Available at
http://pvamu.edu.edu/pages/398.asp
ISSN: 1932-9466
Vol. 3, Issue 1 (June 2008) pp. 113 – 127
(Previously Vol. 3, No. 1)

Applications and Applied Mathematics (AAM):
An International Journal

Remarks on the Stability of some Size-Structured Population Models II: Changes in Vital Rates due to Size and Population size

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Received June 17, 2007; accepted December 20, 2007

Abstract

The stability of some size-structured population dynamics models are investigated. We determine the steady states and study their stability. We also give examples that illustrate the stability results. The results in this paper generalize previous results, for example, see Calsina, et al. (2003) and El-Doma (2006).

Keywords: Population; Stability; Steady state; Size-structure.

MSC 2000: 45K05; 45M10; 35B35; 35L60; 92D25.

1. Introduction

In this paper, we study the following size-structured population dynamics model:

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial t} + \dfrac{\partial}{\partial a}(V(a,P(t))p(a,t)) + \mu(a,P(t))p(a,t) = 0, \ a \in [0,l), l \le +\infty, \ t > 0, \\ V(0,P(t))p(0,t) = \displaystyle\int_0^l \beta(a,P(t))p(a,t)da, \ t \ge 0, \\ p(a,0) = p_0(a), \ a \in [0,l), \\ P(t) = \displaystyle\int_0^l p(a,t)da, \ t \ge 0, \end{cases} \tag{1}$$

where $p(a,t)$ is the density of the population with respect to size $a \in [0,l)$ at time $t \ge 0$; $l \le +\infty$, is the maximum size an individual in the population can attain; $P(t) = \int_0^l p(a,t)da$ is the total population size at time t; $\beta(a,P(t)), \mu(a,P(t))$ are, respectively, the birth rate i.e. the average number of offspring, per unit time, produced by an individual of size a when the population size is $P(t)$, and the mortality rate, i.e., the death rate at size a, per unit population, when the population size is $P(t)$; $0 < V(a,P)$ is the individual growth rate at the population size P; $p(0,t) = \int_0^l \beta(a,P(t))p(a,t)da$ is the number of births, per unit time, when the population size is $P(t)$.

We study problem (1) under the following general assumptions:

- $0 \le p_0(a) \in L^1([0,l)) \cap L_\infty[0,l), \mathbb{R}^+ = [0,\infty)$;
- $V(a,P(t)), \beta(a,P(t))$ and $\mu(a,P(t)) \in C([0,l) \times \mathbb{R}^+)$, and are nonnegative functions;
- $V_P(a,P), V_{Pa}(a,P), \beta_P(a,P), \mu_P(a,P)$ exist $\forall a \ge 0, P \ge 0$; and
- $V_P(.,P), V_{Pa}(.,P), \beta(.,P), \beta_P(.,P), \mu(.,P), \mu_P(.,P) \in C([0,l) : L_\infty(\mathbb{R}^+))$.

Models of size-structured populations were first derived by Sinko et al. (1967), where the population density and the vital rates depend on age, size and time. Due to its complication, this type of model has been ignored by mathematicians, for example, see Metz, et al. (1986). Problem (1) is special case of the classical model given by Sinko, et al. (1967). However problem (1) generalizes those given in Calsina, et al. (2003) and El-Doma (2006), where the vital rates are taken to depend on the population size only.

Mimura, et al. (1988), studied a model that is similar to problem (1) but the dependence on the population size $P(t)$ is changed to a dependence on a weighted population size $r(t)$ i.e., $r(t) = \int_0^l \omega(a)p(a,t)da, \omega \ge 0$, and the growth rate V is of separable form that is special case of problem (1); and they proved the global existence and uniqueness of non-negative solutions, and obtained some stability results when the death rate μ depends on the weighted population size $r(t)$ only. Calsina, et al. (1995), studied problem (1) with the additional assumption that there is an inflow of newborns (of zero-size), like seeds in plants, and proved the existence and uniqueness of solution; and the existence of a global attractor when the inflow is a constant.

Further generalization of size-structured population dynamics models involved the additional assumption of subdividing the population into subgroups based on growth rates, these growth rates can be finite in number leading to a finite number of subgroups, for example, see Ackleh, et al. (2005) or infinitely many different growth rates, for example, see Huyer (1994). These studies proved existence and uniqueness results; and provided numerical results as in Huyer (1994), and numerical and statistical results as in Ackleh, et al. (2005).

In this paper, we study problem (1) and determine its steady states and examine their stability. We prove that the trivial steady state is always a steady state and that there are as many nontrivial steady states P_∞ as the positive solutions of the equation, $R(P_\infty) = 1$, see section 2 for the definition of $R(P)$. Then we study the stability of the trivial steady state and show that if, $R(0) < 1$, then the trivial steady state is locally asymptotically stable and if, $R(0) > 1$, then the trivial steady state is unstable. We also determine sufficient conditions for the local asymptotic stability of a nontrivial steady state P_∞, and show that if, $R'(P_\infty) > 0$, then a nontrivial steady state is unstable.

The organization of this paper as follows: in section 2 we determine the steady states; in section 3 we study the stability of the steady states and give several examples that illustrate our theorems; in section 4 we conclude our results.

2. The Steady States

In this section, we determine the steady states of problem (1). A steady state of problem (1) satisfies the following:

$$\begin{cases} \dfrac{d}{da}[V(a,P_\infty)p_\infty(a)] + \mu(a,P_\infty)p_\infty(a) = 0, \quad a \in [0,l), \\ V(0,P_\infty)p_\infty(0) = \displaystyle\int_0^l \beta(a,P_\infty)p_\infty(a)da, \\ P_\infty = \displaystyle\int_0^l p_\infty(a)da. \end{cases} \tag{2}$$

From (2), by solving the differential equation, we obtain that

$$p_\infty(a) = p_\infty(0)V(0,P_\infty)\frac{\pi(a,P_\infty)}{V(a,P_\infty)}, \tag{3}$$

where $\pi(a,P_\infty)$ is defined as

$$\pi(a,P) = e^{-\int_0^a \frac{\mu(\tau,P)}{V(\tau,P_\infty)}d\tau}.$$

Also, from (2) and (3), we obtain that $p_\infty(0)$ satisfies the following:

$$p_\infty(0) = p_\infty(0)\int_0^l \frac{\beta(a,P_\infty)}{V(a,P_\infty)}\pi(a,P_\infty)da. \tag{4}$$

Accordingly, from (4), we conclude that either $p_\infty(0) = 0$ or P_∞ satisfies the following:

$$1 = \int_0^l \frac{\beta(a, P_\infty)}{V(a, P_\infty)} \pi(a, P_\infty) da. \tag{5}$$

In order to facilitate our writing, we define a threshold parameter $R(P)$ by

$$R(P) = \int_0^l \frac{\beta(a, P)}{V(a, P)} \pi(a, P) da, \tag{6}$$

which when $V \equiv 1$ and a is age (the age-structured case) is interpreted as the number of children expected to be born to an individual, in a life time, when the population size is P.

We note that from equation (3),

$$p_\infty(0) = \frac{P_\infty}{V(0, P_\infty) \displaystyle\int_0^l \frac{\pi(a, P_\infty)}{V(a, P_\infty)} da},$$

and accordingly, either $p_\infty(a) \equiv 0$ or $p_\infty(a)$ is completely determined by a solution $P_\infty > 0$ of equation (5).

In the following theorem, we describe the steady states of problem (1), the proof of the theorem is straightforward and therefore, is omitted.

Theorem 2.1

(1) *Problem (1) has the trivial steady state,* $P_\infty = 0$, *as a steady state.*
(2) *All positive solutions of,* $R(P_\infty) = 1$, *are nontrivial steady states of problem (1).*

3. Stability of the Steady States

In this section, we study the stability of the steady states for problem (1) as given by Theorem 2.1.

To study the stability of a steady state $p_\infty(a)$, which is a solution of (2) and is given by equation (3), we linearize problem (1) at $p_\infty(a)$ in order to obtain a characteristic equation, which in turn will determine conditions for the stability. To that end, we consider a perturbation $\omega(a, t)$ defined by $\omega(a, t) = p(a, t) - p_\infty(a)$, where $p(a, t)$ is a solution of problem (1). Accordingly, we obtain that $\omega(a, t)$ satisfies the following:

$$\begin{cases} \dfrac{\partial \omega(a,t)}{\partial t} + \dfrac{\partial}{\partial a}\Big(V(a, P_\infty)\omega(a,t)\Big) + \Big[\dfrac{\partial}{\partial a}\Big(V_P(a, P_\infty)p_\infty(a)\Big) + p_\infty(a)\mu_P(a, P_\infty)\Big]W(t) \\ +\mu(a, P_\infty)\omega(a,t) = 0, \ a \in [0, l), \ t > 0, \\ \omega(0,t)V(0, P_\infty) = \displaystyle\int_0^l \beta(a, P_\infty)\omega(a,t)da + W(t)\int_0^l \beta_P(a, P_\infty)p_\infty(a)da \\ -p_\infty(0)V_P(0, P_\infty)W(t), \ t \geq 0, \\ \omega(a, 0) = p_0(a) - p_\infty(a), \quad a \in [0, l), \\ W(t) = \displaystyle\int_0^l \omega(a,t)da, \quad t \geq 0. \end{cases} \tag{7}$$

By substituting $\omega(a,t) = f(a)e^{\xi t}$ in (7), where ξ is a complex number, and straightforward calculations, we obtain the following characteristic equation:

$$\begin{aligned}1 &= \frac{1}{V(0,P_\infty)}\int_0^l e^{-\int_0^a E(\tau)d\tau}\beta(a,P_\infty)da \\ &+\frac{\displaystyle\int_0^l e^{-\int_0^a E(\tau)d\tau}da}{V(0,P_\infty)\Big[1+\displaystyle\int_0^l\int_0^a e^{-\int_\sigma^a E(\tau)d\tau}g(\sigma)d\sigma da\Big]} \\ &\times\Big[\int_0^l \beta_P(a,P_\infty)p_\infty(a)da - p_\infty(0)V_P(0,P_\infty) - \int_0^l\int_0^a e^{-\int_\sigma^a E(\tau)d\tau}\beta(a,P_\infty)g(\sigma)d\sigma da\Big],\end{aligned} \tag{8}$$

where $g(\sigma)$ and $E(\sigma)$ are, respectively, given by

$$\begin{aligned} g(\sigma) &= \frac{\frac{\partial}{\partial\sigma}\Big(V_P(\sigma,P_\infty)p_\infty(\sigma)\Big)+p_\infty(\sigma)\mu_P(\sigma,P_\infty)}{V(\sigma,P_\infty)}, \\ E(\sigma) &= \frac{\xi+V_\sigma(\sigma,P_\infty)+\mu(\sigma,P_\infty)}{V(\sigma,P_\infty)}.\end{aligned}$$

In the following theorem, we describe the stability of the trivial steady state, $p_\infty(a)\equiv 0$.

Theorem 3.1 *The trivial steady state,* $p_\infty(a)\equiv 0$, *is locally asymptotically stable if,* $R(0)<1$, *and is unstable if,* $R(0)>1$.

Proof. We note that for the trivial steady state, $p_\infty(a)\equiv 0$, $P_\infty = 0$, and therefore, from the characteristic equation (8), we obtain the following characteristic equation:

$$1 = \int_0^l e^{-\xi\int_0^a \frac{d\tau}{V(\tau,0)}}\frac{\beta(a,0)}{V(a,0)}\pi(a,0)da. \tag{9}$$

To prove the local asymptotic stability of the trivial steady state, we note that if, $R(0)<1$, then equation (9) can not be satisfied for any ξ with, $Re\xi\geq 0$, since

$$\begin{aligned}\Big|\int_0^l e^{-\xi\int_0^a \frac{d\tau}{V(\tau,0)}}\frac{\beta(a,0)}{V(a,0)}\pi(a,0)da\Big| &\leq \int_0^l e^{-Re\xi\int_0^a \frac{d\tau}{V(\tau,0)}}\frac{\beta(a,0)}{V(a,0)}\pi(a,0)da \\ &\leq R(0)<1.\end{aligned}$$

Accordingly, the trivial steady state is locally asymptotically stable if, $R(0)<1$.

To prove the instability of the trivial steady state when, $R(0)>1$, we note that if we define a function $h(\xi)$ by

$$h(\xi) = \int_0^l e^{-\xi\int_0^a \frac{d\tau}{V(\tau,0)}}\frac{\beta(a,0)}{V(a,0)}\pi(a,0)da,$$

and suppose that ξ is real, then we can easily see that $h(\xi)$ is a decreasing function if $\xi>0$, $h(\xi)\longrightarrow 0$ as $\xi\longrightarrow+\infty$, and $h(0)=R(0)$. Therefore, if, $R(0)>1$, then there exists $\xi^*>0$ such that $h(\xi^*)=1$, and hence the trivial steady state is unstable. This completes the proof of the theorem.

In the next theorem, we give a condition for the instability of a nontrivial steady state.

Theorem 3.2 *A nontrivial steady state is unstable if,* $R'(P_\infty) > 0$.

Proof. We note that the characteristic equation (8) can be rewritten as

$$\begin{aligned} 1 &= \frac{1}{V(0,P_\infty)} \int_0^l e^{-\int_0^a E(\tau)d\tau} \beta(a,P_\infty)da \Big(1 + \int_0^l \int_0^a e^{-\int_\sigma^a E(\tau)d\tau} g(\sigma) d\sigma da \Big) \\ &- \int_0^l \int_0^a e^{-\int_\sigma^a E(\tau)d\tau} g(\sigma) d\sigma da + \frac{1}{V(0,P_\infty)} \int_0^l e^{-\int_0^a E(\tau)d\tau} da \times \\ & \Big[\int_0^l \beta_P(a,P_\infty) p_\infty(a) da - p_\infty(0) V_P(0,P_\infty) - \int_0^l \int_0^a e^{-\int_\sigma^a E(\tau)d\tau} \beta(a,P_\infty) g(\sigma) d\sigma da \Big] \\ &:= G(\xi). \end{aligned} \tag{10}$$

Now, suppose that, $R'(P_\infty) > 0$, then from the characteristic equation (10), we obtain that $G(0) = 1 + R'(P_\infty)P_\infty > 1$, and $G(\xi) \longrightarrow 0$ as $\xi \longrightarrow +\infty$. Accordingly, $\exists \xi^* > 0$ such that $G(\xi^*) = 1$, and hence a nontrivial steady state is unstable. This completes the proof of the theorem.

In the next theorem, we prove that, $\xi = 0$, is a root of the characteristic equation (10) iff, $R'(P_\infty) = 0$.

Theorem 3.3 $\xi = 0$, *is a root of the characteristic equation (10) iff,* $R'(P_\infty) = 0$.

Proof. We note that if, $\xi = 0$, then using equation (5), the characteristic equation (10) becomes, $R'(P_\infty) = 0$. This completes the proof of the theorem.

To obtain further stability results, we note that by suitable changes of the variables of the integrations, we can rewrite the characteristic equation (10) in the following form:

$$\begin{aligned} 1 &= \frac{1}{V(0,P_\infty)} \int_0^l e^{-\int_0^a E(\tau)d\tau} \Big[\beta(a,P_\infty) + \int_0^l \beta_P(b,P_\infty) p_\infty(b) db - p_\infty(0) V_P(0,P_\infty) \Big] da \\ &+ \frac{1}{V(0,P_\infty)} \int_0^l \int_0^l \int_0^a e^{-\int_0^b E(\tau)d\tau} e^{-\int_\sigma^a E(\tau)d\tau} g(\sigma) \Big[\beta(b,P_\infty) - \beta(a,P_\infty) \Big] d\sigma da db \\ &- \int_0^l \int_0^a e^{-\int_\sigma^a E(\tau)d\tau} g(\sigma) d\sigma da. \end{aligned} \tag{11}$$

In the next theorem, we give a sufficient condition for the local asymptotic stability of a nontrivial steady state. We note that this result is for the general problem(1), and in the sequel we give other conditions which are for special cases of problem (1).

Theorem 3.4 *Suppose that the following holds:*

$$\begin{aligned} & \int_0^l \frac{\pi(a,P_\infty)}{V(a,P_\infty)} \Big| \Big[\beta(a,P_\infty) + \int_0^l \beta_P(b,P_\infty) p_\infty(b) db - p_\infty(0) V_P(0,P_\infty) \Big] \Big| da \\ + & \int_0^l \int_0^l \int_0^a e^{-\int_\sigma^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)} d\tau} \frac{V(\sigma,P_\infty)\pi(b,P_\infty)}{V(b,P_\infty)V(a,P_\infty)} \Big| g(\sigma) \Big[\beta(b,P_\infty) - \beta(a,P_\infty) \Big] \Big| d\sigma da db \end{aligned}$$

$$+ \quad \int_0^l \int_0^a e^{-\int_\sigma^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)}} \frac{V(\sigma,P_\infty)}{V(a,P_\infty)} \Big| g(\sigma) \Big| d\sigma da < 1. \tag{12}$$

Then a nontrivial steady state is locally asymptotically stable.

Proof. We note that if we assume that, $Re\xi \geq 0$, then from the characteristic equation (11), we obtain

$$1 \leq \int_0^l e^{-Re\xi \int_0^a \frac{d\tau}{V(\tau,P_\infty)}} \frac{\pi(a,P_\infty)}{V(a,P_\infty)} \Big| \Big[\beta(a,P_\infty) + \int_0^l \beta_P(b,P_\infty) p_\infty(b) db - p_\infty(0) V_P(0,P_\infty) \Big] \Big| da$$
$$+ \int_0^l \int_0^l \int_0^a e^{-Re\xi \int_0^b \frac{d\tau}{V(\tau,P_\infty)}} e^{-Re\xi \int_\sigma^a \frac{d\tau}{V(\tau,P_\infty)}} e^{-\int_\sigma^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)} d\tau} \frac{V(\sigma,P_\infty)\pi(b,P_\infty)}{V(b,P_\infty)V(a,P_\infty)} \times$$
$$\Big| g(\sigma) \Big[\beta(b,P_\infty) - \beta(a,P_\infty) \Big] \Big| d\sigma da db$$
$$+ \int_0^l \int_0^a e^{-Re\xi \int_\sigma^a \frac{d\tau}{V(\tau,P_\infty)}} e^{-\int_\sigma^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)}} \frac{V(\sigma,P_\infty)}{V(a,P_\infty)} \Big| g(\sigma) \Big| d\sigma da$$
$$\leq \int_0^l \frac{\pi(a,P_\infty)}{V(a,P_\infty)} \Big| \Big[\beta(a,P_\infty) + \int_0^l \beta_P(b,P_\infty) p_\infty(b) db - p_\infty(0) V_P(0,P_\infty) \Big] \Big| da$$
$$+ \int_0^l \int_0^l \int_0^a e^{-\int_\sigma^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)} d\tau} \frac{V(\sigma,P_\infty)\pi(b,P_\infty)}{V(b,P_\infty)V(a,P_\infty)} \Big| g(\sigma) \Big[\beta(b,P_\infty) - \beta(a,P_\infty) \Big] \Big| d\sigma da db$$
$$+ \int_0^l \int_0^a e^{-\int_\sigma^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)}} \frac{V(\sigma,P_\infty)}{V(a,P_\infty)} \Big| g(\sigma) \Big| d\sigma da < 1.$$

Accordingly, the characteristic equation (11) is not satisfied for any ξ with, $Re\xi \geq 0$, and hence a nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem.

In the next result, we give a corollary to Theorem 3.4, and the proof of this corollary is straightforward, and therefore, is omitted.

Corollary 3.5 *Suppose that the following hold:*

1. $\beta(a,P) = \beta(P), \forall a \geq 0,$
2. $\displaystyle\int_0^l \frac{\pi(a,P_\infty)}{V(a,P_\infty)} \Big| \Big[\beta(P_\infty) + \beta'(P_\infty) P_\infty - p_\infty(0) V_P(0,P_\infty) \Big] \Big| da$

$$+ \int_0^l \int_0^a e^{-\int_\sigma^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)}} \frac{V(\sigma,P_\infty)}{V(a,P_\infty)} \Big| g(\sigma) \Big| d\sigma da < 1.$$

Then a nontrivial steady state is locally asymptotically stable.

In the next result, we obtain stability result for the special case when, $V(a,P) = V(a)$, and also, $\mu(a,P) = \mu(P)$. We note that this result is a generalization of that given in Gurney, et al. (1980) and Weinstock, et al. (1987) for the classical age-structured population dynamics model of Gurtin, et al. (1974), which corresponds to problem (1) when $V \equiv 1$. We also note that if $\mu(A_\infty) = 0$, then from (3), we obtain that $P_\infty = +\infty$. Therefore, we assume that $\mu(A_\infty) > 0$.

Theorem 3.6 *Suppose that,* $l = +\infty, V(a,P) = V(a), \mu(a,P) = \mu(P)$, *and,* $\mu(P_\infty) \displaystyle\int_0^\infty \frac{d\tau}{V(\tau)} = +\infty$. *Then a nontrivial steady state is locally asymptotically stable if,* $\mu'(P_\infty) \geq 0$, *and,* $\displaystyle\int_0^\infty \beta_P(a,P_\infty) p_\infty(a) da \leq 0$, *with both not equal to zero.*

Proof. We note that by straightforward calculations in the characteristic equation (10), we obtain

$$\begin{aligned}0 &= \frac{1}{\xi}\Big[\xi + \frac{p_\infty(0)\mu'(P_\infty)V(0)}{\mu(P_\infty)}\Big]\Big[\frac{1}{V(0)}\int_0^\infty \beta(a,P_\infty)e^{-\int_0^a E(\tau)d\tau}da - 1\Big] \\ &\quad + \frac{1}{[\xi+\mu(P_\infty)]}\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da. \qquad (13)\end{aligned}$$

Now, we note that the characteristic equation (13) is studied in Gurney, et al. (1980) and Weinstock, et al. (1987), and the theorem follows from the results therein. This completes the proof of the theorem.

We note that concerning the assumptions in Theorem 3.6, and in particular the assumption that $\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da \leq 0$, if we assume that $\mu(a,P) = \mu(a)$ and therefore, $\mu_P = 0$, and that $V(a,P) = V(a)$, then $R(P) = \frac{1}{V(0)p_\infty(0)}\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da$, and according to Theorem 3.2 and Theorem 3.3 a nontrivial steady state P_∞ can be locally asymptotically stable only if $R'(P_\infty) < 0$, and this implies that $\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da < 0$, and so in such cases the assumption of Theorem 3.6 is necessary.

In the next result, we obtain stability result for the special case when, $V(a,P) = V(a)$, and, $\mu(a,P) = \mu(a)$. We note in this case the form of the characteristic equation resembles that of cannibalism, for example, see Iannelli (1995), Bekkal-Brikci, et al. (2007) and El-Doma (to appear).

Theorem 3.7 *Suppose that,* $l = +\infty, V(a,P) = V(a), \mu(a,P) = \mu(a)$, *and,* $\int_0^\infty \frac{\mu(\tau)}{V(\tau)}d\tau = +\infty$. *Then a nontrivial steady state is locally asymptotically stable if,* $R'(P_\infty) < 0$, *and,* $V(a)\mu'(a) \leq \mu^2(a)$.

Proof. We note that if we set $\xi = x + iy$ in the characteristic equation (10), we obtain

$$\begin{aligned}1 &= \frac{1}{V(0)}\int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)}d\tau}\beta(a,P_\infty)\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}\Big)da + \qquad (14)\\ &\quad \frac{1}{V(0)}\Big(\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da\Big)\int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)}d\tau}\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}d\tau\Big)da,\\ 0 &= \frac{1}{V(0)}\int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)}d\tau}\beta(a,P_\infty)\sin\Big(y\int_0^a \frac{d\tau}{V(\tau)}\Big)da + label3.9 \qquad (15)\\ &\quad \frac{1}{V(0)}\Big(\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da\Big)\int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)}d\tau}\sin\Big(y\int_0^a \frac{d\tau}{V(\tau)}d\tau\Big)da.\end{aligned}$$

Now, suppose that $x \geq 0$ and $y = 0$, then from equations (5) and (14), we obtain

$$\begin{aligned}1 &= \frac{1}{V(0)}\int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)}d\tau}\beta(a,P_\infty)da + \\ &\quad \Big(\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da\Big)\int_0^\infty \frac{1}{V(a)}e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)}d\tau}da\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{V(0)}\int_0^\infty e^{-\int_0^a \frac{[V'(\tau)+\mu(\tau)]}{V(\tau)}d\tau}\beta(a,P_\infty)da + \\
&\quad \Big(\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da\Big)\int_0^\infty \frac{1}{V(a)}e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)}d\tau}da \\
&= 1+\Big(\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da\Big)\int_0^\infty \frac{1}{V(a)}e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)}d\tau}da \\
&< 1.
\end{aligned}$$

We note that the last inequality is obtained by using, $R'(P_\infty) < 0$. Accordingly, the characteristic equation (10) is not satisfied for any $x \geq 0$ and $y = 0$.

Now, suppose that $x \geq 0$ and $y \neq 0$, and observe that equation (14) can be rewritten in the following form:

$$\begin{aligned}
1 &= \frac{1}{V(0)}\int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)}d\tau}\beta(a,P_\infty)\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}\Big)da + \\
&\quad \frac{1}{y^2}\Big(\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da\Big)\times \\
&\quad \int_0^\infty \frac{1}{V(a)}e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)}d\tau}\Big[(x+\mu(a))^2 - V(a)\mu'(a)\Big]\Big(1-\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}d\tau\Big)\Big)da \\
&= \int_0^\infty e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)}d\tau}\frac{\beta(a,P_\infty)}{V(a)}\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}\Big)da + \\
&\quad \frac{1}{y^2}\Big(\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da\Big)\times \\
&\quad \int_0^\infty \frac{1}{V(a)}e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)}d\tau}\Big[(x+\mu(a))^2 - V(a)\mu'(a)\Big]\Big(1-\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}d\tau\Big)\Big)da.
\end{aligned}$$

We note that the right-hand side of the above equation is strictly less than one because, $R'(P_\infty) < 0$, and therefore, by Theorem 3.3, $(0,0)$ is not a root of the characteristic equation (10), and accordingly,

$$\int_0^\infty e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)}d\tau}\frac{\beta(a,P_\infty)}{V(a)}\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}\Big)da < 1$$

by equation (5), also, $R'(P_\infty) < 0$, and, $V(a)\mu'(a) \leq \mu^2(a)$, and therefore, the second term in the right-hand side is nonpositive. Accordingly, the characteristic equation (10) is not satisfied for any ξ with, $Re\xi \geq 0$. Therefore, a nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem.

We note that if we only assume that $V(a,P_\infty) = V(a), \mu(a,P_\infty) = \mu(a)$, then from Theorem 3.4, we obtain the following condition for the local asymptotic stability of a nontrivial steady state when $l \leq +\infty$:

$$\int_0^l \frac{\pi(a,P_\infty)}{V(a,P_\infty)}\Big|\Big[\beta(a,P_\infty) + \int_0^l \beta_P(b,P_\infty)p_\infty(b)db\Big]\Big|da < 1.$$

Example 1: We consider the special case given in Calsina, et al. (2003) and El-Doma (2006), where $\mu(a,P) = \mu(P), \beta(a,P) = \beta(P), V(a,P) = V(P)$. Then we have the following two

cases:
Case: $l = +\infty$.

In this case we have, $R(P) = \dfrac{\beta(P)}{\mu(P)}$, and therefore, $\beta(P_\infty) = \mu(P_\infty)$, and, $P_\infty = \dfrac{p_\infty(0)V(P_\infty)}{\mu(P_\infty)}$. Now, as in El-Doma (2006) and Calsina, et al. (2003), straightforward calculations in the characteristic equation (10) yield, $\xi = P_\infty\Big(\beta'(P_\infty) - \mu'(P_\infty)\Big)$. Accordingly, a nontrivial steady state is locally asymptotically stable if, $\beta'(P_\infty) < \mu'(P_\infty)$, and unstable if, $\beta'(P_\infty) > \mu'(P_\infty)$.
Case: $l < +\infty$.

In this case, $R(P) = \dfrac{\beta(P)}{\mu(P)}\Big[1 - e^{-\frac{\mu(P)}{V(P)}l}\Big]$, and therefore, $\dfrac{\mu(P_\infty)}{\beta(P_\infty)} = \Big[1 - e^{-\frac{\mu(P_\infty)}{V(P_\infty)}l}\Big]$, and, $P_\infty = \dfrac{p_\infty(0)V(P_\infty)}{\beta(P_\infty)}$. Now, we can use Corollary 3.5 to obtain the following condition for the local asymptotic stability of a nontrivial steady state:

$$\Big|1 + P_\infty\Big(\frac{\beta'(P_\infty)}{\beta(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\Big)\Big| + P_\infty\Big|\frac{\mu'(P_\infty)}{\mu(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\Big|\Big[1 - \frac{l}{V(P_\infty)}\Big(\beta(P_\infty) - \mu(P_\infty)\Big)\Big] < 1.$$

Now, if we assume that, $\dfrac{\mu'(P_\infty)}{\mu(P_\infty)} - \dfrac{V'(P_\infty)}{V(P_\infty)} = 0$, then we obtain

$$\frac{V'(P_\infty)}{V(P_\infty)} - \frac{2}{P_\infty} < \frac{\beta'(P_\infty)}{\beta(P_\infty)} < \frac{V'(P_\infty)}{V(P_\infty)},$$

which is exactly the result obtained in El-Doma (2006).

Also, if we assume that, $1 + P_\infty\Big(\dfrac{\beta'(P_\infty)}{\beta(P_\infty)} - \dfrac{V'(P_\infty)}{V(P_\infty)}\Big) = 0$, then we obviously obtain

$$P_\infty\Big|\frac{\mu'(P_\infty)}{\mu(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\Big|\Big[1 - \frac{l}{V(P_\infty)}\Big(\beta(P_\infty) - \mu(P_\infty)\Big)\Big] < 1,$$

which is also exactly the result obtained in El-Doma (2006).

Example 2: In this example, we consider the case when $\beta(a,P) = \beta(P)e^{-\alpha a}$, where $\alpha > 0$ is a constant, $\mu(a,P) = \mu(P), V(a,P) = V(P), l = +\infty$. We note that the form of $\beta(a,P)$ allows the concentration of reproduction in the smallest sizes, for example, see Gurtin, et al. (1974).

Now, using equation (5), we obtain

$$V(P_\infty)\alpha + \mu(P_\infty) = \beta(P_\infty). \tag{16}$$

If we also assume that $\beta(P), \mu(P), V(P)$ are defined as follows

$$\beta(P) = \frac{c_1}{P^n}, \quad n = 1, 2, ..., \tag{17}$$

$$\mu(P) = c_2 P^m, \quad m = 0, 1, 2, ..., \tag{18}$$

$$V(P) = c_3, \tag{19}$$

where c_1, c_2, c_3 are positive constants, then from equation (16), we obtain

$$c_3\alpha + c_2 P_\infty^m = \frac{c_1}{P_\infty^n}. \tag{20}$$

So, it is clear that if we take $m = 0$ and n arbitrary in (20), then we obtain that P_∞ satisfies

$$P_\infty = \Big(\frac{c_1}{c_3\alpha + c_2}\Big)^{\frac{1}{n}}.$$

Therefore, by Theorem 2.1, P_∞, given as above is a nontrivial steady state of problem (1), and is locally asymptotically stable via Theorem 3.6.

Also, if we take $n = 1, m = 1$, then we obtain

$$P_\infty = \frac{1}{2c_2}\Big[\sqrt{4c_1 + c_3^2\alpha^2} - c_3\alpha\Big].$$

As before this is a steady state of problem (1), and is locally asymptotically stable via Theorem 3.6.

In fact, we obtain the following general result.

Theorem 3.8 *Suppose that $\beta(P), \mu(P), V(P)$ are given, respectively, by (17)-(19), and $l = +\infty$. Then problem (1) has a unique nontrivial steady state which is locally asymptotically stable.*
Proof. From equation (20), we obtain

$$P_\infty^n\Big[c_3\alpha + c_2P_\infty^m\Big] = c_1. \tag{21}$$

Now, if we denote the left-hand side of equation (21) by $f(P_\infty)$, then $f(0) = 0$, and $f'(P_\infty) > 0$ for $P_\infty > 0$, and accordingly equation (21) has a unique solution $P_\infty > 0$ since $c_1 > 0$. This unique solution is a steady state of problem (1) by Theorem 2.1, and is locally asymptotically stable by Theorem 3.6. This completes the proof of the theorem.

Example 3: In this example, we consider Example: 2, and only change $V(P)$ to $V(a)$, and therefore, using equation (5) and integration by parts, we obtain

$$1 = \frac{\beta(P_\infty)}{\mu(P_\infty)} - \frac{\alpha\beta(P_\infty)}{\mu(P_\infty)}\int_0^\infty e^{-\alpha a - \mu(P_\infty)\int_0^a \frac{d\tau}{V(\tau)}}da. \tag{22}$$

Now, we assume that $\beta(P) = \dfrac{c_1}{P^n}, n = 1, 2, \ldots$, and $\mu(P) = c_2P$, where c_1, c_2 are positive constants, and hence using equation (22), we obtain

$$P_\infty^{n+1} = \frac{c_1}{c_2} - \frac{\alpha c_1}{c_2}\int_0^\infty e^{-\alpha a - c_2P_\infty\int_0^a \frac{d\tau}{V(\tau)}}da. \tag{23}$$

If we define $f(P_\infty)$ as the right-hand side of equation (23), then $f(0) = 0$, and $f'(P_\infty) > 0$. Also, if we let $g(P_\infty) = P_\infty^{n+1}$, then $g(0) = 0$, $g'(P_\infty) > 0$ for $P_\infty > 0$, and $g'(0) = 0$. Since $f'(0) > g'(0) = 0$, and $f(P_\infty) \longrightarrow \dfrac{c_1}{c_2}$ as $P_\infty \longrightarrow +\infty$, whereas $g(P_\infty) \longrightarrow +\infty$ as $P_\infty \longrightarrow +\infty$, this implies that equation (23) has a unique positive solution, which corresponds to a unique nontrivial steady state of problem (1) via Theorem 2.1. Furthermore, this steady state is locally asymptotically stable by Theorem 3.6.

Example 4: In this example, we consider the case given in Calsina, et al. (1995), where $\beta(a, P) = \beta(P)\Big[1 - e^{-\alpha a}\Big]$, $\alpha > 0$ is a constant, which corresponds to the case when reproduction is concentrated at large-sizes, $\mu(a, P) = \mu(P), V(a, P) = V(P), l = +\infty$.

By using equation (5), we obtain

$$\mu(P_\infty) = \frac{\alpha\beta(P_\infty)V(P_\infty)}{\Big[\alpha V(P_\infty) + \mu(P_\infty)\Big]}. \tag{24}$$

Now, if we assume that $\beta(P), \mu(P), V(P)$, are given, respectively, by equations (17)-(19), then from equation (24), we obtain

$$c_2 P_\infty^{m+n}\Big[c_3\alpha + c_2 P_\infty^m\Big] = \alpha c_3 c_1. \tag{25}$$

As for equation (21), it is easy to see that equation (25) has a unique positive solution, which corresponds to a unique nontrivial steady state of problem (1), and is locally asymptotically stable by Theorem 3.6.

Example 5: In this example, we consider Example 4, but we change $V(P)$ to $V(a)$, then from equation (5), we obtain

$$1 = \frac{\alpha\beta(P_\infty)}{\mu(P_\infty)}\int_0^\infty e^{-\alpha a - \mu(P_\infty)\int_0^a \frac{d\tau}{V(\tau)}}da. \tag{26}$$

Now, if we assume that $\beta(P), \mu(P)$, satisfy, respectively, equations (17)-(18), then from (26), we obtain

$$P_\infty^{m+n} = \frac{\alpha c_1}{c_2}\int_0^\infty e^{-\alpha a - c_2 P^m\int_0^a \frac{d\tau}{V(\tau)}}da. \tag{27}$$

From equation (27), if we assume that $m = 0$, then it is easy to see that P_∞ satisfies

$$P_\infty = \Big[\frac{\alpha c_1}{c_2}\int_0^\infty e^{-\alpha a - c_2\int_0^a \frac{d\tau}{V(\tau)}}da\Big]^{\frac{1}{n}}, \quad n = 1, 2, ...$$

And if $m \neq 0$, we let $f(P_\infty)$ to be the right-hand side of equation (27), then $f(0) = \frac{c_1}{c_2}$, $f'(P_\infty) < 0$ for $P_\infty > 0$, and $f(P_\infty) \longrightarrow 0$ as $P_\infty \longrightarrow +\infty$. Also, the function $g(P_\infty) = P_\infty^{m+n}$ in an increasing function that satisfies $g(P_\infty) \longrightarrow +\infty$ as $P_\infty \longrightarrow +\infty$. Accordingly equation (27) has a unique positive solution, which by Theorem 2.1 corresponds to a unique nontrivial steady state for problem (1), and is also locally asymptotically stable by Theorem (12).

Example 6: In this example, we look at the case when $l = +\infty$ and $\beta(a, P), \mu(a, P), V(a, P)$ are given as follows, for example, see Weinstock, et al. (1987).

$$\begin{aligned} \beta(a, P) &= c_1 a^n e^{-c_2 P^k a}, \quad n = 1, 2, ..., \quad k = 1, 2, ..., \\ \mu(a, P) &= c_3 P^m, \quad m = 1, 2, ..., \\ V(a, P) &= c_4, \end{aligned}$$

where c_1, c_2, c_3, c_4 are positive constants.

From equation (5), we obtain

$$\Big[c_2 P_\infty^k + \frac{c_3}{c_4}P_\infty^m\Big]^{n+1} = \frac{c_1}{c_4}n!. \tag{28}$$

From equation (28), it is easy to see that there exists a unique nontrivial steady state, and is locally asymptotically stable by Theorem 3.6.

Example 7: In this example, we consider the case when $l = +\infty$ and $\beta(a, P), \mu(a, P), V(a, P)$ are given as follows

$$\begin{aligned}\beta(a, P) &= ce^{-P}(1 + a), \quad c > 0 \; is \; a \; constant,\\ \mu(a) &= (1 + a)(2 + a + a^2),\\ V(a) &= 1 + a.\end{aligned}$$

From equation (5), we obtain

$$e^{P_\infty} = c\int_0^\infty e^{-2a-\frac{a^2}{2}-\frac{a^3}{3}}da.$$

So, if we choose, $c > 0$, such that $c\int_0^\infty e^{-2a-\frac{a^2}{2}-\frac{a^3}{3}}da > 1$, then we obtain a unique positive solution, P_∞, which corresponds to a unique nontrivial steady state, and is locally asymptotically stable by Theorem 3.7.

Finally, if we change $\beta(a, P)$ to $\beta(a, P) = \dfrac{c}{P^n}, n = 1, 2, ...,$ then we obtain

$$P_\infty = \Big(c\int_0^\infty e^{-2a-\frac{a^2}{2}-\frac{a^3}{3}}da\Big)^{\frac{1}{n}}, \quad n = 1, 2, ..., \tag{29}$$

as the unique nontrivial steady state, and is locally asymptotically stable by Theorem 3.7.

Example 8: In this example, we consider the case when $\beta(a, P), \mu(a, P), V(a, P)$ are given as follows

$$\begin{aligned}\beta(a, P) &= \frac{c}{P}, \quad c > 0 \; is \; a \; constant,\\ \mu(a) &= \mu(a),\\ V(a) &= V(a).\end{aligned}$$

By using equation(5), we obtain

$$P_\infty = c\int_0^l \frac{e^{-\int_0^a \frac{\mu(\tau)}{V(\tau)}d\tau}}{V(a)}da.$$

And hence by Corollary 3.5, this steady state is locally asymptotically stable. Of course, we must have $c\int_0^l \frac{e^{-\int_0^a \frac{\mu(\tau)}{V(\tau)}d\tau}}{V(a)}da < +\infty$, for example, $\mu(a) = (1 + a)(2 + a + a^2), V(a) = 1 + a$. Note that in this case the result holds for $l \leq +\infty$.

4. Conclusion

In this paper, we studied a size-structured population dynamics model where the maximum size is either finite or infinite. We determined the steady states of the model and examined their stability. We proved that the trivial steady state is always a steady state and that there are as many nontrivial steady states as the positive solutions of the equation, $R(P_\infty) = 1,$ where $R(P)$ is given by equation (6). Then we studied the stability of the trivial steady state and showed that if, $R(0) < 1,$ then the trivial steady state is locally asymptotically stable and if, $R(0) > 1,$ then the trivial steady state is unstable.

Furthermore, we studied the stability of a nontrivial steady state and we proved a theorem that provided a sufficient condition for the local asymptotic stability of a nontrivial steady state of the general model, and then we proved a corollary to that theorem for the special case when, $\beta(a, P) = \beta(P)$. We also studied two other special cases, the first was when, $V(a, P) = V(a),$ and, $\mu(a, P) = \mu(P),$ and the second was when, $V(a, P) = V(a),$ and, $\mu(a, P) = \mu(a)$. We note that the first special case linked our study of the stability of our size-structured population dynamics model to the study of the classical Gurtin-MaCcamy's age-structured population dynamics model given in Gurtin, et al. (1974), specifically, the studies for the stability given in Gurney, et al. (1980) and Weinstock, et al. (1987), in fact, the characteristic equation for this special case has the same qualitative properties as the characteristic equation of the Gurtin-MaCcamy's age-structured population dynamics model. Also similarly, the second special case linked our study to studies related to cannibalism, for example, see Iannelli (1995), Bekkal-Brikci, et al. (2007) and El-Doma (to appear). We also showed that if, $R'(P_\infty) > 0,$ then a nontrivial steady state is unstable. Finally, we illustrated our stability results by several examples.

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