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On Local Asymptotic Stability of *q*-Fractional Nonlinear Dynamical Systems

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Abstract

In this paper, locally asymptotic stability of q-fractional order nonlinear dynamical systems is introduced and studied. The sufficient conditions for local stability of such dynamical systems are obtained. Also, useful definitions of fractional order q-integrals and q-derivatives are recalled. Finally, a q-fractional order nonlinear dynamical model is considered.

Keywords: q- calculus; fractional q-integral; fractional q-derivative; q-fractional system; stability

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1. Introduction

Fractional calculus, a branch of mathematical analysis, has gained popularity and importance during the last three decades. Fractional differential equations are an important application area of fractional calculus. The first book on fractional calculus is the book written by Oldham and Spanier (1974). One of the most popular books on fractional calculus is the book by Podlubny (1999). Fractional differential equations are very useful and important tools in physics (Hilfer (2011)), engineering (Sun et al. (1984)), mathematical biology (Ahmed and Elgazzar (2007), Ozalp and Demirci (2011)), interpersonal relationships (Ozalp and Koca (2012), Koca and Ozalp (2014)) and finance (Laskin (2000)).

The development of q-analysis started in the 1740s. q-calculus is a methodology centered on deriving q-analogous of the classical calculus results without using limits. The main tool of q-calculus is q-derivative. As a survey about q-calculus we refer to Ernst (2000). Starting from the work of Agarwal (1969) and Al Salam (1966, 1967), the q-fractional calculus is defined and many properties of q-fractional calculus are obtained.

Recently, stability of fractional dynamical systems has attracted increasing interest. In 1966, Matignon (1966) first studied the stability of linear fractional differential systems with Caputo derivative. Since then, many researchers have done further studies on the stability of linear fractional differential systems. In q-calculus the stability of q-fractional dynamical systems was investigated by Abdeljawad and Baleanu (2011) and Jarad et al. (2011). In Jarad et al. (2011), sufficient conditions for the Mittag-Leffler stability of q-fractional nonlinear dynamical systems were obtained. To the best of our knowledge, the local stability of q-fractional order nonlinear dynamical systems has not yet been studied. In this study, some conditions are derived to discuss local stability of q-fractional order nonlinear systems. Also an existence theorem for Caputo q-fractional differential equations is given.

We first recall some useful definitions of fractional order q-integrals and q-derivatives.

2. Preliminaries and Definitions

The definitions can be found in Gasper and Rahman (2000), Kac and Cheung (2002), Stankovic et al. (2009) and Annaby and Mansour (2012).

Definition 1.

Let q be regarded as a real number with $0 < q < 1$. q-number is defined by

$$[x]_q = [x: q] = \frac{1-q^x}{1-q}.$$

We note that $\lim_{q\to 1}[x]_q = x$.

Definition 2.

For any $x > 0$, q-Gamma function is defined by

$$\Gamma_q(x) = \int_0^\infty t^{x-1} E_q^{-qt} d_q t \,. \tag{1}$$

Here, q-analogue of the classical exponential function is defined by

$$E_q^x = \sum_{j=0}^\infty q^{j(j-1)/2} \frac{x^j}{[j]_q!} = (1 + (1-q)x)_q^\infty,$$

where

$$[m]_q! = [1]_q [2]_q \dots [m-1]_q [m]_q.$$

Using the Equation (1), the following is obtained:

For any $x > 0$, $\Gamma_q(x+1) = [x]_q\, \Gamma_q(x)$,

$$\Gamma_q(1) = \int_0^{\infty} E_q^{-qt} d_q t = E_q^0 - E_q^{-\infty} = 1.$$

So, for any nonnegative integer n, $\Gamma_q(n+1) = [n]_q!$.

Definition 3.

Let $\alpha \geq 0$ and f be a function defined on [0,1]. The fractional q-integral of Riemann-Liouville type is:

$$\left(I_{q,a}^{\alpha} f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x-qt)^{\alpha-1} f(t) d_q t\ , \qquad (\alpha > 0, x \in [0,1]),$$

with $\left(I_q^0 f\right)(x) = f(x)$. Here and elsewhere Γ_q denotes the q-Gamma function.

Let $\alpha, \beta \in R^+$, the fractional q-integration has the following property:

$$(I_{q,a}^{\beta} I_{q,a}^{\alpha} f)(x) = \left(I_{q,a}^{\alpha+\beta} f\right)(x) \qquad (a < x).$$

Definition 4.

The Riemann-Liouville type fractional q-derivative of a function $f: (0,\infty) \to R$ is defined by

$$\left(\boldsymbol{D}_{q,a}^{\alpha} f\right)(x) = \begin{cases} \left(I_{q,a}^{-\alpha} f\right)(x), & \alpha \leq 0, \\ \left(\boldsymbol{D}_q^{\lceil\alpha\rceil} I_{q,a}^{\lceil\alpha\rceil-\alpha} f\right)(x), & \alpha > 0, \end{cases}$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater than or equal to α.

Definition 5.

The Caputo type fractional q-derivative of a function $f: (0,\infty) \to R$ is defined by

$$\left(D_{q,a}^{\alpha} f\right)(x) = \begin{cases} \left(I_{q,a}^{-\alpha} f\right)(x), & \alpha \leq 0, \\ \left(I_{q,a}^{\lceil\alpha\rceil-\alpha} D_q^{\lceil\alpha\rceil} f\right)(x), & \alpha > 0\,. \end{cases}$$

Some of the main properties of the Riemann-Liouville type fractional q-integral and fractional q-derivative are given below:

i) Let $\alpha \in R^+$, then for $a < x$,

$$\left(\boldsymbol{D}_{q,a}^{\alpha} I_{q,a}^{\alpha} f\right)(x) = f(x).$$

Also, let $\alpha \in R^+ \backslash N$, then for $a < x$,

$$\left(I_{q,a}^{\alpha} \boldsymbol{D}_{q,a}^{\alpha} f\right)(x) = f(x).$$

ii) Let $\alpha \in R$ and $\beta \in R^+$, then for $a < x$,

$$\left(\boldsymbol{D}_{q,a}^{\alpha} I_{q,a}^{\beta} f\right)(x) = (\boldsymbol{D}_{q,a}^{\alpha-\beta} f)(x).$$

Also, let $\alpha \in R\backslash N$ and $\beta \in R^+$, then for $a < x$,

$$\left(I_{q,a}^{\beta} \boldsymbol{D}_{q,a}^{\alpha} f\right)(x) = (\boldsymbol{D}_{q,a}^{\alpha-\beta} f)(x).$$

iii) Let $\alpha \in R^+$, $\lambda \in (-1, +\infty)$, then

$$I_{q,a}^{\alpha}\left((t-a)^{(\lambda)}\right) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}(t-a)^{(\alpha+\lambda)}, \qquad 0 < a < t < b.$$

iv) Let $\alpha > 0$ and *n* be a positive integer, then

$$\left(I_{q,a}^{\alpha} D_{q,a}^{n} f\right)(x) = \left(D_{q,a}^{n} I_{q,a}^{\alpha} f\right)(x) - \sum_{k=0}^{n-1} \frac{x^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)}\left(D_q^k f\right)(0).$$

3. Locally Asymptotic Stability of *q*-Fractional Order Systems

Let $\alpha \in (0,1]$ and consider the system:

$$\begin{aligned} D_q^{\alpha} x_1(t) &= f_1(x_1, x_2, \dots, x_k), \\ D_q^{\alpha} x_2(t) &= f_2(x_1, x_2, \dots, x_k), \\ &\vdots \\ D_q^{\alpha} x_k(t) &= f_k(x_1, x_2, \dots, x_k), \end{aligned} \tag{2}$$

where the fractional derivative in System (2) is in the sense of q-Caputo. Let the initial values of System (2) be given as

$$x_{q1}(0) = x_{01}, x_{q2}(0) = x_{02}, \dots, x_{qk}(0) = x_{0k}.$$

The equilibrium solutions of (2) are obtained by equating the system to zero.

Let $E^* = (x_1^*, x_2^*, \dots, x_k^*)$ be an equilibrium point of System (2) and

$$x_i(t) = x_i^* + \varepsilon_i(t),$$

then

$$D_q^{\alpha}(x_i^* + \varepsilon_i) = f_i(x_1^* + \varepsilon_1, x_2^* + \varepsilon_2, \dots, x_k^* + \varepsilon_k), \quad i = 1,2, \dots, k.$$

So, one can obtain

$$D_q^{\alpha} \varepsilon_i(t) = f_i(x_1^* + \varepsilon_1, x_2^* + \varepsilon_2, \dots, x_k^* + \varepsilon_k).$$

Using the q-Taylor expansion and the fact that

$$f_i(x_1^*, x_2^*, \dots, x_k^*) = 0,$$

we obtain:

$$D_q^\alpha \varepsilon_i(t) \approx \frac{d_q f_i}{d_q x_1}|_{eq}\varepsilon_1 + \frac{d_q f_i}{d_q x_2}|_{eq}\varepsilon_2 + \cdots + \frac{d_q f_i}{d_q x_k}|_{eq}\varepsilon_k,$$

which reduces to the following system.

$$D_q^\alpha \varepsilon_i = J_q \varepsilon, \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ . \\ . \\ . \\ \varepsilon_k \end{bmatrix}, \quad J_q(E^*) = \begin{bmatrix} \frac{d_q f_1}{d_q x_1} & \frac{d_q f_1}{d_q x_2} & \cdots & \frac{d_q f_1}{d_q x_k} \\ \vdots & \vdots & & \vdots \\ \frac{d_q f_k}{d_q x_1} & \frac{d_q f_k}{d_q x_2} & \cdots & \frac{d_q f_k}{d_q x_k} \end{bmatrix}, \tag{3}$$

where J_q is the q-Jacobian matrix evaluated at the equilibrium point $E^*(x_1^*, x_2^*, \dots, x_k^*)$ and satisfies the following relation:

$$A^{-1}J_q A = B, \quad B = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & 0 & \lambda_k \end{bmatrix}, \tag{4}$$

where $\lambda_1, \lambda_2, \dots, \lambda_k$ are eigenvalues of J_q and A is the matrix of eigenvectors of J_q. System (3) has the initial values

$$\varepsilon_1(0) = x_{q1}(0) - x_1^*, \varepsilon_2(0) = x_{q2}(0) - x_2^*, \dots, \varepsilon_k(0) = x_{qk}(0) - x_k^*. \tag{5}$$

Using (3) and (4), the following equalities are obtained:

$$D_q^\alpha \varepsilon = (ABA^{-1})\varepsilon, \quad D_q^\alpha (A^{-1}\varepsilon) = B.$$

Hence,

$$D_q^\alpha \eta = B\eta, \eta = A^{-1}\varepsilon, \eta = [\eta_1, \eta_2, \dots, \eta_k]^T. \tag{6}$$

Therefore,

$$D_q^\alpha \eta_1 = \lambda_1 \eta_1, D_q^\alpha \eta_2 = \lambda_2 \eta_2, \dots, D_q^\alpha \eta_k = \lambda_k \eta_k. \tag{7}$$

The solutions of (7) are obtained by using q-Mittag-Leffler functions [Annaby and Mansour (2012)]

$$\eta_1(t) = \sum_{n=0}^{\infty} \frac{(\lambda_1)^n t^{n\alpha}}{\Gamma_q(n\alpha+1)} \eta_1(0) =_q E_\alpha(\lambda_1 t^\alpha)\eta_1(0), \tag{8}$$

$$\vdots$$

$$\eta_i(t) = \sum_{n=0}^{\infty} \frac{(\lambda_i)^n t^{n\alpha}}{\Gamma_q(n\alpha+1)} \eta_i(0) =_q E_\alpha(\lambda_i t^\alpha)\eta_i(0),$$

$i = 1,2,\dots,k$. Using the result in Matignon (1996) and via q-generalization of Ahmed et al. (2007), we see that if

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \qquad (i = 1,2,\dots,k), \tag{9}$$

then $\eta_i(t)$ are decreasing and consequently, $\varepsilon_i(t)$ are decreasing.

Corollary 1.

The equilibrium point $E^*(x_1^*, x_2^*, \dots, x_k^*)$ is locally asymptotically stable if condition (9) is satisfied.

4. Main Results

Consider the initial value problem (IVP) given by

$$\begin{gathered} D_q^\alpha x(t) = f(t, x(t)), \qquad (\alpha > 0), \\ D_q^k x(0^+) = b_k \qquad (b_k \in R; k = 0,1,\dots,\lceil\alpha\rceil - 1). \end{gathered} \tag{10}$$

The following theorem is given for (10) by Annaby and Mansour (2012).

Theorem 1.

Let $\alpha > 0, n = \lceil\alpha\rceil$. Let G be an open set in $\mathbb{C}$ and $f: (0, a] \times G \to R$ be a function such that $f(t, x) \in C_\gamma[0, a]$ for any $x \in G, \gamma \le \alpha - n + 1$. If $x \in C_q^n[0, a]$ then $x(t)$ satisfies (10), for all $t \in (0, a]$, if and only if $x(t)$ satisfies the q-integral equation:

$$x(t) = \sum_{k=0}^{n-1} \frac{b_k}{\Gamma_q(k+1)} t^k + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t \left(\frac{q\tau}{t}; q\right)_{\alpha-1} f(\tau, x(\tau)) d_q\tau \tag{11}$$

for all $t \in [0, a]$.

Now consider the following Caputo q-fractional order dynamical system:

$$D_q^\alpha x(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

where $t \ge t_0;\ t_0, t \in T_q; 0 < q < 1$ *and* $f: T_q \times R \to R^n$ is continuous in *x*.

For $0 < q < 1$, let T_q be the time scale (Bohner and Peterson (2001)) defined by

$$T_q = \{q^n : n \in Z\} \cup \{0\}.$$

Theorem 2. (Koca (2015))

Let $\|.\|$ denote any convenient norm on R^n. For $0 < q < 1$, let T_q be the time scale and assume that $f \in C\left[R_1 \in T_q \times R^n, R^n\right]$, where $R_1 = [(t, X): 0 \leq t \leq a \; and \; \|X - X_0\| \leq b]$, $f = (f_1, f_2, \ldots, f_n)^T$, $X = (x_1, x_2, \ldots, x_n)^T$ and let $\|f(t, X)\| \leq M$ on R_1. Then there exists at least one solution for the system of q-fractional differential equation given by

$$D_q^\alpha X(t) = f\big(t, X(t)\big)$$

with the initial conditions:

$$X(0) = X_0,$$

on $0 \leq t \leq \beta$, where $\beta = \min(a, \left[\frac{b\Gamma_q(\alpha+1)}{M}\right]^{\frac{1}{2\alpha-1}}\Big), 0 < \alpha < 1$.

5. Equilibrium Points and Locally Asymptotic Stability of q-Fractional Order Model

Consider the system of q-fractional nonlinear differential equations of order α, $0 < \alpha \leq 1$, that models the interpersonal relationships:

$$\begin{cases} D_q^\alpha x_1(t) = -\alpha_1 x_1 + \beta_1 x_2 (1 - \varepsilon x_2^2) + A_1, \\ D_q^\alpha x_2(t) = -\alpha_2 x_2 + \beta_2 x_1 (1 - \varepsilon x_1^2) + A_2, \\ x_{q1}(0) = 0, \; x_{q2}(0) = 0, \end{cases} \tag{12}$$

where D_q^α is the Caputo type fractional q-derivative. $\alpha_i > 0$, β_i and $A_i, (i = 1,2)$ are real constants. These parameters are oblivion, reaction and attraction constants. In the equations above, we assume that feelings decay exponentially fast in the absence of partners. The parameters specify the romantic style of individuals 1 and 2. In the beginning of the relationships, because they have no feelings for each other, initial conditions are assumed to be zero. Detailed analysis of this model has been considered in Ozalp and Koca (2012) and Koca and Ozalp (2013). Different from the mentioned references, in this paper, the model is discussed with Caputo type fractional q-derivative.

To evaluate the equilibrium points of (12), let

$$\begin{cases} D_q^\alpha x_1(t) = f_1(x_1, x_2) = 0, \\ D_q^\alpha x_2(t) = f_2(x_1, x_2) = 0, \end{cases}$$

where

$$f_1(x_1, x_2) = -\alpha_1 x_1 + \beta_1 x_2 (1 - \varepsilon x_2^2) + A_1$$

and

$$f_2(x_1, x_2) = -\alpha_2 x_2 + \beta_2 x_1 (1 - \varepsilon x_1^2) + A_2.$$

Then the equilibrium point is $E_1 = (x_1^*, x_2^*)$.

q-Jacobian matrix J_q for the system given in (12) is:

$$J_q = \begin{bmatrix} \frac{d_q f_1(x_1,x_2)}{d_q x_1} & \frac{d_q f_1(x_1,x_2)}{d_q x_2} \\ \frac{d_q f_2(x_1,x_2)}{d_q x_1} & \frac{d_q f_2(x_1,x_2)}{d_q x_2} \end{bmatrix} = \begin{bmatrix} \frac{f_1(qx_1,x_2)-f_1(x_1,x_2)}{qx_1-x_1} & \frac{f_1(x_1,qx_2)-f_1(x_1,x_2)}{qx_2-x_2} \\ \frac{f_2(qx_1,x_2)-f_2(x_1,x_2)}{qx_1-x_1} & \frac{f_2(x_1,qx_2)-f_2(x_1,x_2)}{qx_2-x_2} \end{bmatrix}.$$

Therefore,

$$J_q(E_1) = \begin{bmatrix} -\alpha_1 & \beta_1(1-[3]_q \varepsilon x_2^{*2}) \\ \beta_2(1-[3]_q \varepsilon x_1^{*2}) & -\alpha_2 \end{bmatrix}.$$

To discuss the local stability of the equilibrium $E_1 = (x_1^*, x_2^*)$ of the system given with (12), we consider the linearized system at E_1. The characteristic equation of the linearized system is of the form

$$C(\lambda) = \lambda^2 + (\alpha_1 + \alpha_2)\lambda + \alpha_1\alpha_2 - \beta_1\beta_2(1-[3]_q \varepsilon x_2^{*2})(1-[3]_q \varepsilon x_1^{*2}) = 0 \qquad (13)$$

Theorem 3.

The equilibrium point $E_1 = (x_1^*, x_2^*)$ of the system given in (12) is asymptotically stable if one of the following conditions holds for all eigenvalues of $J_q(E_1)$:

(i) $1 < \frac{-4\beta_1\beta_2}{(\alpha_1-\alpha_2)^2}(1-[3]_q \varepsilon x_2^{*2})(1-[3]_q \varepsilon x_1^{*2})$,

(ii) $\frac{\beta_1\beta_2}{\alpha_1\alpha_2}(1-[3]_q \varepsilon x_2^{*2})(1-[3]_q \varepsilon x_1^{*2}) < 1$.

Proof:

The equilibrium point $E_1 = (x_1^*, x_2^*)$ of the system given in (12) is asymptotically stable if all of the eigenvalues, λ_i, $i = 1,2$, of $J_q(E_1)$, satisfy the condition

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \qquad (i = 1,2).$$

These eigenvalues can be determined by solving the characteristic equation

$$\det\left(J_q(E_1) - \lambda I\right) = 0.$$

Thus, we have the following equation:

$$\lambda^2 + K\lambda + L = 0,$$

where

$$K = (\alpha_1 + \alpha_2),$$

$$L = \alpha_1\alpha_2 - \beta_1\beta_2(1-[3]_q \varepsilon x_2^{*2})(1-[3]_q \varepsilon x_1^{*2}).$$

The roots of the characteristic equation are

$$\lambda_{1,2} = \frac{-K}{2} \pm \frac{\sqrt{K^2 - 4L}}{2}.$$

It is obvious that $K = (\alpha_1 + \alpha_2) > 0$. If $K^2 - 4L < 0$, then all of the eigenvalues, $\lambda_{1,2}$, have negative real parts and satisfy the condition given by (i). If $K^2 > K^2 - 4L$, then both of the eigenvalues are negative and satisfy the condition given by (ii).

6. Conclusion

It is worth nothing that, q-derivative is the q-analogue of ordinary derivative. The q-analogues in the literature find application areas in many fields of science, for example the multi-fractal measures, entropy of chaotic dynamical systems and fractals. The connection between dynamical systems and fractals is due to the fact that numerous fractal designs do have the symmetries of Fuchsian group, generally speaking. The interpersonal relationships based upon two individuals or three are represented by nonlinear dynamical systems. Therefore, in order to include the entropy of chaotic dynamics into the mathematical model portraying the dynamical relationship between two or three individuals, we use the q-derivative in this paper and extend the model of interpersonal relationships to the scope of q-analogues. In this paper, local asymptotic stability conditions in q-fractional order nonlinear systems have been studied. These conditions have been applied to the system of q-fractional order nonlinear dynamical model of interpersonal relationships of order $\alpha, 0 < \alpha \leq 1$.

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