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On the Growth of Solutions of the Generalized Axially Symmetric, Reduced Wave Equation in $(n+1)$ Variables

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Abstract

In this paper we have investigated the growth properties of solutions of the generalized axially symmetric, reduced wave equation in $(n+1)$ variables. Results analogus to those for order and type found in the theory of entire functions of several complex variables, of solutions, in terms of their expansion coefficients have been obtained. Our study is essential to a detailed understanding of the scattering of waves by central potentials and may be applied for generalized $(n+2)-$dimensional problems of potential scattering in quantum mechanics.

Keywords: Axially symmetric potentials, potential scattering, order and type, Bessel function and several complex variables.

MSC 2010 No.: 30E10, 30E15

1. Introduction

Functions of the form

$$\varphi(X) = \left(\frac{\lambda r}{2}\right)^{-(n+s-1)/2} \sum_{\mu=0}^{\infty} \sum_{|M|=\mu} \mu! b_M J_{\mu+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi), \tag{1}$$

where $X = (x_1, \dots, x_n, \rho), r^2 = x_1^2 + \dots + x_n^2 + \rho^2, \rho = r\left(1 - \sum_{i=1}^n \xi_i^2\right)^{1/2}, \lambda \neq 0$ is real, $s > -1, \mu = m_1 + m_2 + \dots + m_n, M \equiv m_1, \dots, m_n, V_M^{(s)}(\xi) \equiv V_M^{(s)}(\xi_1, \dots, \xi_n)$ and $J_\nu(x)$ is a

Bessel function of first kind and ν^{th} order, arise as solutions of the generalized axially symmetric, reduced wave equation in $(n+1)$ variables, namely,

$$L_{\lambda,s}^{(n)} \equiv \frac{\partial^2\varphi}{\partial x_1^2} + \cdots + \frac{\partial^2\varphi}{\partial x_n^2} + \frac{\partial^2\varphi}{\partial\rho^2} + \frac{s}{\rho}\frac{\partial\varphi}{\partial\rho} + \lambda^2\varphi = 0. \tag{2}$$

When $\lambda = 0$ this equation is known as the equation of generalized axially symmetric potential theory (GASPT). The special case where $n = 1$ $(\lambda = 0)$ has also been investigated by Erdélyi (1956, 1965), Gilbert (1960, 1962, 1964, 1965), Ranger (1965), Henrici (1953, 1957, 1960), and Fryant (1979). For $n = 1$, the reduced wave equation (2) has been studied by Colton (1967), Henrici (1953, 1957, 1960), Gilbert (1967), Gilbert and Howard (1965), Nautiyal (1983), Kumar and Rajbir (2013), and for $n > 1$ it was investigated by Gilbert and Howard (1965). For the latest related work see Kumar and Arora (2010), Kumar and Harfaoui (2012), Kumar and Basu (2014), Kumar (2012, 2014). Although Mishra (2007) studied some problems on approximations of functions in Banach spaces by using different operators, Mishra et al. (2013) have obtained the inverse results in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators. In her Ph.D. thesis, Deepmala (2014) also studied some results on fixed point theorems for nonlinear contractions with applications, but our results are different from these authors.

In this paper we develop, for certain series representations of solutions of Equation (2), results analogus to those for order and type found in the theory of entire functions of several complex variables. Our results may be applied for generalized $(n+2)-$dimensional problems of potential scattering in quantum mechanics. Our study concerning the entire function solutions of Equation (2) is essential to a detailed understanding of the scattering of waves by central potentials.

The well-known properties involving the functions $V_m^{(s)}(\xi)$ are given in the following lemma (Gilbert, 1967).

Lemma 1.1.

If the Dirichlet data for the partial differential equation (2) given on the hypersphere $r = R_0$ is integrable, then this boundary value problem has a $C^{(\infty)}$ solution for $r < R_0$ which may be expanded in terms of the functions

$$\varphi_{n,M}(X) = r^{-(n+s-1)/2} J_{\mu+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi);$$

furthermore, each solution $C^{(\infty)}$ in a domain containing the origin has a unique representation in terms of these solutions.

R.P. Gilbert (1967) constructed an integral operator which maps the class of monomials $\Im^M$ onto the solutions

$$r^{-(n+s-1)/2} J_{\mu+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi).$$

Let us discuss some more properties regarding the solutions of partial differential equation (2).

Using Cauchy's integral formula in terms of formal sum, we get

$$\left(\frac{1}{2\pi i}\right)^n \int_{e_k} \left[\left(\frac{\lambda r}{2}\right)^{-(n+s-1)/2} \sum_{\mu=0}^{\infty} \sum_{|M|=\mu} \mu! \Im^M J_{\mu+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi) \right] \Im^{-M} \frac{d\Im}{\Im}$$
$$= \frac{\lambda r}{2}^{-(n+s-1)/2} \mu! J_{\mu+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi),$$

or

$$J_{\mu+(n+s-1)/2}(\lambda(r) V_M^{(s)}(\xi)$$
$$= \left(\frac{\lambda}{2}\right)^{-(n+s+1)/2} \frac{1}{\mu!} \left(\frac{1}{2\pi i}\right)^n \int_{e_k} \Im^{-M} \left[\sum_{\mu=0}^{\infty} \sum_{|M|=\mu} \mu! \Im^M J_{\mu+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi) \right] \frac{d\Im}{\Im}, \tag{3}$$

where $\Im^{-k} \equiv \Im_1^{-k_1}, \ldots, \Im_n^{-k_n}$, $\frac{d\Im}{\Im} = \frac{d\Im_1}{\Im_1} \ldots \frac{d\Im_n}{\Im_n}$ and $e_k, k = 1, 2, \ldots, n$, are suitably chosen contours (homologous to zero) such that $\{\prod_{k=1}^n e_k\}$ is outside a sufficiently large poly-cylinder. If we consider for each fixed r, ξ the series [*] in the right hand side of (3) is a power series in $\Im^M$ where the coefficients are given by

$$a_M \equiv \frac{\lambda r^{-(n+s-1)/2}}{2} \mu! J_{\mu+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi),$$

its domain of convergence for $\Im \in C^{(n)}$ can be studied by considering the associated radii of convergence, $r_1, \ldots, r_n$, which satisfy Fuks (1963)

$$\lim_{\mu\to\infty} (|a_M| r^M)^{1/\mu} \to \infty, r^M = r_1^{M_1}, \ldots, r_n^{M_n}.$$

It is given (Gilbert (1968), p. 36) that if $f(\Im) = \sum_{\mu=0}^{\infty} \sum_{|M|=\mu} b_M \Im^{-M}$, then the solution $\varphi(X)$ corresponding to it is

$$\varphi(X) = \left(\frac{\lambda r}{2}\right)^{-(n+s-1)/2} \sum_{\mu=0}^{\infty} \sum_{|M|=\mu} \mu! b_M J_{\mu+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi).$$

2. Auxiliary Results

First we compute the radius of convergence of the series (1) in terms of the coefficients b_M.

Theorem 2.1.

The series

$$\left(\frac{\lambda r}{2}\right)^{-(n+s-1)/2} \sum_{\mu=0}^{\infty} \sum_{|M|=\mu} \mu! b_M J_{\mu+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi) \tag{4}$$

converges absolutely and uniformly on compact subsets of the open disk centered at the origin of radius R given by

$$\frac{1}{R} = \limsup_{\mu\to\infty} |b_M|^{1/\mu}.$$

Further, such convergence is not obtained on any larger disk.

Proof:

Let ρ^* denote the radius of the largest disk centered at the origin in which the series (4) will converge uniformly on compact subsets. Let φ denote the limit function. Then by using biorthogonal relation

$$\begin{aligned}\int_{1\geq\|\xi\|\geq 0}(1-\|\xi\|^2)^{(s-1)/2}V_M^{(s)}(\xi)U_L^{(s)}(\xi)d^n\xi &= \delta_{l_1}^{m_1}\ldots\delta_{l_n}^{m_n}\\ &\times\frac{2\pi^{n/2}\Gamma(\frac{s}{2}+1)\Gamma(s+m)}{(2m+n+s-1)\Gamma(\frac{n+s-1}{2})\Gamma(s)M!}\end{aligned} \tag{5}$$

or

$$\int_{1\geq\|\xi\|\geq 0}(1-\|\xi\|^2)^{(s-1)/2}[V_M^{(s)}(\xi)]^2d^n\xi = \frac{2\pi^{n/2}\Gamma(\frac{s}{2}+1)\Gamma(s+m)}{(2m+n+s-1)\Gamma(\frac{n+s-1}{2})\Gamma(s)M!}$$

and termwise integration gives

$$\begin{aligned}&\int_{1\geq\|\xi\|\geq 0}(1-\|\xi\|^2)^{(s-1)/2}\varphi(r,\xi)V_M^{(s)}(\xi)d^n\xi\\ &= \frac{2\pi^{n/2}\Gamma(\frac{s}{2}+1)\Gamma(s+m)\mu!b_M J_{\mu+(n+s-1)/2}(\lambda r)(\lambda r)^{\frac{(n+s-1)}{2}}}{(2m+n+s-1)\Gamma(\frac{n+s-1}{2})\Gamma(s)M!}.\end{aligned} \tag{6}$$

Since $J_{\mu+(n+s-1)/2}(\lambda r) \geq \frac{1}{2\Gamma\mu+(\frac{n+s+1}{2})}(\frac{\lambda r}{2})^{\mu+(n+s-1)/2}$, now (6) gives

$$\begin{aligned}&\frac{\pi^{n/2}\Gamma(\frac{s}{2}+1)\Gamma(s+m)\mu!b_M}{(2m+n+s-1)\Gamma(\frac{n+s-1}{2})\Gamma(s)M!\Gamma\mu+(\frac{n+s+1}{2})}(\frac{\lambda r}{2})^{\mu}\\ \leq\ &\int_{1\geq\|\xi\|\geq 0}(1-\|\xi\|^2)^{(s-1)/2}\varphi(r,\xi)V_M^{(s)}(\xi)d^n\xi.\end{aligned}$$

Using Cauchy-Schwartz inequality, we get

$$\begin{aligned}|b_M|\left(\frac{\lambda r}{2}\right)^{\mu} &\leq C^*\left[\frac{\Gamma(\mu+(\frac{n+s+1}{2}))(2m+n+s-1)\Gamma(\frac{n+s-1}{2})\Gamma(s)C_1M!}{\pi^{n/2}\Gamma(\frac{s}{2}+1)\Gamma(s+m)\mu!}\right]\\ &\times M[\varphi,D_R],\end{aligned} \tag{7}$$

where

$$C_1 = \int_{1\geq\|\xi\|\geq 0}\left|(1-\|\xi\|^2)^{(s-1)/2}\right|^2 d^n\xi, \quad C^* = \max_{1\geq\|\xi\|\geq 0}\left|V_M^{(s)}(\xi)\right|,$$

$M[\varphi,D_r] = \sup_{\|X\|_e=\alpha R}|\varphi(X)|$, $\|X\|_e^2 = x_1^2+x_2^2+\cdots+x_n^2+\rho^2$, $D_R \equiv \left\{(z) : \left(\frac{z}{R}\right) \in D \subset C^{(n)}\right\}$, D is an arbitrary, $n-$circular domain and α is a constant.

Noting that from Gilbert (1967),

$$\lim_{\mu\to\infty}\left[\frac{(2\mu+n+s-1)M!}{\Gamma(s+\mu)}\right]^{\frac{1}{\mu}}\to 1,$$

and taking $|Z_\nu^o|$ to be fixed, from [11] we have

$$\frac{1}{R}<\frac{|Z_\nu^o|}{\lambda r}=\frac{\alpha}{R},$$

and this yields from (7) that

$$\limsup_{\mu\to\infty}|b_M|^{1/\mu}\le\frac{2}{|Z_\nu^o}\left(\frac{\alpha}{R}\right),$$

and since the choice of $\left(\frac{|Z_\nu^o|R}{2\alpha}\right)<\rho^*$ was arbitrary,

$$\limsup_{\mu\to\infty}|b_M|^{1/\mu}\le\frac{1}{\rho^*}.$$

On the other hand, suppose

$$\limsup_{\mu\to\infty}|b_M|^{1/\mu}=\frac{1}{R}. \tag{8}$$

Using $J_{\mu+(n+s-1)/2}(\lambda r)\le\frac{(\lambda r/2)^{\mu+(n+s-1)/2}}{\Gamma(\mu+(n+s+1)/2)}$, we find the series (4) is dominated by

$$\sum_{\mu=0}^{\infty}\sum_{|M|=\mu}\mu!|b_M|\left(\frac{|Z_\nu^0|R}{2\alpha}\right)^{\mu}\frac{C^*}{\Gamma\left(\mu+\left(\frac{n+s+1}{2}\right)\right)}.$$

Thus (8) implies that the series (4) converges absolutely and uniformly on compact subsets of the disk centered at the origin of radius R, from which the proof of the theorem is complete. □

In particular, $\varphi(X)$ is entire if and only if

$$\limsup_{\mu\to\infty}|b_M|^{1/\mu}=0.$$

Now we introduce the growth parameters order $\tau_\varphi(D)$ and type $\sigma_\varphi(D)$ of an entire function solution $\varphi(X)$ of Equation (2) following the usual function theoretic definitions of several complex variables (Fuks (1963)),

$$\tau_\varphi(D)=\limsup_{R\to\infty}\frac{\log\log M[\varphi,D_R]}{\log R} \tag{9}$$

$$\sigma_\varphi(D)=\limsup_{R\to\infty}\frac{\log M[\varphi,D_R]}{R^{\tau_\varphi(D)}}. \tag{10}$$

We have the following theorem analogus of the Goldberg result and Gilbert (1968, Thm. 5.2).

Theorem A.

Let $f(\Im) = \sum_{\mu=0}^{\infty} \sum_{|M|=\mu} b_M \Im^{-M}$ be an entire function of $(1/\Im) \in C^{(n)}$ about $(1/\Im) = (0)$. Furthermore, let us define the quantities

$$\tau_f(D) = \limsup_{R\to\infty} \frac{\log\log M[f, D_R]}{\log R}, \tag{11}$$

$$\sigma_f(D) = \limsup_{R\to\infty} \frac{\log M[f, D_R]}{R^{\tau_f(D)}}, \tag{12}$$

$$\tau_f(D) = \limsup_{|k|\to\infty} \frac{\log k}{\log[|a_k|d_k]^{-1/|k|}}, \tag{13}$$

and

$$(e\tau_f(D)\sigma_f(D))^{1/\tau_f(D)} = \limsup_{|k|\to\infty} \left\{ |k|^{1/\tau_f(D)}[|a_k|d_k]^{1/|k|} \right\}, \tag{14}$$

where

$$|k| = k_1 + \cdots + k_n, d_k = \sup_{(z)\in D} |z^k| \ \ \texttt{and} \ \ M[f, D_R] = \sup_{z\in D_R} |f(z)|.$$

3. Main Results

In this section we shall characterize the order and type of $\varphi(X)$ in terms of coefficients occurring in series development (1).

Theorem 3.1.

Let $\varphi(X)$ be a solution of (2) with a series development (1). Furthermore, let $\varphi(X)$ be an entire function solution of order $\tau_\varphi(D)$. Then

$$\tau_\varphi(D) = \limsup_{\mu\to\infty} \frac{\log \mu}{\log |b_M|^{-1/\mu}},$$

where $\{b_M\}$ are the coefficients of $\varphi(X)$ in its expansion (1).

Proof:

In view of (7) and definition (9) of order, we have for R sufficiently large,

$$|b_M| \leq C^* \left[\frac{\Gamma(\mu + (n+s-1)/2)(2m+n+s-1)\Gamma((n+s-1)/2)\Gamma(s)C_1 M!}{\pi^{n/2}\Gamma(\frac{s}{2}+1)\Gamma(s+m)\mu!} \right]^{1/2} \times \left(\frac{2\alpha}{|Z_\nu^o|R} \right)^{\mu} \exp(R^{(\tau_\varphi(D)+\varepsilon)}).$$

The minimum value of $\left(\frac{2\alpha}{|Z_\nu^o|R}\right)^\mu \exp(R^{(\tau_\varphi(D)+\varepsilon)})$ is attained at

$$\frac{R|Z_\nu^o|}{2\alpha} = \left(\frac{\mu}{\tau_\mu(D)+\varepsilon}\right)^{1/(\tau_\varphi(D)+\varepsilon)}.$$

Thus for μ sufficiently large,

$$\begin{aligned}|b_M| \leq &\\ & C^* \left[\frac{\Gamma(\mu+(n+s-1)/2)(2m+n+s-1)\Gamma((n+s-1)/2)\Gamma(s)C_1 M!}{\pi^{n/2}\Gamma(\frac{s}{2}+1)\Gamma(s+m)\mu!}\right]^{1/2}\\ & \times \exp\left(\left(\frac{2\alpha}{|Z_\nu^o|}\right)^{(\tau_\varphi(D)+\varepsilon)} \frac{\mu}{\tau_\varphi(D)+\varepsilon}\right) \left(\frac{\tau_\varphi(D)+\varepsilon}{\mu}\right)^{\frac{\mu}{(\tau_\varphi(D)+\varepsilon)}}\\ & = C^* \left[\frac{\Gamma(\mu+(n+s-1)/2)(2m+n+s-1)\Gamma((n+s-1)/2)\Gamma(s)C_1 M!}{\pi^{n/2}\Gamma(\frac{s}{2}+1)\Gamma(s+m)\mu!}\right]^{1/2}\\ & \times \exp\left(\left(\frac{2\alpha}{|Z_\nu^o|}\right)^{(\tau_\varphi(D)+\varepsilon)} \left[\frac{(\tau_\varphi(D)+\varepsilon)e}{\mu}\right]^{\frac{\mu}{(\tau_\varphi(D)+\varepsilon)}}\right),\end{aligned}$$

and now we can easily get

$$\limsup_{\mu\to\infty} \frac{\log\mu}{\log|b_M|^{-1/\mu}} \leq \tau_\varphi(D). \tag{15}$$

In order to prove reverse inequality, let us choose our D as the unit hypersphere in the maximum norm, i.e., $\triangle \equiv \{(z) : \|z\|_m < 1\}$, where $\|z\|_m = \max_{1\leq k\leq n}|z_k|$. Since the function $f(\Im)$ is expanded in the negative power monomials $\Im^{-M}$, hence, if $1/(\Im) \in \triangle_R$, then $|1/\Im_k| < R$ or $(\Im) \notin \triangle_{1/R}$. Now we define

$$\widetilde{M}[f, \triangle_R^o] \equiv \left(\sup_{(\Im)\notin\triangle_{1/R}} |f(\Im)|\right) R^{(n+s-1)/2} \tag{16}$$

and fix R large. Since $\|\Im\|_m \geq 1/R$, taking the domain of integration the set $\prod_{k=1}^n \{\Im_k = e^{i\theta k}/R\}$, we have, for $|\Im_\nu| = \frac{1}{R} < |Z_\nu^o|/(\lambda r) = \frac{\alpha}{R}$, that (see Gilbert (1968))

$$\varphi(X) \leq \frac{A_3}{\rho^{(n+s-1)/2}} \widetilde{M}[f, \triangle_R^o] \ \texttt{for}\ \|X|_e < \alpha R, \tag{17}$$

where R sufficiently large, $\rho > 0, A_3$ and α are the suitably chosen constants.

Using (16) in (17) with the definitions of order (9) and (11), we get

$$\tau_\varphi(D) \leq \tau_f(D).$$

The formula expressing the order of an entire function of several complex variables in terms of its Taylor coefficients by Equation (13) then yields

$$\tau_\varphi(D) \leq \limsup_{\mu\to\infty} \frac{\log\mu}{\log|b_M|^{-1/\mu}}. \tag{18}$$

Inequalities (15) and (18) together completes the proof. □

Theorem 3.2.

Let $\varphi(X)$ be a solution of (2) with a series development (1). Furthermore, let $\varphi(X)$ be an entire function solution of order $\tau_\varphi(D)$ and type $\sigma_\varphi(D)$. Then

$$(e\tau_\varphi(D)\sigma_\varphi(D))^{1/\tau_\varphi(D)} = \limsup_{\mu\to\infty}\{\mu^{1/\tau_\varphi(D)}|b_M|^{1/\mu}\}.$$

Proof:

In view of the result of Theorem 3.1 and expression of the order (13) of an entire function of several complex variables in terms of its Taylor coefficients, it follows that the order of $\varphi(X)$ equals the order of $f(\Im)$. Thus for the inequality expression given for the type of an entire function of several complex variables, we have

$$\sigma_\varphi(D) \leq \sigma_f(D) = \frac{1}{e\tau_f(D)}\limsup_{\mu\to\infty}\{\mu|b_M|^{\tau_f(D)/\mu}\}. \tag{19}$$

Using (7) and the definition (10) of type, we have for sufficiently large R,

$$|b_M| \leq C^*\left[\frac{\Gamma(\mu+(n+s-1)/2)(2m+s+s-1)\Gamma((n+s-1)/2)\Gamma(s)C_1 M!}{\pi^{n/2}\Gamma(\frac{s}{2}+1)\Gamma(s+m)\mu!}\right]^{1/2} \times \frac{\exp\left[(\sigma_\varphi(D)+\varepsilon)R^{\tau_\varphi(D)}\right]}{R^\mu}.$$

The minimum value of $R^{-\mu}\exp\left[(\sigma_\varphi(D)+\varepsilon)R^{\tau_\varphi(D)}\right]$ is attained at

$$R = \left[\frac{\mu}{(\sigma_\varphi(D)+\varepsilon)\tau_\varphi(D)}\right]^{\frac{1}{\tau_\varphi(D)}}.$$

For sufficiently large μ, we have

$$|b_M| \leq C^*\left[\frac{\Gamma(\mu+(n+s-1)/2)(2m+s+s-1)\Gamma((n+s-1)/2)\Gamma(s)C_1 M!}{\pi^{n/2}\Gamma\left(\frac{s}{2}+1\right)\Gamma(s+m)\mu!}\right]^{1/2} \times \left[\frac{(\sigma_\varphi(D)+\varepsilon)e\tau_\varphi(D)}{\mu}\right]^{\frac{\mu}{\tau_\varphi(D)}},$$

which gives

$$\frac{1}{e\tau_\varphi(D)}\limsup_{\mu\to\infty}\mu|b_M|^{\frac{\tau_\varphi(D)}{\mu}} \leq \sigma_\varphi(D). \tag{20}$$

(19) and (20) together completes the proof. □

4. Conclusion

The potential play an important role in many aspects of mathematical physics, in particular to an understanding of compressible flow in the transonic region. Although axially symmetric potential theory is a well developed subject with many applications to the physical sciences, it is, perhaps, not fully appreciated that certain biological problems suggest the use of this theory. The problem of steady-state differential flow through a cylindrical structure arises frequently. Not surprisingly, the physiological situation may provide motivation for solving problems and seeking techniques that are different from those arising from purely mathematical or physical considerations.

The solutions of the form (1.1) of the generalized axially symmetric reduced wave equation (1.2) in $(n+1)$ variables for $\lambda = 0$ are called the generalized axially symmetric potentials. The Euler-Poisson-Darboux equation, arising in gas dynamics, is viewed in terms of Equation (1.2) for $\lambda = 0$ after a transformation and have a variety of physical interpretations. Our results concerning the growth of entire function solutions of the equation (1.2) is applied for the detailed study of generalized $(n+2)$-dimensional problem of potential scattering in quantum mechanics.

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