, and this is a suitable domain for the Sinc-Galerkin method for boundary value problems of the form (1.0.1). The basis functions on (0, 1) are taken to be the composite translated Sinc functions

$$S_j(z) = S(j,h) \circ \phi(z) = \operatorname{sinc}\left(\frac{\phi(z) - jh}{h}\right),\ z \in D_E.$$

The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \psi(w) = \left(\frac{\exp(w)}{1+\exp(w)}\right)$$

Thus, we may define the inverse images of the real line and of the evenly spaced nodes $\{jk\}_{j=-\infty}^{\infty}$ as

$$\Gamma = \left\{ \psi(t) \in D_E : -\infty < t < \infty \right\} = (0,1)$$

and,

$$x_j = \psi(jh) = \left(\frac{\exp(jh)}{1+\exp(jh)}\right),\ j = 0, \pm 1, \pm 2, \ldots,$$

respectively. To proceed with the development of the approximate solution of equation (1.0.1), simplify equation (1.0.1) as

$$y''(x) + \tau(x) y'(x) + \lambda R(x) y(x) + Q(x) y(x) = F(x), \tag{2.3.2}$$

where $\tau(x) = p'(x)/p(x)$, $R(x) = r(x)/p(x)$, $Q(x) = q(x)/p(x)$, and $F(x) = f(x)/p(x)$. The most direct development of the discrete system for equation (2.3.2) is obtained by substituting

$$y_T(x) = \sum_{j=-N}^{N} y_j S_j \circ \phi(x) \tag{2.3.3}$$

onto equation (2.3.2), where $y_T(x)$ is the approximated solution and y_j are unknowns to be determined.

Define the residual

$$R_T = \boldsymbol{L} u_T - F \tag{2.3.4}$$

and the weighted inner product <.,.> is taken to be

$$\langle f, g \rangle = \int_0^1 f(x) g(x) w(x) dx \ . \tag{2.3.5}$$

Here, $w(x)$ plays the role of a weight function, which may be chosen for a variant of reasons. Although other reasons exist, a choice we make here is due to the requirement that the boundary condition vanish. For the case of second order boundary value problems, it is convenient to take $w(x) = 1/\phi'(x)$.

A complete discussion on the choice of the weight function can be found in Norman et al. (1987) and Stenger (1993). Orthogonalizing the residual with respect to

$$\vec{S} = \begin{bmatrix} S_{-N} \circ \phi \\ S_{-N+1} \circ \phi \\ \vdots \\ S_N \circ \phi \end{bmatrix} \tag{2.3.6}$$

leads to the system

$$\langle y_T'', \vec{S} \rangle + \langle \tau(x) y_T', \vec{S} \rangle + \langle \lambda R(x) y_T, \vec{S} \rangle + \langle Q(x) y_T, \vec{S} \rangle = \langle F, \vec{S} \rangle . \tag{2.3.7}$$

For ease of notation, from now on we replace y_T by y. Following the standard Sinc-Galerkin method, the first term in equation (2.3.7) is integrated by parts twice and the second term once. Assuming the boundary conditions $y(0) = 0 = y(1)$, equation (2.3.7) leads to

$$\int_0^1 y(x)[\vec{S}w]''(x)dx - \int_0^1 y(x)[\tau\vec{S}w]'(x)dx + \int_0^1 y(x)[\lambda R\vec{S}w](x)dx + \int_0^1 y(x)[Q\vec{S}w](x)dx = \int_0^1 y(x)[F\vec{S}w](x)dx. \tag{2.3.8}$$

To construct an approximate solution via the Sinc-Galerkin method, we need to evaluate the integrals in (2.3.8) and hence to derive a linear system, the Sinc quadrature rule (2.1.3) will be used (for details of the quadrature rule and conditions governing its error bounds, see Stenger 1993). We only state that if an integral of a function $G(x)$ over the interval (0, 1) satisfies the hypothesis of the quadrature rule, then for $x_k = \phi^{-1}(kh)$ we have

$$\left| \int_0^1 G(x)dx - h\sum_{k=-N}^{N} \frac{G(x_k)}{\phi'(x_k)} \right| \le \mathrm{O}\left(e^{-M\sqrt{N}}\right).$$

The Sinc-Galerkin method requires derivatives of composite Sinc functions evaluated at the nodes. For a one-to-one conformal mapping ϕ of the simply connected domain D_E onto D, we need the following expressions required for the present work:

$$\delta_{i,j}^{(0)} \equiv \begin{cases} 1, & i = j, \\ 0, & i \ne j, \end{cases} \tag{2.3.9}$$

$$\delta_{i,j}^{(1)} \equiv h\frac{d}{d\phi}\left[S_i \circ \phi(x)\right]\downarrow_{x=x_j} = \begin{cases} 1, & i = j, \\ \dfrac{(-1)^{j-i}}{j-i}, & i \ne j, \end{cases} \tag{2.3.10}$$

$$\delta_{i,j}^{(2)} \equiv h^2\frac{d^2}{d\phi^2}\left[S_i \circ \phi(x)\right]\downarrow_{x=x_j} = \begin{cases} \dfrac{-\pi^2}{3}, & i = j, \\ \dfrac{-2(-1)^{j-i}}{(j-i)^2}, & i \ne j. \end{cases} \tag{2.3.11}$$

Therefore, the terms in the ith equation of (2.3.8) are approximated by

$$\int_0^1 y(x)[S_i \circ \phi w]''(x)dx \approx h\sum_{j=-N}^{N}\left[\frac{1}{h^2}\delta_{i,j}^{(2)}\phi'(x_j)w(x_j) + \frac{1}{h}\delta_{i,j}^{(1)}\left(\frac{\phi''(x_j)}{\phi'(x_j)}w(x_j) + 2w'(x_j)\right) + \delta_{i,j}^{(0)}\frac{w''(x_j)}{\phi'(x_j)}\right]y(x_j),$$

while

$$\int_0^1 y(x)[\tau S_i \circ \phi w]'(x)dx \approx h\sum_{j=-N}^{N}\left[\frac{1}{h}\delta_{i,j}^{(1)}\tau(x_j)w(x_j)+\delta_{i,j}^{(0)}\frac{(\tau w)'(x_j)}{\phi'(x_j)}\right]y(x_j).$$

Also,

$$\int_0^1 y(x)[\lambda R S_i \circ \phi w](x)dx \approx \lambda h\sum_{j=-N}^{N}\delta_{i,j}^{(0)}R(x_j)\frac{w(x_j)}{\phi'(x_j)}y(x_j),$$

$$\int_0^1 y(x)[Q S_i \circ \phi w](x)dx \approx h\sum_{j=-N}^{N}\delta_{i,j}^{(0)}Q(x_j)\frac{w(x_j)}{\phi'(x_j)}y(x_j),$$

and

$$\int_0^1 F(x)[S_i \circ \phi w](x)dx \approx h\sum_{j=-N}^{N}\delta_{i,j}^{(0)}\frac{w(x_j)}{\phi'(x_j)}F(x_j).$$

The following notations will be necessary for writing down the system.

Define the $(2N+1)\times(2N+1)$ matrices:

$$I^{(p)} = \left[\delta_{i,j}^{(p)}\right],\ p=1,2. \tag{2.3.12}$$

For example, the matrix $I^{(2)}$ is the $(2N+1)\times(2N+1)$ is the matrix whose i, j^{th} entry is given by (2.3.11).

Let $D(y)$ be the $(2N+1)\times(2N+1)$ diagonal matrix defined by:

$$D(y) = diag[y(z_{-N}),...,y(z_N)]^T,$$

where the superscript "T" denotes the transpose of the matrix. The discretized Sinc-Galerkin system corresponding to (2.3.8) has the more compact matrix representation as

$$\left(A+B+\lambda D(R)D\left(\frac{w}{\phi'}\right)+D(Q)D\left(\frac{w}{\phi'}\right)\right)\vec{Y}_{2N+1} = D\left(\frac{w}{\phi'}\right)\vec{F}_{2N+1}, \tag{2.3.13}$$

where the matrix A can be regarded as the Sinc discretization of y'', is given by

$$A = \left[\frac{1}{h^2}I^{(2)}D(\phi' w)+\frac{1}{h}I^{(1)}D\left(\frac{\phi''}{\phi'}w+2w'\right)+D\left(\frac{w''}{\phi'}\right)\right] \tag{2.3.14}$$

and

$$B = -\left[\left(\frac{1}{h} I^{(1)} D(w) + D\left(\frac{w'}{\phi'}\right)\right) D(\tau) + D\left(\frac{w}{\phi'}\right) D(\tau')\right].$$

With this in mind, we have arrived at the following theorem:

Theorem 2.3:

If the assumed approximate solution of the Sturm-Liouville eigenvalue problem (1.0.1) subject to the conditions in equation (1.0.2) is equation (2.3.3), then the discrete Sinc-Galerkin system for the determination of the unknown coefficients $\{y_j\}, j = -N, ..., N$ is given by (2.313).

Now we have a linear system of $2N+1$ equations of the $2N+1$ unknown coefficients, namely, $\{y_j\}_{j=-N}^{N}$. We can obtain the coefficient of the approximate solution by solving the linear system (2.3.13) by the LU decomposition method. The solution $\vec{Y} = (y_{-N}, ..., y_N)$ gives the coefficients in the approximate Sinc-Galerkin solution $y_T(x)$ of $y(x)$.

3. Differential Transform Method

The differential transform technique, which was first proposed by Zhou (1986), is one of the numerical methods for ordinary and partial differential equations, which use the form of polynomials as the approximation to the exact solutions that are sufficiently differentiable. The differential transform technique provides an iterative procedure to obtain higher-order series solution. Basic definitions and operations of differential transformation are introduced as follows:

An arbitrary function $y(x)$ can be expanded in Taylor series about a point $x = 0$ as

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k y}{dx^k}\right]_{x=0}. \tag{3.0.1}$$

The differential transformation of $y(x)$ is defined Zhou (1986) as

$$Y(k) = \frac{1}{k!} \left[\frac{d^k y}{dx^k}\right]_{x=0}. \tag{3.0.2}$$

Then, the inverse differential transform is

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k). \tag{3.0.3}$$

In actual application, the function $y(x)$ is expressed by a finite series

$$y(x) = \sum_{k=0}^{n} x^k Y(k). \tag{3.0.4}$$

Table 1: Fundamental operations of differential transform method

Original Function	Transformed function
$f(x) = c_1 g(x) \pm c_2 h(x)$	$F(k) = c_1 G(k) \pm c_2 H(k)$
$f(x) = g^n(x)$	$F(k) = \frac{(k+n)!}{k!} G(k+n)$
$f(x) = g(x)h(x)$	$F(k) = \sum_{k_1=0}^{k} G(k_1) H(k-k_1)$
$f(x) = x^n$	$F(k) = \delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$
$f(x) = \sin(wx + \alpha)$	$F(k) = \frac{w^k}{k!} \sin\left(\frac{\pi k}{2} + \alpha\right)$
$f(x) = \cos(wx + \alpha)$	$F(k) = \frac{w^k}{k!} \cos\left(\frac{\pi k}{2} + \alpha\right)$

Equation (3.0.3}) implies that $\sum_{k=n+1}^{\infty} x^k Y(k)$ is negligible small. In Table 1 the fundamental operations related to one-dimensional problems are listed. The proofs are well known in the literature Ugour (2006), Zhou (1986) and will not be proven here. To solve equation (1.0.1) subject to the boundary conditions in (1.0.2) using DTM. Following Abdel-Halim (2002), by applying Differential Transformations to equation (1.0.1), and using Table 1, we obtain

$$\sum_{i=0}^{k} (i+1) P(i+1)(k-i+1) Y(k-i+1) + \sum_{i=0}^{k} P(i)(k-i+1)(k-i+2) Y(k-i+2) + \sum_{i=0}^{k} \left[\lambda R(i) - Q(i)\right] Y(k-i) = F(k), \tag{3.0.5}$$

where the upper case symbols $P(k)$, $Q(k)$, $R(k)$, $Y(k)$, and $F(k)$ are used to denote the differential transformed function of $p(x)$, $q(x)$, $r(x)$, $y(x)$, and $f(x)$, respectively. Taking the differential transform of the first equation in (1.0.2) yields

$$Y(0) = 0, \tag{3.0.6}$$

while the differential transform of the second equation in (1.0.2) becomes

$$\sum_{k=0}^{n} Y(k) = 0. \tag{3.0.7}$$

Putting $Y(1) = c$ at $k = 0$ into (3.0.5) yields

$$Y(2) = \frac{F(0) - cP(1)}{2P(0)}. \tag{3.0.8}$$

Following the same procedure as above, we calculate the nth term $Y(n)$ and substituting $Y(1),...,Y(n)$ into (3.0.7) yields a polynomial of λ corresponding to n denoted by $f_n(\lambda)$. Solving $f_n(\lambda) = 0$ we get $\lambda = \lambda_i$, $i = 1, 2, ...$, as the nth estimated eigenvalue corresponding to n. Substituting λ_i into $Y(0), Y(1), ..., Y(n)$ and using $y(x) = \sum_{k=0}^{n} x^k Y(k)$, we obtain the eigenfunctions $y_i(x) = \sum_{k=0}^{n} x^k Y_{\lambda_i}$.

4. Numerical Applications

To show the efficiency of the two methods described in the previous sections, four examples in this section will be tested using the two methods. The examples reported in this section were selected from a large collection of problems to which the Sinc-Galerkin and DTM could be applied. For purposes of comparison, examples with known solutions were chosen. For the Sinc-Galerkin method, d is taken to be $\pi/2$ and $\alpha = 1/2$. The step size and the summation limit N are selected so that the error is asymptotically balanced. Once N is chosen, the step size $h = \pi\sqrt{N}$. We use the absolute error which is defined as $E_s = |y_{exact} - y_{\text{sinc}}|$ and $E_T = |y_{exact} - y_{transform}|$.

Some of the following considered examples can be reduced to the standard Sturm-Liouville eigenvalue problem and written in the form of equation (2.3.2).

Example 4.1: Singular Problem

Consider the singular two-point boundary value problem [Ravi and Aruna (2008)]

$$\left(1-\frac{x}{2}\right)\frac{d^2y}{dx^2}+\frac{3}{2}\left(\frac{1}{x}-1\right)\frac{dy}{dx}+\left(\frac{x}{2}-1\right)y(x)=5-\frac{29x}{2}+\frac{13x^2}{2}+\frac{3x^3}{2}-\frac{x^4}{2}, \tag{4.0.1}$$

subject to the following boundary conditions

$$y(0)=0\ ,\ \ y(1)=0\ . \tag{4.0.2}$$

The exact solution for this problem is $y(x)=x^2-x^3$. For solution using Sinc-Galerkin method, $\tau(x)=x-2$, $R(x)=-1$, $Q(x)=0$, and $F(x)$ is the right hand side of equation (4.0.1), and since the interval is (0,1), our conformal mapping will be $\phi(z)=\ln\left(\frac{z}{1-z}\right)$. The matrix A in (2.3.13) was set up by means of calculating the functions $\phi'\phi$, $\frac{\phi''}{\phi'}\phi$, and $\frac{\phi''}{\phi'}$. Upon passing simple *Mathematica* rules, we solve the discrete system in equation (2.3.13) for $N=32$. The results in Table 2 indicate that the exponential rate is maintained even with the presence of singularities. For solution using DTM, the transformed version of equation (4.0.1) is

$$\sum_{i=0}^{k}\delta(i-1)(k-i+1)Y(k-i+2)+\frac{1}{2}\sum_{i=0}^{k}\delta(i-2)(k-i+1)(k-i+2)Y(k-i+2)$$

$$+\frac{3}{2}(k+1)Y(k+1)-\frac{3}{2}\sum_{i=0}^{k}\delta(i-1)(k-i+1)Y(k-i+1)+\frac{1}{2}\sum_{i=0}^{k}\delta(i-2)Y(k-i)$$

$$-\sum_{i=0}^{k}\delta(i-1)Y(k-i)-5\delta(i-1)+\frac{29}{2}\delta(i-2)-\frac{13}{2}\delta(i-3)-\frac{3}{2}\delta(i-4)+\frac{1}{2}\delta(i-5)=0\ .$$

The transformed boundary conditions are $Y(0)=0$ and $\sum_{k=0}^{n}Y(k)=0$. Using the transformed equation and boundary conditions for $n=7$, we obtain 8×8 system of algebraic equations with unknowns $Y(0), Y(1), ..., Y(7)$. Solving this system and using the inverse transformation rule (3.0.3), we get the closed form solution $y(x)=x^2-x^3$, which is our exact solution for the given problem. This shows excellent performance of the DTM.

Table 2: The absolute error using Sinc-Galerkin method when solving the discrete system (2.3.13) when $N = 32$ for Example 4.1

x	E_s
0	6.02×10^{-10}
0.1	2.05×10^{-8}
0.2	4.82×10^{-8}
0.3	4.94×10^{-8}
0.4	3.21×10^{-8}
0.5	6.05×10^{-9}
0.6	4.85×10^{-8}
0.7	3.36×10^{-8}
0.8	7.05×10^{-8}
0.9	1.23×10^{-8}
1.0	4.82×10^{-10}

Example 4.2: Titchmarch Equation

We consider the Titchmarch model

$$\frac{d^2y(x)}{dx^2}+\left(\lambda-x^{2m}\right)y(x)=0,\quad y(0)=y(1)=0\,, \tag{4.0.3}$$

where m is a nonnegative integer. For the Sinc solution, we follow the procedure outlined in subsection 2.3 to find numerical solution for (4.0.3). The first eigenvalue listed in Table 3 is computed from the matrix system (2.3.13) when $N = 32$ for three different values of the parameter m .

Table 3: An estimate to the first eigenvalue λ_1 for Example 4.2

m	Sinc-Galerkin ($N = 32$)	Differential Transform ($n = 10$)
0	10.933	10.932
1	10.702	10.701
2	10.345	10.345

For the solution using DTM, using Table 1 and by taking Differential Transform for both sides of (4.0.3), we have

$$Y(k+2) = \frac{k!}{(k+2)!}\sum_{i=0}^{k}\delta(i-2m)Y(k-i) - \lambda Y(k). \tag{4.0.4}$$

The differential transform of the boundary conditions become $Y(0)=0$ and $\sum_{k=0}^{n} Y(k)=0$. For the first eigenvalue, substituting $Y(0)=0$, and putting $Y(1)=c$ at k = 0, 1, 2, ..., 10 into (4.0.4), we get the values of $Y(2), Y(3), ..., Y(10)$. Then, substituting $Y(0), ..., Y(10)$ into $\sum_{k=0}^{10} Y(k)=0$, we get $f_{10}(\lambda)=0$, for some function *f*. Solving for λ_1, we obtain an estimate for the first eigenvalue for different values of the parameter m. These results are shown in Table 3.

The first eigenvalue of the comparison equation $y''(x)+\lambda y(x)=0,\ \ y(0)=y(1)=0$ is π^2. Table 3 predicts an estimate for the least eigenvalue λ_1 that satisfies $\pi^2 < \lambda_1 < 11$, which is consistent with the results obtained in Bujurke, Salimath and Shiralashetti (2009). The corresponding eigenfunction to the first eigenvalue λ_1 for equation (4.0.3) when $m=2$ was computed at some points in its domain. The results are listed in Table 4.

Table 4: Comparison of solutions corresponding to the first eigenvalue λ_1 for Example 4.2, when $m=2$ using the two methods.

x	Sinc-Galerkin ($N=32$)	Differential Transform ($n=10$)
0	0	0
0.125	0.5461	0.5477
0.250	1.002	1.006
0.375	1.321	1.326
0.500	1.422	1.416
0.625	1.321	1.326
0.750	1.002	1.006
0.875	0.5301	0.5312
1.000	0	0

Example 4.3:

Consider the second order differential equation

$$-\frac{d^2y(x)}{dx^2} + \left(\cos^2 x\right)y(x) = \lambda y(x),\quad y(0)=y(\pi)=0. \tag{4.0.5}$$

For the Sinc method, the computed eigenvalues λ_i are obtained from the solution of the $(2N+1)\times(2N+1)$ system (2.3.13) when $N=32$, where in this case $\phi(x)=\ln\left(\frac{x}{\pi-x}\right)$, and $x_k=\frac{\pi e^{kh}}{e^{kh}+1}$, $p(x)=-1$, $\tau(x)=0$, $Q(x)=\cos^2 x$, $R(x)=-1$, $F(x)=0$. For the solution using DTM, we follow the procedure outlined in the previous two examples and obtain numerical values for eigenvalues of equation (4.0.5). The errors listed in Table 5 were computed and compared with the exact eigenvalues listed in Eggert et al. (1987) for equation (4.0.5).

Table 5: The absolute error in computing eigenvalues using Sinc-Galerkin ($N=32$) and Differential Transform ($n=10$) compared with the exact eigenvalues listed in Eggert et al. (1987) for example 4.3

True eigenvalue [5]	E_s	E_T
$\lambda_1=1.24242$	1.42×10^{-5}	3.19×10^{-5}
$\lambda_2=4.49479$	4.85×10^{-4}	1.70×10^{-4}
$\lambda_3=9.50366$	9.91×10^{-3}	1.83×10^{-3}
$\lambda_4=16.50208$	1.31×10^{-1}	3.81×10^{-1}

Example 4.4: Non-Fickian diffusion problem

The following example is taken from Ramos (2005). We consider a linear, one-dimensional non-Fickian diffusion problem in composite media with a potential field in Cartesian coordinates. Such a problem is governed by a telegraph equation for the mass concentration which, upon applying the Laplace transform in time, yields the following second-order ordinary differential equation

$$\frac{d^2c}{dx^2}+\frac{d}{dx}(pc)-\beta^2c=0\,,\ c(0)=1\,,\ c(1)=0\,, \tag{4.0.6}$$

where $x: 0<x<1$ is the Cartesian coordinate, c is the concentration, β is related to the exponent of the kernel of the Laplace transform, and p is the quadratic drift force given by

$$p(x)=a_p(x-x_p)^2+b_p(x-x_p)+c_p \tag{4.0.7}$$

and a_p, b_p, c_p are constants. The reader is referred to Ramos (2005) for detailed discussions about the model in equation (4.0.6)). Equation (4.0.6) is linear, but has variable coefficients. Therefore, it is in general impossible to obtain analytical solutions for this equation. In this example $c(0)=0$, so that the development leading to equation (2.3.13) is not applicable for the Sinc solution.

Therefore, for the nonhomogeneous boundary condition $c(0)=1$, a change of variable $y(x)=c(x)-(1-x)$ is employed to transform the differential equation with nonhomogeneous conditions (4.0.6) to one with homogeneous boundary conditions. For the Sinc-Galerkin solution, set

$$c(x)=\sum_{j=-N}^{N} c_j S_j(x)+(1-x).$$

The coefficient matrix for the Sinc solution of this problem is identical to that in (2.3.13), whereas the right-hand side differs in another function $\tilde{F}$, which replaces F. To show the effects of the drift force $p(x)$ on the concentration profiles, we have applied both methods to equation (4.0.6) with quadratic $p(x)$ of equation (4.0.7). Some numerical results are computed and tabulated in Table 6. The results presented in Table 6 show that the steepness of the concentration profiles near $x=0$ decreases as $x_p : 0 \le x_p \le 1$ is increased. This is consistent with the results obtained by Ramos (2005).

Table 6: Concentration $c(x)$ of equation (4.0.6)

x	$x_p=0$	$x_p=0.5$	$x_p=1$
0	1	1	1
0.1	0.973	0.761	0.433
0.2	0.912	0.556	0.235
0.3	0.778	0.486	0.102
0.4	0.609	0.390	0.092
0.5	0.515	0.315	0.073
0.6	0.322	0.248	0.068
0.7	0.198	0.192	0.052
0.8	0.108	0.127	0.030
0.9	0.052	0.086	0.018
1.0	0	0	0

Using Sinc-Galerkin method ($N=32$) and DTM ($n=10$) for the case $\beta=1$, $a_p=10$, $b_p=c_p=0$, $x=0,0.5,1$ give identical results shown in Table 6.

5. Conclusion

In this study, we have introduced two methods: Sinc-Galerkin method and DTM to solve Sturm-Liouville eigenvalue problems (1.0.1)-(1.0.2). The results of the previous section indicate that our procedures can be applied to obtain accurate numerical solutions. The accuracy of the methods depends on the value of n for DTM, and N for Sinc-Galerkin method. The DTM is simple in applicability as it does not require discretization.

Acknowledgement

I would like to thank the Editor and the anonymous referees for their in-depth reading, criticism of, and insightful comments on an earlier version of this paper.

REFERENCES

Abdel-Halim Hassan, I. H. (2002). Different Applications for the Differential Transform in the differential equations. Applied Math. Comput., Vol 129, pp. 183-201.

Adel, Mohsen, El-Gamel, Mohamed (2009). On the Galerkin and collocation methods for two-point boundary value problems using Sinc bases, Comp & Math with Applications.

Bender, C. M., Orszag, S. A. (1987). Advanced Mathematical Methods for Scientific and Engineers, McGraw-Hill.

Bujurke, N., Salimath, C., Shiralashetti, S. (2009). Computation of eigenvalues and solutions of regular Sturm-Liouville problems using Haar wavelets, J. Comp. Appl. Math.

Chanane, B. (2007). Computing the eigenvalues of singular Sturm-Liouville problems using the regularized sampling method. Appl. Math. Comput., Vol 184, Issue 2, pp. 972-978.

Eggert, Norman, Jarratt, Mary, Lund, L. (1987). Sinc function computation of the eigenvalues of Sturm-Liouville problems. Journal of Comput. Phys., Vol. 69, pp. 209-229.

El-Gamel, M., Zayed, A. I. (2004). Sinc-Galerkin method for solving nonlinear boundary- value problems. Computers & Mathematics with Applications, Vol. 48, pp. 1285-1298.

Farlow, J. S. (1982). Partial Differential Equations for Scientists and Engineers, Wiley, New York, p. 53.

Guseinov, G., Karaca, I. (2004). Instability intervals of a Hill's equation with piecewise constant and alternating coefficient, Computers & Mathematics with Applications, Vol. 47, Issues 2-3, pp. 319-326.

Ibrahim, Celik (2005). Approximate calculation of eigenvalues with the method of weighted residuals-collocation method, Appl. Math. Comput., Vol. 160, pp. 401-410.

Kadakal, M., Mukhtarov, O. (2007). Sturm-Liouville problems with discontinuities a two-points, Computers & Mathematics with Applications, Vol. 54, pp. 1367-1379.

McArthur, Kelly Marie (1989). A collocative variation of the Sinc-Galerkin method for second order boundary value problems, Computation and Control Theory, pp. 253-261.

Ramos, J. I. (2005). On the numerical treatment of an ordinary differential equation arising in one-dimensional non-Fickian diffusion problems, Comput. Phys. Communications, Vol. 170, Issue 3, pp. 231-238.

Ravi Kanth, A.S.V., Aruna, K. (2008). Solution of singular two-point boundary value problems using differential transformation method, Physics Letters A, Vol. 372, Issue 26, pp. 4671-4673.

Saadatmandi, A., Razzaghi, M., Dehghan, M. (2005). Sinc-Galerkin solution for nonlinear two-point boundary value problems with applications to chemical reactor theory, Math & Comp Modeling, Vol. 42, Issues 11-12, pp. 1237-1244.

Siraj-Ul, Islam, Sirajul, Haq, Javid, Ali. (2009). Numerical solution of special 12th-order boundary value problems using differential transform method, Communications in Nonlinear Science and Numerical Simulation.

Stenger, F. (1993). Numerical Methods Based on Sinc and Analytic Functions, Springer-Verlag, NY.

Trim, W. D. (1990). Applied Partial Differential Equations, PWS-Kent Publishing Company, Boston, MA, p. 149.

Ugour, Yucel. (2006). Approximations of Sturm-Liouville eigenvalues using differential quadrature (DQ) method. Appl. Math. & Comput., Vol. 192, Issue 2, pp. 310-319.

Vedat Suat, Erturk, Shaher, Momani. (2007). Comparing numerical methods for solving fourth-order boundary value problems, Appl. Math. & Comput., Vol. 188, pp 1963-1968.

Zhou, J. K. (1986). Differential Transform and its application for electrical circuits, Wuuhahn, China: Huarjung University Press.