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A New HPM for Integral Equations

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Abstract

Homotopy perturbation method is an effective method for obtaining exact solutions of integral equations. However, it might perform poorly on ill-posed integral equations. In this paper, we introduce a new version of the homotopy perturbation method that efficiently solves ill-posed integral equations. Finally, several numerical examples, including a system of integral equations, are presented to demonstrate the efficiency of the new method.

Keywords: Homotopy Perturbation Method; Integral Equations; Ill-Posed Problems

MSC (2000) No.: 45A0, 45F05

1. Introduction

Integral equations frequently arise in modeling various real world physical problems [Delves and Mohamed (1988) and Hansen (1994)]. Solving such integral equations are of great interest and several numerical and analytical methods have been developed [Abbasbandy (2006, 2007), Biazar and Ghazvini (in press, 2008), Bellman and Kabala (1962), Delves and

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Mohamed (1988) and Hansen (1994)]. Classical numerical methods, such as finite difference and finite element methods are computationally more expensive and usually affected by round-off errors, which may result to inaccurate results. Analytical methods that are widely used for solving functional equations are very restrictive and can be used in very special cases [Delves and Mohamed (1988)].

Based on homotopy, which is a basic concept in topology, a general analytic method (namely the homotopy analysis method (HAM)) is established by Liao (1992) to obtain solutions of nonlinear differential equations [Abbasbandy (2006) and Sajid et al. (2007)]. Another widely used method based on the homotopy, is the so called homotopy perturbation method (HPM) introduced by He (1999, 2000, 2004, 2003, 2004, 2005, 2006, 2005). It continuously deforms a difficult problem into another one, which is easy to solve. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to the exact solution.

One should note that in HPM one homotopy parameter is introduced, and first order or second order approximate solution is searched for, while HAM includes another homotopy parameter to adjust convergence of the obtained series (He, in press). It can be said that He's HPM is a universal one that is able to solve various kinds of nonlinear functional equations.
For examples, it was applied to nonlinear Schrödinger equations [Biazar and Ghazvini (2007)], nonlinear equations arising in heat transfer [Ganji (2006)], the functional integral equations [Abbasbandy (2006, 2007)], the quadratic Ricatti differential equation (Odibat and Momani, in press), and many others [Siddiqui et al. (2008), Cveticanin (2006), Biazar and Ghazvini (in press, 2008), and He (2008, 2006, 2006, 2008)].

Although HPM is an efficient method to solve many integral equations, but it might perform poorly on ill-posed problems. Since many real world modeling problems might lead to ill-posed integral equations [Delves and Mohamed (1988)], thus it is essential to have alternative approaches which can deal with such problems. In this paper, we propose a new homotopy perturbation method (NHPM) to obtain exact solutions of some ill-posed integral equations [Delves and Mohamed (1988) and Hansen (1994)]. Several examples are presented to illustrate the efficiency of the proposed approach.

2. NHPM

To illustrate the basic ideas of this method, let us consider the following nonlinear differential equation

$$A(u(x)) - f(r(x)) = 0, \quad r(x) \in \Omega, \tag{1}$$

with the boundary conditions

$$B\left(u(x), \frac{\partial u(x)}{\partial n}\right) = 0, \quad r(x) \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r(x))$ is a known analytical function and Γ is the boundary of the domain Ω.

The operator A can be divided into two parts, L and N, where L is linear and N is a nonlinear operator. Therefore, equation (1) becomes

$$L(u(x)) + N(u(x)) - f(r(x)) = 0. \tag{3}$$

By the homotopy technique, we construct a homotopy $U(r(x),p):\Omega\times[0,1]\to\mathbb{R}$, which satisfies

$$H(U(x),p) = (1-p)[L(U(x)) - L(u_0(x))] + p[A(U(x)) - f(r(x))] = 0, \quad p\in[0,1],\ r(x)\in\Omega, \tag{4}$$

or equivalently,

$$H(U(x),p) = L(U(x)) - L(u_0(x)) + pL(u_0(x)) + p[N(U(x)) - f(r(x))] = 0, \tag{5}$$

where $p\in[0,1]$ is an embedding parameter, $u_0(x)$ is an initial approximation of solution of equation (1). From equations (4) and (5) we have

$$H(U(x),0) = L(U(x)) - L(u_0(x)) = 0, \tag{6}$$

$$H(U(x),1) = A(U(x)) - f(r(x)) = 0. \tag{7}$$

Now, according to the HPM, we assume that the embedding parameter p is a small parameter, and the solutions of equations (4) and (5) are representable as a power series in p as

$$U(x) = \sum_{n=0}^{\infty} p^n U_n . \tag{8}$$

Let us write equation (5) in the following form

$$L(U(x)) = L(u_0(x)) + p[f(r(x)) - L(u_0(x)) - N(U(x))]. \tag{9}$$

By applying the inverse operator, L^{-1}, to both sides of equation (9), we have

$$U(x) = u_0(x) + p[L^{-1}f(r(x)) - u_0(x) - L^{-1}N(U(x))]. \tag{10}$$

Moreover, suppose that the initial approximation of equation (1) has the form

$$u_0(x) = \sum_{n=0}^{\infty} a_n P_n(x), \tag{11}$$

where $a_0, a_1, a_2, \ldots$ are unknown coefficients and $P_0(x), P_1(x), P_2(x), \ldots$ are specific functions depending on the problem. By substituting (8) and (11) into the equation (10), we get

$$\sum_{n=0}^{\infty} p^n U_n(x) = \sum_{n=0}^{\infty} a_n P_n(x) + p[L^{-1}f(r(x)) - \sum_{n=0}^{\infty} a_n P_n(x) - L^{-1}N(\sum_{n=0}^{\infty} p^n U_n(x))]. \tag{12}$$

Comparing coefficients of terms with identical powers of p , leads to

$$\begin{aligned} p^0 &: U_0(x) = \sum_{n=0}^{\infty} a_n P_n(x), \\ p^1 &: U_1(x) = L^{-1}f(r(x)) - \sum_{n=0}^{\infty} a_n P_n(x) - L^{-1}N(u_0(x)), \\ p^2 &: U_2(x) = -L^{-1}N(U_0(x), U_1(x)), \\ &\vdots \\ p^j &: U_j(x) = -L^{-1}N(U_0(x), U_1(x), \ldots, U_{j-1}(x)). \end{aligned} \tag{13}$$

Now, if we solve these equations in such a way that $U_1(x) = 0$, then $U_2(x) = U_3(x) = \cdots = 0$. Therefore, the exact solution of original problem becomes

$$u(x) = U_0(x) = \sum_{n=0}^{\infty} a_n P_n(x).$$

It is worthwhile to note that if $f(r(x))$ and $u_0(x)$ are analytic at $x = x_0$, then their Taylor series

$$u_0(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, f(r(x)) = \sum_{n=0}^{\infty} a_n^* (x - x_0)^n,$$

can be used in Equation (12) , where $a_0^*, a_1^*, a_2^*, \ldots$ are known coefficients and $a_0, a_1, a_2, \ldots$ are unknown ones, which must be determined.

3. Illustrative Examples

In this section, we present three examples that are taken from [Biazar and Ghazvini (in press), Delves and Mohamed (1988) and Hansen (1994)]. Since Examples 1 and 2 are ill-posed, then the HPM can not give the exact solution. For example 3 as it is shown in [Biazar and Ghazvini (in press)], the classical HPM again fails to give the exact solution.

Example 1.

Consider the following ill-posed integral equation taken from [Delves and Mohamed (1988)]

$$\int_0^1 (x - s)^2 f(s) ds = \frac{x^2}{2} - \frac{2x}{3} + \frac{1}{4}, \tag{14}$$

where the exact solution is $f(x) = x$.

Let us consider the homotopy introduced in [Abbasbandy (2006)]:

$$(1-p)F(x)-p\left(\frac{x^2}{2}-\frac{2x}{3}+\frac{1}{4}\right)+p\int_0^1 (x-s)^2F(s)\,ds=0. \tag{15}$$

Suppose that the solution of equation (15) has the form

$$F(x)=F_0(x)+pF_1(x)+p^2F_2(x)+\dots . \tag{16}$$

By substituting (16) into (15), and equating the terms with identical powers of p gives

$$p^0: F_0(x)=0,$$

$$p^1: F_1(x)=\frac{x^2}{2}-\frac{2x}{3}+\frac{1}{4},$$

$$p^j: F_j(x)=F_{j-1}(x)-\int_0^1 (x-s)^2F_{j-1}(s)\,ds,\quad j=2,3,\dots .$$

Now, suppose that $f(x)\approx\sum_{j=0}^{10}F_j(x)$. Then the approximate solution of Example 1 becomes

$$f(x)\approx 2.652682384-6.377275521x+2.705651356x^2$$

However, to solve equation (14) by the NHPM, we construct the following homotopy:

$$(1-p)\big(F(x)-f_0(x)\big)+p\left(\int_0^1 (x-s)^2F(s)\,ds-\frac{x^2}{2}+\frac{2x}{3}-\frac{1}{4}\right)=0. \tag{17}$$

Now, suppose that the solution of equation (17) has the form (16). Substituting (16) into (17), and comparing coefficients of terms with identical powers of p, leads to:

$$p^0: F_0(x)=f_0(x),$$

$$p^1: F_1(x)=-\int_0^1 (x-s)^2F_0(s)\,ds+\frac{x^2}{2}-\frac{2x}{3}+\frac{1}{4},$$

$$p^2: F_2(x)=F_1(x)-\int_0^1 (x-s)^2F_1(s)\,ds,$$

$$\vdots$$

$$p^j: F_j(x)=F_{j-1}(x)-\int_0^1 (x-s)^2F_{j-1}(s)\,ds.$$

Now, assume that $f_0(x)=\sum_{n=0}^{\infty}a_nx^n$ and $F_1(x)=0$, then we have

$$-\int_0^1 (x-s)^2(a_0+a_1s+a_2s^2+\dots)\,ds+\frac{x^2}{2}-\frac{2x}{3}+\frac{1}{4}=0.$$

Therefore,

$$\frac{1}{4}-\frac{1}{3}a_0-\frac{1}{4}a_1-\frac{1}{5}a_2-\frac{1}{6}a_3-\frac{1}{7}a_4-\cdots+\left(-\frac{2}{3}+a_0+\frac{2}{3}a_1+\frac{2}{5}a_3+\frac{1}{3}a_4+\cdots\right)x$$
$$+\left(\frac{1}{2}-a_0-\frac{1}{2}a_1-\frac{1}{3}a_2-\frac{1}{4}a_3-\frac{1}{5}a_4-\cdots\right)x^2=0.$$

This further implies that

$$\frac{1}{3}a_0+\frac{1}{4}a_1+\frac{1}{5}a_2+\frac{1}{6}a_3+\frac{1}{7}a_4+\cdots=\frac{1}{4},$$
$$a_0+\frac{2}{3}a_1+\frac{2}{5}a_3+\frac{1}{3}a_4+\cdots=\frac{2}{3},$$
$$a_0+\frac{1}{2}a_1+\frac{1}{3}a_2+\frac{1}{4}a_3+\frac{1}{5}a_4-\cdots=\frac{1}{2}.$$

Thus,

$$a_1=1, a_0=a_2=a_3=\cdots=0,$$

and subsequently $f(x)=F_0(x)=x$, which is an exact solution of (14).

Example 2.

Consider the following ill-posed integral equation taken from [Delves and Mohamed (1988) and Hansen (1994)]

$$l\int_a^b k(x,s)f(s)\,ds=g(x),$$

where

$$l=1, k(x,s)=\begin{cases} x(s-1), & x<s, \\ s(x-1), & x\geq s, \end{cases}$$

and

$$g(x)=\begin{cases} g_1(x)=\dfrac{x^3-x}{6}, \\ g_2(x)=\mathrm{e}^x+(1-\mathrm{e})x-1. \end{cases}$$

If $g(x)=g_1(x)$, then the homotopy considered in [Abbasbandy (2006)] becomes

$$(1-p)F(x)-p\left(\frac{x^3-x}{6}\right)+p\left(\int_0^x x(s-1)F(s)\,ds+\int_x^1 s(x-1)F(s)\,ds\right)=0. \tag{18}$$

Now, suppose that the solution of equation (18) has the form (16). Substituting (16) into (18), and equating the terms with identical powers of p, results to

$$p^0 : F_0(x) = 0,$$
$$p^1 : F_1(x) = \frac{x^3 - x}{6},$$
$$p^j : F_j(x) = F_{j-1}(x) - \int_0^x x(s-1)F_{j-1}(s)ds - \int_x^1 s(x-1)_{j-1}F(s)ds, \quad j = 2, 3, \cdots.$$

Let $f(x) \approx \sum_{j=0}^{10} F_j(x)$, then the approximate solution of Example 2 is

$$f(x) \approx -3.251089917 + 1.584423249x - 2.826206432x^2 + 1.358735477x^3 - \cdots.$$

However, the homotopy of the NHPM for $g(x) = g_1(x)$ is

$$(1-p)\big(F(x) - f_0(x)\big) + p\left(\int_0^x x(s-1)F(s)ds + \int_x^1 s(x-1)F(s)ds - \frac{x^3 - x}{6}\right) = 0.$$

Suppose that $F(x) = \sum_{n=0}^{\infty} p^n F_n(x)$, then by equating the coefficients of the terms with the identical powers of p, we have

$$p^0 : F_0(x) = f_0(x),$$
$$p^1 : F_1(x) = -\int_0^x x(s-1)F_0(s)ds - \int_x^1 s(x-1)F_0(s)ds + \frac{x^3 - x}{6},$$
$$p^2 : F_2(x) = F_1(x) - \int_0^x x(s-1)F_1(s)ds - \int_x^1 s(x-1)F_1(s)ds,$$
$$\vdots$$
$$p^j : F_j(x) = F_{j-1}(x) - \int_0^x x(s-1)F_{j-1}(s)ds - \int_x^1 s(x-1)F_{j-1}(s)ds.$$

Furthermore, assume that $f_0(x) = \sum_{n=0}^{\infty} a_n x^n$ and $F_1(x) = 0$. Then, we have

$$-\int_0^x x(s-1)F_0(s)ds - \int_x^1 s(x-1)F_0(s)ds + \frac{x^3 - x}{6} = 0.$$

Thus,

$$\left(-\frac{1}{6} + \frac{1}{2}a_0 + \frac{1}{6}a_1 + \frac{1}{12}a_2 + \frac{1}{20}a_3 + \frac{1}{30}a_4\right)x - \frac{1}{2}a_0x^2 + \left(\frac{1}{6} - \frac{1}{6}a_1\right)x^3$$
$$-\frac{1}{12}a_2x^4 - \frac{1}{20}a_3x^5 - \frac{1}{30}a_4x^6 - \frac{1}{42}a_7x^7 - \cdots = 0.$$

Now, it can be easily shown

$$a_1 = 1, a_0 = a_2 = a_3 = \cdots = 0.$$

This further implies that $f(x) = F_0(x) = x$, which is the exact solution of Example 2. Now, let us consider the case where $g(x) = e^x + (1-e)x - 1$. Then, the homotopy considered in [Abbasbandy (2006)] is

$$(1-p)F(x) - p\,g(x) + p\left(\int_0^x x(s-1)F(s)\,ds + \int_x^1 s(x-1)F(s)\,ds\right) = 0. \tag{19}$$

Suppose that the solution of equation (19) has the form (16). By substituting (16) into (19), and equating the terms with identical powers of p, we have

$$\begin{aligned}
p^0 &: F_0(x) = 0,\\
p^1 &: F_1(x) = e^x + (1-e)x - 1,\\
p^j &: F_j(x) = F_{j-1}(x) - \int_0^x x(s-1)F_{j-1}(s)\,ds - \int_x^1 s(x-1)_{j-1}F(s)\,ds, \quad j = 2,3,\ldots .
\end{aligned}$$

Now, if $f(x) \approx \sum_{j=0}^{5} F_j(x)$, then the approximate solution of Example 2 is

$$f(x) \approx 31e^x - 32.06967896 - 33.52173018x - 13.43911800x^2 - 5.384097045x^3 - \ldots$$

However, the homotopy of NHPM is

$$\begin{aligned}
(1-p)\left(G(x) - \sum_{n=0}^{\infty} a_n x^n\right) + p\Bigg(&\int_0^x x(s-1)G(s)\,ds\\
&+ \int_x^1 s(x-1)G(s)\,ds - e^x - (1-e)x + 1\Bigg) = 0.
\end{aligned} \tag{20}$$

Now by assuming that the solution of equation (20) has the form

$$G(x) = G_0(x) + pG_1(x) + p^2G_2(x) + \ldots . \tag{21}$$

and substituting (21) into (20), analogously we get

$$\begin{aligned}
p^0 &: G_0(x) = \sum_{n=0}^{\infty} a_n x^n,\\
p^1 &: G_1(x) = -\int_0^x x(s-1)G_0(s)\,ds - \int_x^1 s(x-1)G_0(s)\,ds + e^x + (1-e)x - 1,\\
p^2 &: G_2(x) = G_1(x) - \int_0^x x(s-1)G_1(s)\,ds - \int_x^1 s(x-1)G_1(s)\,ds,\\
&\vdots\\
p^j &: G_j(x) = G_{j-1}(x) - \int_0^x x(s-1)G_{j-1}(s)\,ds - \int_x^1 s(x-1)G_{j-1}(s)\,ds.
\end{aligned}$$

If we set $G_1(x) = 0$, then

$$-\int_0^x x(s-1)G_0(s)\,ds - \int_x^1 s(x-1)G_0(s)\,ds + e^x + (1-e)x - 1 = 0$$

$$\left(2 - e + \frac{1}{2}a_0 + \frac{1}{6}a_1 + \frac{1}{12}a_2 + \frac{1}{20}a_3 + \frac{1}{30}a_4 + \cdots\right)x + \left(\frac{1}{2} - \frac{1}{2}a_0\right)x^2 + \left(\frac{1}{6} - \frac{1}{6}a_1\right)x^3$$

$$+ \left(\frac{1}{24} - \frac{1}{12}a_2\right)x^4 + \left(\frac{1}{120} - \frac{1}{20}a_3\right)x^3 + \left(\frac{1}{720} - \frac{1}{30}a_4\right)x^5 + \cdots = 0.$$

From here, it easily follows that

$$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2!}, a_3 = \frac{1}{3!}, a_4 = \frac{1}{4!}, \ldots .$$

Therefore,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,$$

which is the exact solution of Example 2.

Example 3.

In this example we consider a system of integral equations, which an approximate solution has been obtained by the HPM in [Biazar and Ghazvini (in press)]:

$$f_1(x) + \int_0^x e^{-(s-x)} f_1(s)\,ds + \int_0^x \cos(s-x) f_2(s)\,ds = \cosh x + x \sin x,$$

$$f_2(x) + \int_0^x e^{s+x} f_1(s)\,ds + \int_0^x x \cos s\, f_2(s)\,ds = 2\sin x + x(\sin^2 x + e^x).$$

The exact solutions are $f_1(x) = e^{-x}$ and $f_2(x) = 2\sin x$. However, as we will see the NHPM gives the exact solution.

For solving this system by the NHPM we consider the following homotopy

$$\begin{aligned}(1-p)\Big(F_1(x) - \sum_{n=0}^{\infty} a_n x^n\Big) + p\Big(F_1(x) &+ \int_0^x e^{-(s-x)} F_1(s)\,ds \\ &+ \int_0^x \cos(s-x) F_2(s)\,ds - \cosh x - x\sin x\Big) = 0, \\ (1-p)\Big(F_2(x) - \sum_{n=0}^{\infty} b_n x^n\Big) + p\Big(F_2(x) &+ \int_0^x e^{s+x} F_1(s)\,ds \\ &+ \int_0^x x\cos s\, F_2(s)\,ds - 2\sin x - x(\sin^2 x + e^x)\Big) = 0.\end{aligned} \tag{22}$$

Now suppose that the solution of system (6) is in the following form:

$$F_i(x) = F_{i,0}(x) + pF_{i,1}(x) + p^2F_{i,2}(x) + \cdots, \quad i = 1,2, \tag{23}$$

where $F_{i,j}(t)$, $i = 1,2$, and $j = 0,1,2,\ldots$, are unknown functions that should be determined. Substituting (23) into (22) and equating the coefficients of p with the same power lead to

$$p^0: \begin{cases} F_{1,0}(x) = \sum_{n=0}^{\infty} a_n x^n, \\ F_{2,0}(x) = \sum_{n=0}^{\infty} b_n x^n, \end{cases}$$

$$p^1: \begin{cases} F_{1,1}(x) = -\sum_{n=0}^{\infty} a_n x^n - \int_0^x e^{-(s-x)} F_{1,0}(s)ds - \int_0^x \cos(s-x) F_{2,0}(s)ds + \cosh x + x\sin x, \\ F_{2,1}(x) = -\sum_{n=0}^{\infty} b_n x^n - \int_0^x e^{s+x} F_{1,0}(s)ds - \int_0^x x\cos s\, F_{2,0}(s)ds + 2\sin x + x(\sin^2 x + e^x)), \end{cases}$$

$$p^2: \begin{cases} F_{1,2}(x) = -\int_0^x e^{-(s-x)} F_{1,1}(s)ds - \int_0^x \cos(s-x) F_{2,1}(s)ds, \\ F_{2,2}(x) = -\int_0^x e^{s+x} F_{1,1}(s)ds - \int_0^x x\cos s\, F_{2,1}(s)ds, \end{cases}$$

$$\vdots$$

$$p^j: \begin{cases} F_{1,j}(x) = -\int_0^x e^{-(s-x)} F_{1,j-1}(s)ds - \int_0^x \cos(s-x) F_{2,j-1}(s)ds, \\ F_{2,j}(x) = -\int_0^x e^{s+x} F_{1,j-1}(s)ds - \int_0^x x\cos s\, F_{2,j-1}(s)ds. \end{cases}$$

Now, if we set $F_{1,1}(x) = 0$, then

$$(1 - a_0) - (a_0 + b_0 + a_1)x + \left(\frac{3}{2} - \frac{a_0}{2} - \frac{a_1}{2} - \frac{b_1}{2} - a_2\right)x^2$$

$$- \left(\frac{a_0}{6} + \frac{a_1}{6} + \frac{a_2}{3} + a_3 - \frac{b_0}{6} + \frac{b_2}{3}\right)x^3 + \left(\frac{1}{8} - \frac{a_0}{24} - \frac{a_1}{24} - \frac{a_2}{12} - \frac{a_3}{4} - a_4 + \frac{b_1}{24} - \frac{b_3}{4}\right)x^4 + \cdots = 0$$

and if we set $F_{2,1}(x) = 0$, then

$$-b_0 + (3 - a_0 - b_1)x + \left(1 - b_0 - b_2 - \frac{3a_0}{2} - \frac{a_1}{2}\right)x^2$$

$$+ \left(\frac{7}{6} - \frac{b_1}{6} - b_3 - \frac{7a_0}{6} - \frac{5a_1}{6} - \frac{a_2}{3}\right)x^3 + \left(\frac{1}{6} - \frac{b_0}{6} - b_4 - \frac{5a_0}{8} - \frac{17a_1}{24} - \frac{7a_2}{12} - \frac{a_3}{4}\right)x^4 + \cdots = 0.$$

From here, it easily follows that

$$a_0 = 1, a_1 = -1, a_2 = \frac{1}{2!}, a_3 = -\frac{1}{3!}, a_4 = \frac{1}{4!}, a_5 = -\frac{1}{5!}, \ldots$$

$$b_0 = 0, b_1 = 2, b_2 = 0, b_3 = -\frac{2}{3!}, b_4 = 0, b_5 = \frac{2}{5!}, b_6 = 0, \ldots.$$

Therefore, the exact solution of the system of integral equation can be expressed as

$$f_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n} = e^{-x},$$

$$f_2(x) = 2\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = 2\sin x .$$

4. Conclusions

In this article, we have introduced a new homotopy perturbation method for solving linear integral equation, specially ill-posed problems. Several examples, including some ill-posed problems, are presented to show the ability of the new method compared to the classical HPM.

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