Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 11, Issue 1 (June 2016), pp. 184 - 191

Applications and
Applied Mathematics:
An International Journal
(AAM)

Approximate Analytical Solution of Boussinesq Equation in Homogeneous Medium with Leaky Base

Rajeev K Bansal
Department of Mathematics
National Defence Academy
Khadakwasla, Pune-411023 India
bansal_rajeev31@hotmail.com

Received: November 17, 2015; Accepted: March 30, 2016

Abstract

Approximate analytical solutions of Boussinesq equation are widely used for approximation of subsurface seepage flow in confined and unconfined aquifers under varying hydrological conditions. In this paper, we use a 2-dimensional linearized Boussinesq equation to simulate the water table fluctuations in an isotropic aquifer overlying a semi pervious bed under multiple localized recharge and withdrawal. The unconfined aquifer is considered to be in contact with two water bodies of constant water head along opposite cost lines, while the remaining two faces have no flow condition. The mathematical model is solved analytically using finite Fourier sine transform and the application of the results is illustrated with a numerical example. It is observed that the vertical flow through the base of the aquifer is an important factor in the determination of groundwater mound and cone of depression.

Keywords: Approximate analytical solutions; Boussinesq equation; Groundwater; Fourier transform; isotropic aquifer

MSC 2010 No.: 76S05; 76R10; 76R50; 35Q86

1. Introduction

Estimation of surface-groundwater interaction is important to scientists and engineers for its central role in the conjunctive management of water resources. Judicious distribution and optimal utilization of subsurface water resources is undoubtedly an area of major concern due to its multiple socioeconomic impacts on mankind. Accurate estimation of water table fluctuations in an aquifer under the combined action of pumping and recharge is mandatory for proper groundwater management. Since the experimental studies in groundwater hydrology are extremely expensive and time consuming, preliminary knowledge of water table fluctuations based on analytical modeling is gaining vast popularity due its cost effectiveness and ability to easily handle the variations in hydraulic parameters. Thus, there is a need of developing analytical models that can efficiently predict spatial and temporal variation of groundwater table and the capture zones associated with water sources or withdrawals. Socioeconomic impact of water resources management has also been analyzed by several researchers.

Most of the subsurface seepage models for approximation of hydro-interaction between stream and unconfined aquifers are based on Boussinesq equation [Bear (1972)]. The Boussinesq equation is a second order nonlinear partial differential equation which is analytically intractable. However, approximate analytical solutions of the Boussinesq equation are vastly used to understand the flow processes at various spatial and temporal scales [Hantush (1965), Hunt (1999), Moench and Barlow (2000), Taghizade and Neirameh, (2009), Mohyud-Din et al. (2010)] and analyze the transient behavior of water table in response to controlled activities such as artificial recharge and withdrawal from wells [Rai and Manglik (2006), (2012); Bansal (2012), (2013)]. An extensive survey of the literature reveals that most of the existing models are based on a simplified assumption that the aquifer is underlain by a perfectly impervious bed. In our opinion, this assumption is the main weakness of the existing models. Such solutions may not be applicable to several natural systems consisting of leaky aquifers, e.g. multilayered aquifer in deep sedimentary basins where the recharge and withdrawal mechanism of a layer is partially controlled by the hydrological properties of the underlying aquitard, and thus, applying the results of the existing studies might lead to underestimation or overestimation of the actual results.

In this paper, a new analytical solution of 2-dimensional linearized Boussinesq equation is developed. The hydrological setting of the model consists of an unconfined isotropic aquifer overlaying a semi pervious (leaky) base, subjected to recharge and withdrawal activities through multiple recharge basins and extraction/injection wells. The 2-dimensional linearized Boussinesq equation is solved using finite Fourier transform to obtain the closed form expressions for hydraulic head in the aquifer. Implementation of the new results is illustrated with a numerical example. Sensitivity of the hydraulic head based on variation in aquifer parameters is analyzed.

2. Mathematical Formulation and Analytical Solution

Definition sketch of the problem is given in Figure 1. We consider an isotropic aquifer of dimension A x B, underlain by a semi pervious bed. The hydraulic conductivity of the aquifer is denoted by K. The aquifer is in contact with two water bodies along the coastlines $x = 0$ and $x = A$ that maintains a constant water head h_0 along these coastlines. The other two boundaries, viz. $y = 0$ and $y = B$ of the aquifer are impervious and thus no flow condition is imposed across these boundaries. Due to hydrologic equilibrium between the aquifer and adjacent water bodies, the initial water table of the aquifer is considered to be h_0. The aquifer domain is subjected to multiple recharge and withdrawal through rectangular recharge basins and point-sized wells.

If $h(x, y, t)$ denotes the variable water table measured from horizontal datum, then the flow of groundwater in the unconfined aquifer with semi pervious base is governed by the following 2-dimensional partial differential equation:

$$K\left[\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)\right]+P(x,y,t)=S\frac{\partial h}{\partial t}+\frac{k}{b}(h-h_0), \tag{1}$$

where S denotes the specific yield of the aquifer, and k and b denote the hydraulic conductivity and thickness of the semi-pervious bed of the unconfined aquifer. The term $P(x, y, t)$ simulates the combined effects of recharge and discharge. The term $P(x, y, t)$ broadly consists of arbitrarily located rectangular basins of varying dimensions a_i x b_i, and injection/extraction wells of relatively lesser dimensions. Recharge is considered at time-varying rate, whereas the extraction/injection is considered at constant rate. All in all, the term $P(x, y, t)$ is given by

$$P(x,\ y,\ t)=\left[\sum_{i=1}^{p_1}R_i(x,\ y,\ t)+\sum_{j=1}^{p_2}\omega_j\,Q_j\,\delta(x-x_j)\delta(y-y_j)\right], \tag{2}$$

where, p_1 and p_2 denote the number of rectangular basin and wells respectively. $R_i(x, y, t)$ denotes the transient recharge rate in the i^{th} basin ($i = 1, 2, \ldots, p_1$) extending from $x_i \le x \le x_i + a_i$; $y_i \le y \le y_i + b_i$. The term ω_j is 1 or –1 just as the i^{th} well corresponds to an injection or extraction well. Q_j is the constant rate of injection/extraction in the j^{th} well ($j = 1, 2, \ldots, p_2$). δ is the Dirac delta function. The rate of recharge typically depends on several hydrologic parameters. However, the recession limb of a recharge hydrograph can be approximated by an exponentially decaying function of time to approximate the recharge rate. That is,

$$R_i(x,y,t)=\begin{cases}N_{i0}+N_{i1}\,e^{-\lambda_i t}, & x_i\le x\le x_i+a_i\,;\ y_i\le y\le y_i+b_i,\\ 0, & \text{otherwise,}\end{cases} \tag{3}$$

where λ_i is a positive constant, determining the rate at which the recharge in the i^{th} basin reduces to a final value $N_{i\,0}$ from an initial value $N_{i\,0} + N_{i\,1}$.

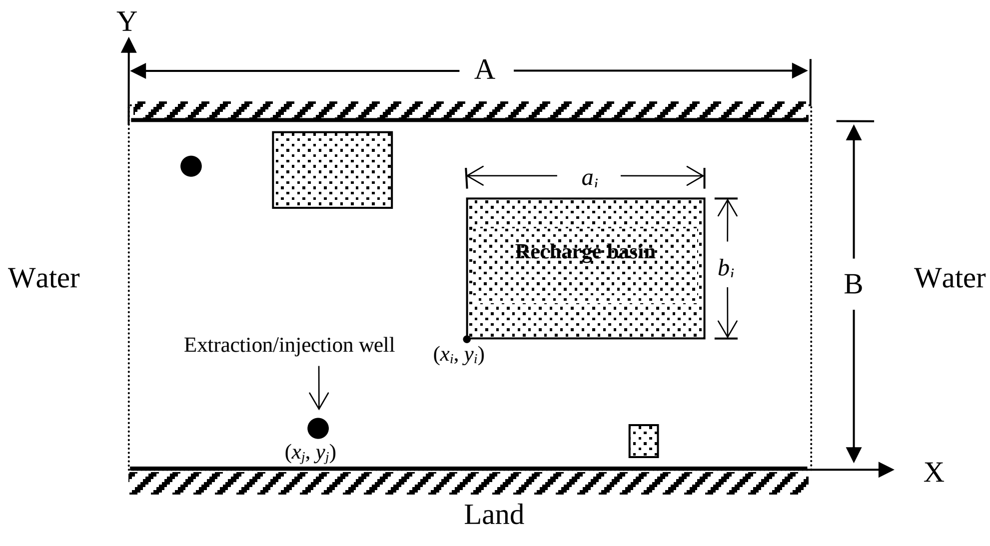

Figure 1. Overview of a two-dimensional anisotropic aquifer with multiple recharge basins, injection and extraction wells.

The initial and the boundary conditions are prescribed as follows:

$$h(x,y,t=0)=h_0\ ; \tag{4}$$

$$\left(\frac{\partial h}{\partial y}\right)_{y=0}=0\ ; \qquad \left(\frac{\partial h}{\partial y}\right)_{y=B}=0; \tag{5}$$

$$h(x=0,y,t)=h_0;\ \ h(x=a,y,t)=h_0. \tag{6}$$

Equation (1) is the second order two-dimensional Boussinesq equation of parabolic nature, and analytically intractable. In order to find approximate analytical solution of Equation (1), we first rewrite it in the form

$$K\left[\frac{\partial^2 h^2}{\partial x^2}+\frac{\partial^2 h^2}{\partial y^2}\right]+2P(x,y,t)=S\left(\frac{1}{h}\frac{\partial h^2}{\partial t}\right)+\frac{2k}{b}\frac{\left(h^2-h_0^2\right)}{\left(h+h_0\right)}. \tag{7}$$

Equation (5) is now linearized by replacing the term h associated with $\partial h^2/\partial$t in the first bracket and the term $(h + h_0)/2$ of the right-hand side by the mean depth of the saturation $\hbar$. The value of $\hbar$ is obtained by successive application of the relation $\hbar = (h_0 + h_t)/2$ where h_0 is the initial water head and h_{t} is the water head at the current moment (Marino 1973). The initial approximation of $\hbar$ is taken as h_0. We obtain

$$\frac{\partial^2 h^2}{\partial x^2}+\frac{\partial^2 h^2}{\partial y^2}+\frac{2}{K}P(x,\ y,\ t)=\frac{S}{K\hbar}\left(\frac{\partial h^2}{\partial t}\right)+\frac{k}{Kb\hbar}\left(h^2-h_0{}^2\right). \tag{8}$$

Now, define $H(x, y, t) = h^2 - h_0^2$, we get

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+\frac{2}{K}P(x,\ y,\ t)=\frac{S}{K\hbar}\frac{\partial H}{\partial t}+\frac{k}{Kb\hbar}H. \tag{9}$$

The initial and boundary conditions are

$$H(x,\ y,\ 0)=0; \tag{10}$$

$$\left(\frac{\partial H}{\partial y}\right)_{y=0}=0\ ; \qquad \left(\frac{\partial H}{\partial y}\right)_{y=B}=0\ ; \tag{11}$$

$$H(x=0,\ y,\ t)=0\ ; \quad H(x=A,\ y,\ t)=0. \tag{12}$$

Equation (9) along with the conditions (10) – (12) is solved by using finite mixed Fourier transform (Sneddon 1974) Define

$$\xi(m,\ n,\ t)=F_{sc}\left\{H(x,y,t);(x,\ y)\to(m,\ n)\right\}=\int_{x=0}^{A}\int_{y=0}^{B}H(x,\ y,\ t)\sin\left(\frac{m\pi x}{A}\right)\cos\left(\frac{n\pi y}{B}\right)dy\,dx. \tag{13}$$

The finite Fourier cosine transform reduces Equation (9) to the following form

$$-\beta_m{}^2\xi-\gamma_n{}^2\xi+\frac{2}{K}\overline{P}\left(m,\,n,\,t\right)=\frac{1}{\nu}\frac{d\xi}{dt}+c\,\xi. \tag{14}$$

where

$$\beta_m=\frac{m\pi}{A}\;;\;\;\gamma_n=\frac{n\pi}{B}\;;\;\;c=\frac{k}{K\,b\hbar};\;\;\nu=\frac{K\hbar}{S}\;; \tag{15}$$

$$\overline{P}(m,n,t)=\left[\sum_{i=1}^{p_1}\Omega_i\left(N_{i0}+N_{i1}\,e^{-\lambda_i t}\right)+\sum_{j=1}^{p_2}\omega_j\,\eta_j\,Q_j\right]; \tag{16}$$

$$\Omega_i=-\frac{1}{\beta_m\gamma_n}\left[\cos\left\{\beta_m\left(x_i+a_i\right)\right\}-\cos\left(\beta_m\,x_i\right)\right]\left[\sin\left\{\gamma_n\left(y_i+b_i\right)\right\}-\sin\left(\gamma_n\,y_i\right)\right]; \tag{17}$$

$$\eta_j=\sin\left(\beta_m\,x_j\right)\cos\left(\gamma_n\,y_j\right). \tag{18}$$

Equation (14), subjected to condition (10) can be solved using ordinary methods. Its solution is

$$\begin{aligned}\xi(m,n,t)=\frac{2\nu}{K}\Bigg[\sum_{i=1}^{p_1}\Omega_i\left\{\frac{N_{i0}}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)+\frac{N_{i1}}{\alpha+\nu c-\lambda_i}\left(e^{-\lambda_i t}-e^{-(\alpha+\nu c)t}\right)\right\}\\+\sum_{j=1}^{p_2}\frac{\omega_j\,\eta_j\,Q_j}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)\Bigg],\end{aligned} \tag{19}$$

where $\alpha=\nu\left(\beta_m{}^2+\gamma_n{}^2\right)$ and τ is a variable of integration. *H*(*x*, *y*, *t*) can now be obtained by inverting the finite Fourier transform as

$$H(m.n,t)=\frac{4}{AB}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\xi(m,n,t)\sin\left(\frac{m\pi x}{A}\right)\cos\left(\frac{n\pi y}{B}\right). \tag{20}$$

Thus, the solution of Equation (6) is

$$\begin{aligned}h^2=h_0{}^2+\frac{8\nu}{ABK}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\left(\beta_m\,x\right)\cos\left(\gamma_n\,y\right)\Bigg[\sum_{j=1}^{p_2}\frac{\omega_j\,\eta_j\,Q_j}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)\\+\sum_{i=1}^{p_1}\Omega_i\left\{\frac{N_{i0}}{\alpha+\nu c}\left(1-e^{-(\alpha+\nu c)t}\right)+\frac{N_{i1}}{\alpha+\nu c-\lambda_i}\left(e^{-\lambda_i t}-e^{-(\alpha+\nu c)t}\right)\right\}\Bigg].\end{aligned} \tag{21}$$

3. Shanks Transformation for Acceleration of Convergence

The rate of convergence of the series obtain in the preceding section is a major concern. Numerical experimentations reveal that one has to choose the indices m, n in the range 150-200 to achieve the convergence up to second place of decimal. This requires a great deal of computer time and memory. In order to accelerate the convergence of the series in the right-hand side of Equation (21), we use a nonlinear transformation known as Shanks transformation [Shank (1965)]. The Shanks transformation $S\{A_n\}$ of a series $\{A_n\}_{n=1,2,\ldots}$ of partial sums A_n is defined as follows:

$$S\{A_n\} = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} - 2A_n + A_{n-1}}. \tag{22}$$

The sequence $S\{A_n\}$ converges more rapidly than the original series $\{A_n\}$. Acceleration of convergence can be obtained by successive application of the Shanks transformation, i.e. by forming sequences $S^2\{A_n\}$, $S^3\{A_n\}$ etc.

3. Discussion of Results

In order to illustrate the implementation of the results developed in this study, we consider an example with aquifer parameters as indicated in Table 1. The model's domain is considered to have a rectangular basin of dimension 20 m x 10 m centered at (40m, 50m), and an extraction well centered at (120m, 50m). The dimensions of the well are small compared to that of the recharge basin. Transient recharge is applied to the rectangular basin at the rate 0.5 + 0.8 $e^{-0.2t}$ mm/d. At the same time, water is pumped out from the extraction well at a constant rate 150 m^3/d. The average saturated depth of the aquifer is determined using an iterative relation $\hbar = (h_0 + h_t)/2$ where h_0 is the initial water head and h_t is the water head at the current moment (Marino 1973). The initial approximation of $\hbar$ is taken as h_0.

Table 1. Values of various aquifer parameters used in the numerical example

Parameter	Value
A x B	150 m x 100 m
K	10 m/d
h_0	15 m
S	0.25
b	1 m
k	0.25 m/d

Transient profiles of the water table fluctuations are determined for various values of time t. Distribution of water head along the line y = 50 m (line passing through the centers of recharge basin and extraction well) is presented in Figure 2. It can be seen from this figure that the groundwater mound and cone of depression are symmetrically distributed about the centers of the basin and well, respectively. While the height of mound increases with time, depth of cone remains almost unchanged. If the base of the aquifer is of lesser hydraulic conductivity (Figure 2(a) for k = 0), the downward leakage through its base is lesser, and thus, such aquifers exhibit higher level of water table in response to recharge. Water table depletion induced by pumping from the well is also affected by the ratio b/k. When the aquifer's base is leaky aquifers, withdrawal from the well is supplemented by the leakage induced vertical flow from hydraulically connected sources. Consequently, the depth of the cone of depression is mitigated by the bed leakage. Three dimensional view of the groundwater mound and the cone of depression is shown in Figures 3 (a) and (b) for $t = 1$ and 10 days, respectively by considering $k = 0.25$ m/d.

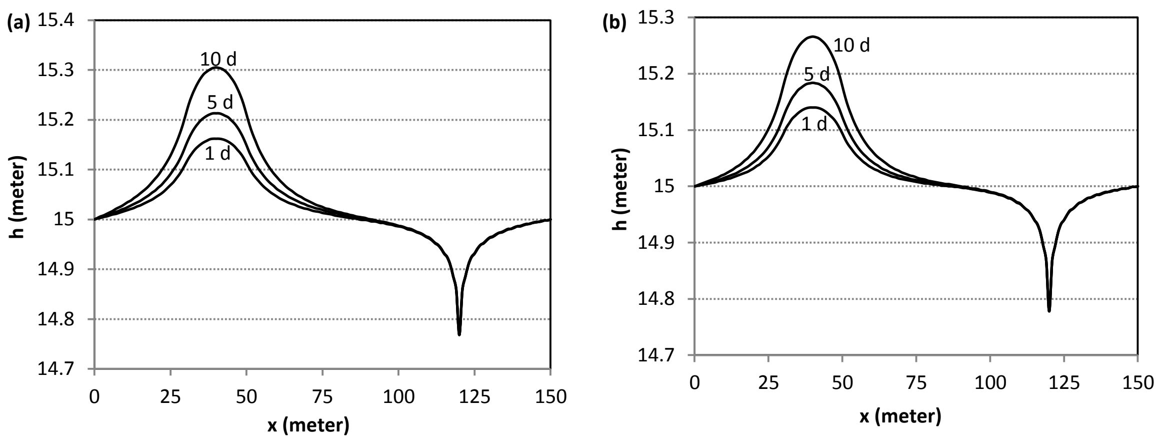

Figure 2. Development of transient water table in the aquifer along the line $y = 50$ m for $t = 1$, 5 and 10 d when (a) $k = 0$, and (b) $k = 0.25$ m/d

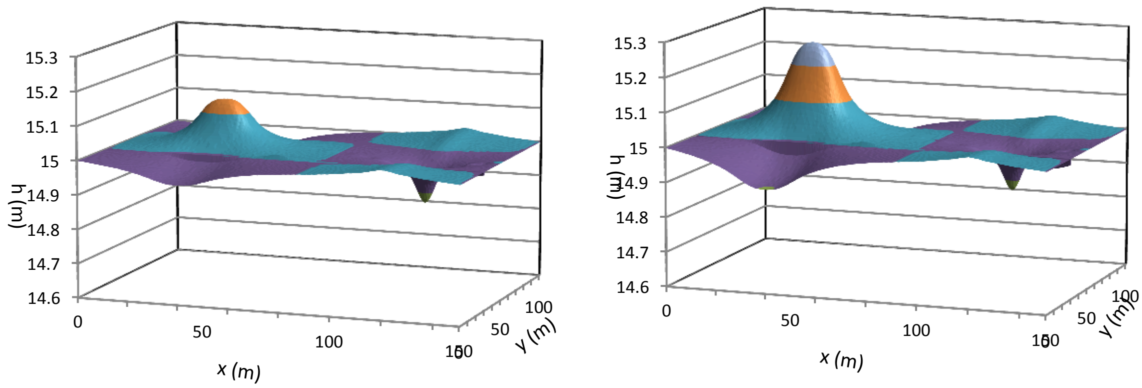

Figure 3. Three dimensional overview of the water table in the aquifer with leaky base ($k = 0.25$ m/d) for (a) $t = 1$ d, and (b) $t = 10$ d

4. Conclusion

New analytical solution of the 2-dimensional linearized Boussinesq equation characterizing flow in isotropic and leaky porous medium is developed. The solution has the ability to predict the fluctuations in water table in unconfined aquifer due to multiple recharge and withdrawal. It is demonstrated in the numerical example that the formation of groundwater mound and cone of depression beneath recharge basin and extraction well are significantly affected by the hydraulic resistance of the base of the aquifer. The results derived in this study can be used for calibration of experimental studies; nevertheless, model prediction can be further enhanced by considering heterogeneity and anisotropy of the unconfined porous medium.

REFERENCES

Bansal R. K. (2012). Groundwater Fluctuations in Sloping Aquifers Induced by Time-varying Replenishment and Seepage from a Uniformly Rising Stream, Transport in Porous Media, Vol. 92, No. 2, pp. 817-836

Bansal R. K. (2013). Groundwater Flow in Sloping Aquifer under Localized Transient Recharge: Analytical Study. Journal of Hydraulic Engineering, Vol. 139, No. 11, pp. 1165–1174

Bear J. (1972). Dynamics of Fluids in Porous Media, Elsevier, New York

Hantush M. S.(1965). Wells Near Streams with Semi-pervious Beds, Journal of Geophysical Research, Vol. 70, No. 12, pp. 2829–2838

Hunt B. (1999). Unsteady Stream Depletion from Ground water Pumping, Ground Water, Vol. 37, pp. 98–102

Marino M. (1973). Water-table Fluctuation in Semipervious Stream–unconfined Aquifer Systems, Journal of Hydrology, Vol. 19, pp. 43–52

Moench A. F., Barlow P. M. (2000). Aquifer Response to Stream-stage and Recharge Variations. I. Analytical Response Functions, Journal of Hydrology, Vol. 230, pp. 192–210

Mohyud-Din S.T., Mustafa I, Cavlak E. (2010). On Numerical Solutions of Two-Dimensional Boussinesq Equations by Using Adomian Decomposition and He's Homotopy Perturbation Method, Applications and Applied Mathematics, Spl Issue, Vol. 1, pp. 1-11

Rai S. N., Manglik A., Singh V. S. (2006). Water Table Fluctuation Owing to Time Varying Recharge, Pumping and Leakage, Journal of Hydrology, Vol. 324, pp. 350–358

Rai S. N., Manglik A. (2012). An Analytical Solution of Boussinesq Equation to Predict Water Table Fluctuations due to Time Varying Recharge and Withdrawal from Multiple Basins, Wells and Leakage Sites, Water Resources Management, Vol. 26, pp. 243–252

Shank (1955). Non-linear transformation of divergent and slowly convergent sequences. Journal of Mathematics and Physics 34: 1-42

Sneddon, I.N. (1974). The Use of Integral Transform, Tata McGraw-Hill, New Delhi

Taghizade N., Neirameh A. (2009). Some Applications of the (G'/G)-Expansion Method for Solving the Nonlinear Partial Differential Equations in Mathematical Physics, Applications and Applied Mathematics, Spl Issue, Vol. 4, No. 2, pp. 290-300