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Variational Iteration Method for Solving Telegraph Equations

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Abstract

In this paper, we apply the variational iteration method (VIM) for solving telegraph equations, which arise in the propagation of electrical signals along a telegraph line. The suggested algorithm is more efficient and easier to handle as compare to the decomposition method. Numerical results show the efficiency and accuracy of the proposed VIM.

Keywords: Variational Iteration Method; Lagrange Multiplier; Telegraph Equations

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1. Introduction

The telegraph equations appear in the propagation of electrical signals along a telegraph line, digital image processing, telecommunication, signals and systems, [see Abdou and Soliman (2005), Wazwaz (2002, 2006)]. The standard form of the telegraph equation is given as

$$u_{xx}(x,t) = a\,u_{tt}(x,t) + b\,u_t(x,t) + cu(x,t),$$

where a, b, c are constants related to resistance, inductance, capacitance and conductance of the cable. The basic motivation of this paper is the application of variational iteration method (VIM) for solving telegraph equations. The VIM was developed and formulated by He for solving various physical problems, [see He (1999, 2000, 2006, 2007, 2008)]. The method has been extremely useful for diversified initial and boundary value problems and has the potential to cope with the versatility of the complex nature of physical problems, [see Abbasbandy (2007), Abdou and Soliman (2005), He (1999, 2000, 2006, 2007, 2008), Noor and Mohyud-Din (2007, 2008), Mohyud-Din et al. (2008, 2009), Xu (2007)].

A wide class of initial and boundary value problems including Riccati differential equations, unsteady flow through a porous medium, Burger's and coupled Burger's equations, nonlinear oscillators, higher-order boundary value problems of various order, seepage flow, autonomous ordinary differential systems, Fisher's equations, Evolution equations, diffusion equations, singular problems, KdVs, parabolic equations, integro-differential equations, chemistry problems, Boussinesq equations, Schrödinger equations, Helmholtz equations, Sine-Gordon equations has been tackled successfully in accordance with their physical nature by the proposed variational iteration method (VIM), [see Abbasbandy (2007), Abdou and Soliman (2005), He (1999, 2000, 2006, 2007, 2008), Noor and Mohyud-Din (2007, 2008), Mohyud-Din et al. (2008, 2009), Xu (2007)].

It is to be highlighted that the use of Lagrange multiplier in variational iteration method (VIM) reduces the successive applications of the integral operator, minimizes the computational work to a tangible level while still maintaining a very high level of accuracy and hence is a clear advantage of this technique over the decomposition method. The VIM is also independent of the small parameter assumption (which is either not there in the physical problems or difficult to locate) and hence is more convenient to apply as compare to the traditional perturbation method. It is worth mentioning that the VIM is applied without any discretization, restrictive assumption or transformation and is free from round off errors.

We apply the proposed VIM for all the nonlinear terms in the problem without discretizing either by finite difference or spline techniques at the nodes, involves laborious calculations coupled with a strong possibility of the ill-conditioned resultant equations which is a complicated problem to solve. Moreover, unlike the method of separation of variables that requires initial and boundary conditions, the VIM provides the solution by using the initial conditions only, [see Abbasbandy (2007), Abdou and Soliman (2005), He (1999, 2000, 2006, 2007, 2008), Noor and Mohyud-Din (2007, 2008), Mohyud-Din et al. (2008, 2009), Xu (2007)].

The proposed variational iteration method (VIM) can be applied to a number of physical problems related to fluid mechanics including boundary layer flow with exponential or algebraic properties, Von Karman swirling viscous flow, nonlinear progressive waves in deep water, porous medium, financial mathematics, deep shallow water waves, electrical signals along a telegraph line, digital image processing, telecommunication, signals and systems, beam deflection theory, quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, astrophysics, nonlinear optics, engineering and applied sciences, [see Mohyud-Din et al. (2009)]. Numerical results show the complete reliability of the proposed technique.

2. Variational Iteration Method (VIM)

To illustrate the basic concept of the He's VIM, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term. According to variational iteration method, [see Abbasbandy (2007), Abdou and Soliman (2005), He (1999, 2000, 2006, 2007, 2008), Noor and Mohyud-Din (2007, 2008), Mohyud-Din et al. (2008, 2009), Xu (2007)], we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + N\tilde{u}_n(s) - g(s))\, ds, \tag{2}$$

where λ is a Lagrange multiplier, [see He (1999, 2000, 2006, 2007, 2008)], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta \tilde{u}_n = 0$; (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [see He (1999, 2000, 2006, 2007, 2008)]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n \to \infty} u_n$.

3. Numerical Applications

In this section, we apply the variational iteration method (VIM) for solving telegraph equations. Numerical results are very encouraging.

Example 3.1.

Consider the following telegraph equation

$$u_{xx} = u_{tt} + u_t - u,$$

with boundary conditions

$$u(0,t)=e^{-2t}, \quad u_x(0,t)=e^{-2t},$$

and the initial conditions

$$u(x,0)=e^{x}, \quad u_t(x,0)=-2e^{x}.$$

The correction functional is given by

$$u_{n+1}(x,t)=e^{-2t}+xe^{-2t}+\int_0^t \lambda(s)\left(\frac{\partial^2 u_n}{\partial s^2}-\frac{\partial^2 \tilde{u}_n}{\partial t^2}-\frac{\partial \tilde{u}_n}{\partial t}+\tilde{u}_n\right)ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s)=(s-t)$, we obtain the following iterative formula

$$u_{n+1}(x,t)=e^{-2t}+xe^{-2t}+\int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}-\frac{\partial^2 u_n}{\partial t^2}-\frac{\partial u_n}{\partial t}+u_n\right)ds.$$

Consequently, following approximants are obtained

$$u_0(x,t)=e^{-2t}(1+x),$$
$$u_1(x,t)=\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3\right)e^{-2t},$$
$$u_2(x,t)=\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5\right)e^{-2t},$$
$$u_3(x,t)=\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7\right)e^{-2t},$$
$$u_4(x,t)=\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8+\frac{1}{9!}x^9\right)e^{-2t},$$
$$\vdots$$

The series solution is given by

$$u(x,t)=\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8+\frac{1}{9!}x^9+\cdots\right)e^{-2t},$$

and the closed form solution is given as

$$u(x,t)=e^{x-2t},$$

which is the exact solution.

Example 3.2.

Consider the following telegraph equation

$$u_{xx} = u_{tt} + 4u_t + 4u,$$

with boundary conditions

$$u(0,t) = 1 + e^{-2t}, \quad u_x(0,t) = 2,$$

and the initial conditions

$$u(x,0) = 1 + e^{2x}, \quad u_t(x,0) = -2.$$

The correction functional is given by

$$u_{n+1}(x,t) = 1 + e^{-2t} + 2x + \int_0^t \lambda(s)\left(\frac{\partial^2 u_n}{\partial s^2} - \frac{\partial^2 \tilde{u}_n}{\partial t^2} - 4\frac{\partial \tilde{u}_n}{\partial t} - 4\tilde{u}_n\right)ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = (s-t)$, we obtain the following iterative formula

$$u_{n+1}(x,t) = 1 + e^{-2t} + 2x + \int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2} - \frac{\partial^2 u_n}{\partial t^2} - 4\frac{\partial \tilde{u}_n}{\partial t} - 4u_n\right)ds.$$

Consequently, following approximants are obtained

$$u_0(x,t) = e^{-2t} + (1 + 2x),$$
$$u_1(x,t) = e^{-2t} + \left(1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3\right),$$
$$u_2(x,t) = e^{-2t} + \left(1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \frac{1}{5!}(2x)^5\right),$$
$$u_3(x,t) = e^{-2t} + \left(1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \frac{1}{5!}(2x)^5 + \frac{1}{6!}(2x)^6 + \frac{1}{7!}(2x)^7\right),$$
$$\vdots$$

The series solution is given by

$$u(x,t) = e^{-2t} + \left(1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \frac{1}{5!}(2x)^5 + \frac{1}{6!}(2x)^6 + \frac{1}{7!}(2x)^7 + \cdots\right),$$

and the closed form solution is given as

$$u(x,t)=e^{-2t}+e^{2x},$$

which is the exact solution.

Example 3.3.

Consider the following telegraph equation

$$u_{xx}=u_{tt}+u_t+u,$$

with boundary conditions

$$u(0,t)=e^{-t}, \quad u_x(0,t)=e^{-t},$$

and the initial conditions

$$u(x,0)=e^{x}, \quad u_t(x,0)=-e^{x}.$$

The correction functional is given by

$$u_{n+1}(x,t)=e^{-t}+xe^{-t}+\int_0^t \lambda(s)\left(\frac{\partial^2 u_n}{\partial s^2}-\frac{\partial^2 \tilde{u}_n}{\partial t^2}-\frac{\partial \tilde{u}_n}{\partial t}-\tilde{u}_n\right)ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s)=(s-t)$, we obtain the following iterative formula

$$u_{n+1}(x,t)=e^{-t}+xe^{-t}+\int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}-\frac{\partial^2 u_n}{\partial t^2}-\frac{\partial u_n}{\partial t}-u_n\right)ds.$$

Consequently, following approximants are obtained

$$u_0(x,t)=e^{-t}+xe^{-t},$$

$\vdots$

Proceeding as before, the exact solution is given as

$$u(x,t)=e^{x-t}.$$

4. Conclusion

In this paper, we applied variational iteration method (VIM) for solving telegraph equations. The method is applied in a direct way without using linearization, transformation, perturbation, discretization or restrictive assumptions. The fact that the proposed VIM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this algorithm over the decomposition method.

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