Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 9, Issue 2 (December 2014), pp. 573-591

A Semiparametric Estimation for Regression Functions in the Partially Linear Autoregressive Time Series Model

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Received: May 19, 2014; Accepted: October 2, 2014

Abstract

In this paper, a semiparametric method is proposed for estimating regression function in the partially linear autoregressive time series model. Here, we consider a combination of parametric forms and nonlinear functions, in which the errors are independent. Semiparametric and nonparametric curve estimation provides a useful tool for exploring and understanding the structure of a nonlinear time series data set to make for a more efficient study in the partially linear autoregressive model. The unknown parameters are estimated using the conditional nonlinear least squares method, and the nonparametric adjustment is also estimated by defining and minimizing the local L2-fitting criterion with respect to the nonparametric adjustment and, with smooth-kernel method, these estimates are corrected. Then, the autoregression function estimators, which can be calculated with the sample and simulation data, are obtained. In this case, some strong and weak consistency and simulated results for the semiparametric estimation in this model are presented. The root mean square error and the average square error criterions are also applied to verify the efficiency of the suggested model.

Keywords: Semiparametric estimation; Nonparametric adjustment; Conditional nonlinear least squares method; Partially linear autoregressive model; Smooth kernel approach; Autocorrelation function; Simulation

MSC 2010 No.: C14, C15, C52, C53

1. Introduction

The nonlinear autoregressive models are the most popular models for the nonlinear time series analysis. Over the past two decades, there has been a growing interest in the time series literature for nonlinear models [Tong (1990)]. Several authors have constructed nonlinear time series models shown to be useful in some applications. Haggen and Ozaki (1981) propose the exponential autoregressive model to apply in the modeling of sound vibration. Cai and Masry (2000) propose an additive nonlinear ARX time series model to consider the estimation and identification of components, both endogenous and exogenous. Chen and Tsay (1993) propose a class of nonlinear additive autoregressive models with exogenous variable for nonlinear time series analysis. Farnoosh and Mortazavi (2011) propose the first-order nonlinear autoregressive model with dependent errors to estimate the yearly amount of deposits in an Iranian Bank.

In time series analysis with the focus on autoregressive models, one faces, in particular, three problems: model identification, i.e., lag selection, model estimation and prediction. Many methods have been proposed to cope with these problems. For nonlinear models, there are results showing the strong (or weak) consistency and asymptotic normality of the estimators. Semiparametric and nonparametric curve estimation provides a useful tool for exploring and understanding the structure of a nonlinear time data set, especially when classic time series models are inappropriate. Auestad and Tjostheim (1990) applied a multivariate kernel smoothing method to estimate the conditional mean and conditional variance of a nonlinear autoregression.

Our interest is the estimation of parameters in a nonlinear time series model which is usually performed by conditional least squares. Achievement of nice asymptotic properties of these estimators is not automatic because of the diverse possibilities in the choice of the model. Results for conditional least squares estimators are proved in Klimko and Nelson (1978) in a general set up. In this paper, we intend to propose a combination of parametric forms and nonlinear functions to make a more efficient study in the following partially linear autoregressive model

$$Y_t = \beta Y_{t-1} + f(Y_{t-2}) + \varepsilon_t, \quad t = 2, \dots, n, \ |\beta| < 1. \tag{1.1}$$

At first, we suppose that *f*(.) has a parametric framework, namely parametric model as

$$f(x) \epsilon \{g(x,\gamma); \gamma \epsilon \Theta\}, \tag{1.2}$$

where $g(x,\gamma)$ is a known smooth function of x, and $\Theta \epsilon R^m$ is a parametric space. Both γ and β are unknown parameters.

We suggest a semiparametric form $g(x,\gamma)\xi(x)$ for the unknown autoregression function *f*(.), where $\xi(x)$ is a nonparametric adjustment. This model is a simple generalization of the first-order nonlinear autoregressive model of Jones (1978) and Zhuoxi et al. (2009), and is a time series counterpart of the generalized additive model of Hastie and Tibshirani (1990) in regression analysis that was introduced by Gao (1998), Gao and Yee (2000).

The goal of this paper is to extend the work of Zhuoxi et al. (2009) in the semiparametric estimation for the nonlinear autoregressive model. There are a number of practical motivations behind it; these include the search for population biology models and the Mackey-Glass system [see Glass and Mackey (1988))]. In addition, the recent development in partially linear (semiparametric) regression (see Heckman (1986); Hardle et al. (1997)) has established a solid foundation for studying model (1.1).

We want to estimate *f*(*x*) and $\xi(x)$ under the following form

$$f(x) = \xi(x)g(x,\gamma). \tag{1.3}$$

Hence, we use a combination of parametric method and nonparametric adjustment. The parameters and nonparametric adjustment are estimated, using a conditional nonlinear least squares method and then, with the smooth-kernel method, these estimates are corrected, so it will be considered as

$$\hat{f}(x) = g(x,\hat{\gamma})\hat{\xi}(x). \tag{1.4}$$

The contents of this paper are organized as follows. In section 2, a conditional nonlinear least squares method is presented to estimating parameters γ and β. Also, in this section, the semiparametric estimator is introduced by a natural consideration of the local L2-fitting criterion for the partially linear autoregressive time series model. The strong and weak consistencies of the semiparametric estimations are investigated in section 3. The performance of this method is assessed by simulation in section 4. Finally, section 5 illustrates an application of this model to predict annual ring width (ARW), of Kelardasht site in the north of Iran, from 1974 to 2008.

2. Semiparametric estimation in the partially linear autoregressive time series model

We consider the following model

$$Y_t = \beta Y_{t-1} + f(Y_{t-2}) + \varepsilon_t, \quad t = 2, \dots, n, \ |\beta| < 1, \tag{2.1}$$

where $\{\varepsilon_t\}$ is a sequence of independent and identically-distributed (i.i.d) random variables with mean zero and variance σ^2. Also, ε_t and Y_t are independent for each t and β is unknown parameter. We want to estimate the regression function *f*(*x*) that can be formed as $g(x,\gamma)$, where $g(x,\gamma)$ is a function of x with $\gamma \epsilon \Theta$ as an unknown parameter of the model. For the model (2.1), γ and β should be well estimated with conditional nonlinear least squares errors method as follows

$$Q_n(\gamma,\beta) = \sum_{j=2}^{n} \left\{ \left(Y_j - \beta Y_{j-1} - g\left(Y_{j-2},\gamma\right) \right)^2 \right\}, \tag{2.2}$$

and

$$\left(\hat{\gamma},\hat{\beta}\right) = argminQ_n(\gamma,\beta), \gamma \epsilon \Theta, |\beta| < 1. \tag{2.3}$$

In fact, $\hat{\gamma}$ and $\hat{\beta}$ are the common conditional least squares estimators based on $Y_0, Y_1, \dots, Y_{n-1}$, for n successive observations from the model.

Now, we estimate $\xi(x)$ in $f(x) = g(x,\hat{\gamma})\xi(x)$ by using a similar idea of Hjort and Jones (1996), Zhuoxi et al. (2009) and Farnoosh and Mortazavi (2011).

We define the local L2-fitting criterion as the following form

$$q_n(x,\xi) = \sum_{j=2}^{n} k\left(\frac{Y_{j-2}-x}{h_n}\right)\{f(Y_{j-2}) - g(Y_{j-2},\hat{\gamma})\xi\}^2, \tag{2.4}$$

where *f*(.) is an unknown autoregression function with sample size *n*, *k* is a kernel and h_n is band- width. So we obtain the estimator $\hat{\xi}(x)$ of $\xi(x)$ by minimizing the above criterion with respect to $\xi(x)$. Therefore, we get a nonparametric estimator with smooth kernel method of $\xi(x)$ as

$$\hat{\xi}(x) = \frac{\sum_{j=2}^{n}\left[k\left(\frac{Y_{j-2}-x}{h_n}\right)g(Y_{j-2},\hat{\gamma})f(Y_{j-2})\right]}{\sum_{j=2}^{n}\left[k\left(\frac{Y_{j-2}-x}{h_n}\right)g^2(Y_{j-2},\hat{\gamma})\right]}, \tag{2.5}$$

then the estimator of *f*(*x*) could be

$$\hat{f}(x) = g(x,\hat{\gamma})\hat{\xi}(x). \tag{2.6}$$

Unfortunately, the formula $\hat{\xi}(x)$ contains the unknown function *f*(*x*), therefore by using

$$\varepsilon_t = Y_t - \beta Y_{t-1} - f(Y_{t-2}),$$

and with regard to the fact the errors of model are small values, we have

$$\sum_{j=2}^{n} k\left(\frac{Y_{j-2}-x}{h_n}\right)g(Y_{j-2},\hat{\gamma})f(Y_{j-2}) \approx \sum_{j=2}^{n} k\left(\frac{Y_{j-2}-x}{h_n}\right)g(Y_{j-2},\hat{\gamma})(Y_j - \hat{\beta}Y_{j-1}).$$

Therefore, one can obtain

$$\tilde{\xi}(x) = \frac{\sum_{j=2}^{n}\left[k\left(\frac{Y_{j-2}-x}{h_n}\right)g(Y_{j-2},\hat{\gamma})(Y_j - \hat{\beta}Y_{j-1})\right]}{\sum_{j=2}^{n}\left[k\left(\frac{Y_{j-2}-x}{h_n}\right)g^2(Y_{j-2},\hat{\gamma})\right]}. \tag{2.7}$$

Finally, the autoregression function estimators which can be calculated with the sample and simulation data, are

$$\tilde{f}(x) = g(x,\hat{\gamma})\tilde{\xi}(x). \tag{2.8}$$

3. The assumptions and properties of asymptotic behaviors

In this section, some properties and the asymptotic behaviors of the estimator are investigated. The assumptions A1-A10 are considered as follows

(A1) *f*(.) is Lipschitz continuous and all moments of ε_t are finite and the density function of errors is Lipschitz and bounded total variation, also, the sequence $\{Y_t\}$ is stationary ergodic sequence of integrable random variable. See Hayashi (2000) and Taniguchi and Kakizawa (2000).

(A2) $\frac{\partial g}{\partial \gamma_i}, \frac{\partial^2 g}{\partial \gamma_i \, \partial \gamma_j}, \frac{\partial^3 g}{\partial \gamma_i \, \partial \gamma_j \, \partial \gamma_k}$ exist and are continuous, for all $\gamma \epsilon \Theta$,

where *i*,*j*,*k* =1,..., *m*.

(A3) $E(Y_t | Y_{t-1}, Y_{t-2}, \dots, Y_0) = E(Y_t | Y_{t-1}, Y_{t-2}, \dots, Y_{t-k}),\ \text{a.s}, t \geq k \geq 1,$

where *k* is a constant.

(A4) We define

$$U_t(\gamma_0, \beta_0) = Y_t - E(Y_t | Y_{t-1}, Y_{t-2}) = Y_t - \beta_0 Y_{t-1} - f(Y_{t-2}),$$

and the following matrices

$$B = E\left(\frac{\partial g(\gamma_0, Y_{t-2})}{\partial \gamma_i} . \frac{\partial g(\gamma_0, Y_{t-2})}{\partial \gamma_j}\right), \quad i, j = 1, \dots, m,$$

$$D = E\left(U_t^2(\gamma_0, \beta_0)(\frac{\partial g(\gamma_0, Y_{t-2})}{\partial \gamma_i} . \frac{\partial g(\gamma_0, Y_{t-2})}{\partial \gamma_j})\right), \quad i, j = 1, \dots, m.$$

We will assume throughout that *B* and *D* are positive definite and finite.

(A5) The sequence $(Y_t)_{t \epsilon Z}$ is α-mixing (Yu (1973)). The sufficient conditions are introduced. See Rosenblatt (1971), Bradley (2007), Masry and Tjostheim (1995) and Robinson (1983).

(A6) Y_0 and Y_1 have the same distribution π(.), such that the density φ(.) of π(.) exists, bounded, continuous and strictly positive in a neighborhood of the point *x*.

(A7) $f(x)$ and $g(x, \gamma)$ are bounded and continuous with respect to *x*, away from 0 in a neighborhood of the point *x*. Set $g(x, \gamma_0) = g_{\gamma_0}(x)$.

(A8) $g(x, \gamma)$ has continuous derivative with respect to γ and the derivative at the point γ_0 is uniformly bounded with respect to *x*.

(A9) The kernel $k: R^1 \to R^+$ is a compactly symmetric bounded function, such that $k(.) > 0$ for x in a set of positive Lebesgue measures.

(A10) $h_n = \lambda n^{-1/5}$, where $\lambda > 0$.

If we accept assumptions (A1)-(A10), we have the following lemma and theorems:

Lemma 3.1.

Under the conditions (A1)-(A10), We have the following results when $n \to \infty$.

$$(a)\ n^{-4/5} \sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) f(Y_{j-2}) g(Y_{j-2}, \hat{\gamma}_n) \xrightarrow{p} \lambda \varphi(x) f(x) g_{\gamma_0}(x),$$

$$(b)\ \ n^{-4/5} \sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) g^2(Y_{j-2}, \hat{\gamma}_n) \xrightarrow{p} \lambda \varphi(x) g_{\lambda_0}^2(x),$$

where $g_{\gamma_0}(x)$ is defined as in (A7) and $\varphi(x)$ is the density of Y_0 or Y_1, which is bounded, continuous and strictly positive in a neighborhood of the point x.

***Proof* (a):**

It can be calculated as

$$n^{-\frac{4}{5}} \sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) f(Y_{j-2}) g(Y_{j-2}, \hat{\gamma}_n)$$
$$= n^{-\frac{4}{5}} \left[\sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) \left(g(Y_{j-2}, \hat{\gamma}_n) - g(Y_{j-2}, \gamma_0)\right) f(Y_{j-2})\right]$$
$$+ n^{-\frac{4}{5}} \sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) g(Y_{j-2}, \gamma_0) f(Y_{j-2}) = A_n + B_n.$$

Using (A1)-(A4), and strong consistency of the conditional least squares method Klimko and Nelson(1978), and (A7)-(A9), we have

$$max_{1 \le j \le n} \left|\left(g(Y_{j-2}, \hat{\gamma}_n) - g(Y_{j-2}, \gamma_0)\right) f(Y_{j-2})\right| = O\left((log_2^n / n)^{1/2}\right), \text{as } n \to \infty.$$

Since, $f(x)$ and $k(x)$ are bounded and continuous with respect to x, one can claim: there exists $C_0 > 0$ such that

$$n^{-\frac{4}{5}} \left|\sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) \left(g(Y_{j-2}, \hat{\gamma}_n) - g(Y_{j-2}, \gamma_0)\right) f(Y_{j-2})\right.$$

$$\leq n^{-\frac{4}{5}} \sum_{j=2}^{n} C_0 \left| g\left(Y_{j-2}, \hat{\gamma}_n\right) - g\left(Y_{j-2}, \gamma_0\right) \right| \Bigg|$$

$$\leq n^{-\frac{4}{5}} n\, C_0\, O\left((log_2^n/n)^{1/2}\right) = O\left((log_2^n)/n^{\frac{3}{10}}\right) \text{a.s.}$$

Thus, $A_n \to 0$ a.s. as $n \to \infty$.

Due to the sequence $\{Y_t\}_{t\epsilon N}$ is α-mixing, then

$$n^{-\frac{4}{5}} \sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) g\left(Y_{j-2}, \gamma_0\right) f\left(Y_{j-2}\right) - n^{\frac{1}{5}} E\left\{k\left(\frac{Y_0 - x}{h_n}\right) g(Y_0, \gamma_0) f(Y_0)\right\} \xrightarrow{p} 0,$$

as $n \to \infty$.

According to (A6)-(A7), we can show (Put $u=\frac{y-x}{h_n}$ also, $\int k(u)\, du = \{1 + o(1)\}$)

$$n^{\frac{1}{5}} E\left\{k\left(\frac{Y_0 - x}{h_n}\right) g(Y_0, \gamma_0) f(Y_0)\right\}$$

$$= \frac{\lambda}{h_n} \int k\left(\frac{Y - x}{h_n}\right) g(y, \gamma_0) f(y) \varphi(y)\, dy$$

$$= \lambda \int k(u) g(x + uh_n, \gamma_0) f(x + uh_n) \varphi(x + uh_n)\, du$$

$$= \lambda f(x) \varphi(x) g_{\gamma_0}(x) \int k(u)\, du\, .$$

Therefore,

$$B_n = n^{-4/5} \sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) f\left(Y_{j-2}\right) g\left(Y_{j-2}, \hat{\gamma}_n\right) \xrightarrow{p} \lambda \varphi(x) f(x) g_{\gamma_0}(x),$$

as $n \to \infty$.

***Proof* (*b*):**

Since, $g(x, \gamma)$ is bounded and continuous with respect to *x*, we get

$$n^{-4/5} \sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) g^2\left(Y_{j-2}, \hat{\gamma}_n\right)$$

$$= n^{-\frac{4}{5}} \left[\sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) \left(g\left(Y_{j-2}, \hat{\gamma}_n\right) - g\left(Y_{j-2}, \gamma_0\right)\right) g\left(Y_{j-2}, \hat{\gamma}_n\right)\right]$$

$$+ n^{-\frac{4}{5}} \sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) g\left(Y_{j-2}, \gamma_0\right) g\left(Y_{j-2}, \hat{\gamma}_n\right) = C_n + D_n.$$

Using (A8)-(A10), there exists $k_0^* > 0$, such that

$$|C_n| \leq n^{-\frac{4}{5}} \sum_{j=2}^{n} k_0^* \left| g\left(Y_{j-2}, \hat{\gamma}_n\right) - g\left(Y_{j-2}, \gamma_0\right) \right|$$

$$\leq n^{-\frac{4}{5}} n\, k_0^*\, O\left((log_2^n/n)^{1/2}\right)$$

$$= O\left((log_2^n)/n^{\frac{3}{10}}\right), \text{a.s.}$$

Thus, $C_n \to 0$ a.s. as $n \to \infty$.

According to (A5), the sequence $\{Y_j\}_{j\epsilon N}$ is α-mixing, therefore

$$D_n = n^{-\frac{4}{5}} \sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) g\left(Y_{j-2}, \gamma_0\right) g\left(Y_{j-2}, \hat{\gamma}_n\right) \to n^{\frac{1}{5}} E\left\{k\left(\frac{Y_0 - x}{h_n}\right) g^2(Y_0, \gamma_0)\right\} \xrightarrow{p} 0.$$

According to (A6)-(A7) ,we can show (Put $u = \frac{y-x}{h_n}$, also, $\int k(u)\, du = \{1 + o(1)\}$)

$$n^{\frac{1}{5}} E\left\{k\left(\frac{Y_0 - x}{h_n}\right) g^2(Y_0, \gamma_0)\right\}$$

$$= \frac{\lambda}{h_n} \int k\left(\frac{y - x}{h_n}\right) g^2(y, \gamma_0) f(y) \varphi(y)\, dy$$

$$= \lambda \int k(u) g^2(x + uh_n, \gamma_0) \varphi(x + uh_n)\, du$$

Finally,

$$D_n = n^{-4/5} \sum_{j=2}^{n} k\left(\frac{Y_{j-2} - x}{h_n}\right) g\left(Y_{j-2}, \gamma_0\right) g\left(Y_{j-2}, \hat{\gamma}_n\right) \xrightarrow{p} \lambda \varphi(x) g_{\lambda_0}^2(x),$$

as $n \to \infty$.

Therefore, the proof is complete.

Theorem 3.2.

If we accept the assumptions (A1)-(A10) and $\hat{f}(x)$ be the introduced estimator in (2.6), then

$\hat{f}(x) \xrightarrow{p} f(x)$, as $n \to \infty$.

Proof:

Substituting Lemma 3.1 in (2.6) and using the strong consistency of $\hat{\gamma}_n$ and $\hat{\beta}_n$, we can prove Theorem 3.2.

Theorem 3.3.

If we accept the assumptions (A1)-(A10) and $\tilde{f}(x)$ be the defined autoregression function estimator in (2.8), then $\tilde{f}(x) - \hat{f}(x) \xrightarrow{p} 0$, as $n \to \infty$.

Proof:

One may obtain the equality

$$\tilde{f}(x) - \hat{f}(x) = g(x, \hat{\gamma}_n) \frac{\sum_{j=2}^{n} \left[k\left(\frac{Y_{j-2} - x}{h_n}\right) \varepsilon_j g\left(Y_{j-2}, \hat{\gamma}_n\right) \right]}{\sum_{j=2}^{n} \left[k\left(\frac{Y_{j-2} - x}{h_n}\right) \left(Y_{j-2}, \hat{\gamma}_n\right) \right]}.$$

On the other hand, we have

$$\begin{aligned} n^{-\frac{4}{5}} \sum_{j=2}^{n} & \left[k\left(\frac{Y_{j-2} - x}{h_n}\right) \varepsilon_j g\left(Y_{j-2}, \hat{\gamma}_n\right) \right] \\ &= n^{-\frac{4}{5}} \sum_{j=2}^{n} \left[k\left(\frac{Y_{j-2} - x}{h_n}\right) \varepsilon_j \left(g\left(Y_{j-2}, \hat{\gamma}_n\right) - g\left(Y_{j-2}, \gamma_0\right) \right) \right] \\ &\quad + n^{-\frac{4}{5}} \sum_{j=2}^{n} \left[k\left(\frac{Y_{j-2} - x}{h_n}\right) \varepsilon_j g\left(Y_{j-2}, \gamma_0\right) \right] = C_n + D_n. \end{aligned}$$

To complete the proof, it is known that

$$max_{1 \le j \le n} \left| \varepsilon_j \right| = O((logn)^{1/2})$$

a.s., as $n \to \infty$, it is enough to show $C_n \xrightarrow{p} 0$ and $D_n \xrightarrow{p} 0$, as $n \to \infty$.

It follows that

$$\begin{aligned} & \left| n^{-\frac{4}{5}} \sum_{j=2}^{n} \left[k\left(\frac{Y_{j-2}-x}{h_n}\right) \varepsilon_j \left(g\left(Y_{j-2}, \hat{\gamma}_n\right) - g\left(Y_{j-2}, \gamma_0\right) \right) \right] \right| \\ &\le n^{-\frac{4}{5}} \sum_{j=2}^{n} \left[k_0 \left|\varepsilon_j\right| \left| g\left(Y_{j-2}, \hat{\gamma}_n\right) - g\left(Y_{j-2}, \gamma_0\right) \right| \right] \\ &\le n^{-\frac{4}{5}} n \, O((logn)^{\frac{1}{2}}) O(\left(log_2^n / n \right)^{\frac{1}{2}}) = O((logn \; log_2^n)^{1/2}) / n^{3/10}, \end{aligned}$$

which implies $C_n \to 0$ a.s., as $n \to \infty$.

Since

$$E\left[n^{-\frac{4}{5}}\sum_{j=2}^{n}\left[k\left(\frac{Y_{j-2}-x}{h_n}\right)\varepsilon_j\, g\left(Y_{j-2},\gamma_0\right)\right]\right]=0,$$

and

$$\begin{aligned}
&E\left[n^{-\frac{4}{5}}\sum_{j=2}^{n}\left[k\left(\frac{Y_{j-2}-x}{h_n}\right)\varepsilon_j\, g\left(Y_{j-2},\gamma_0\right)\right]\right]^2\\
&\quad= n^{-\frac{8}{5}}E\left\{\sum_{j=2}^{n}\left[k\left(\frac{Y_{j-2}-x}{h_n}\right)\varepsilon_j\, g\left(Y_{j-2},\gamma_0\right)\right]\right\}^2\\
&\quad= n^{-\frac{8}{5}}E\left\{\sum_{j=2}^{n}\left[k^2\left(\frac{Y_{j-2}-x}{h_n}\right)\varepsilon_j^2 g^2\left(Y_{j-2},\gamma_0\right)\right]\right\}\\
&\qquad+2\, n^{-\frac{8}{5}}\sum_{2\le i\le j\le n}^{n} E\left[k\left(\frac{Y_{j-2}-x}{h_n}\right)\varepsilon_j\, g\left(Y_{j-2},\gamma_0\right)k\left(\frac{Y_{i-2}-x}{h_n}\right)\varepsilon_i\, g\left(Y_{i-2},\gamma_0\right)\right]\\
&\le n^{-\frac{8}{5}}\, n\, k_1\sigma^2 = O\left(\frac{1}{n^{\frac{3}{5}}}\right),
\end{aligned}$$

where $k_1 > 0$ is a constant, we get $D_n \xrightarrow{p} 0$, as $n \to \infty$. By the strong consistency of $\hat{\gamma}_n$ and $\hat{\beta}_n$ and lemma 3.1, we complete the proof.

4. Simulation Study

We investigate the appropriateness of semiparametric method by estimating parameters and regression function in the following model

$$Y_t = 0.45Y_{t-1} + f(Y_{t-2}) + \varepsilon_t, \quad t = 2, \dots, n,$$

with $\varepsilon_t \sim N(0, (0.125)^2)$ where $\{\varepsilon_t\}$ is a sequence of independent identically-distributed (i.i.d) random variables. We generate the data with sample sizes *n*=400,600,800 using the following nonlinear functions

$$f_1(x) = 5\, exp(-2x^2),$$

and assume

$$g_1(x,\gamma) = \gamma\, exp(-x^2),$$

$$f_2(x) = 2\,exp(-3x) + 0.1co\,s(x)\,,$$

and assume

$$g_2(x, \gamma) = \gamma_1\,exp(\gamma_2 x)\,.$$

Finally, we compute the average square error (ASE) for the efficiency of the proposed estimation method

$$ASE = \frac{1}{n}\sum_{i=1}^{n}\left\{\tilde{f}(x_i) - f(x_i)\right\}^2.$$

The square root of the ASE is denoted by RMSE (Root Mean Square Error).

Tables 1 and 2 show the descriptive statistics indices for simulation data and errors in the above mentioned models 1 and 2, respectively. The Kolmogorov-Smirnov statistic is obtained to test the normality of the errors of model. The test statistic and p-value confirm the normality of the residuals. Also, the RMSE estimation for the above mentioned models 1 and 2 are provided in Tables 3 and 4.

Figures 1 and 2 show the curves of *f*(*x*) and its semiparametric estimator under the two types of the above mentioned models with selected bandwidth and different sample sizes. The red and blue lines are the regression function *f*(*x*) and the semiparametric estimator respectively. Also, Figures 3 and 4 show the autocorrelation function (ACF) errors of the model with *f*(*x*) and its semiparametric estimator under the two types of above mentioned models with selected bandwidth and different sample sizes.

These two figures, show that the ACF errors of the model are almost uncorrelated. The simulation results show that the semiparametric estiamator of the autoregression function performs well.

Table 1. Descriptive statisticsfor the simulation data and errors of the model $f(x) = 5e^{-x^2}$

var	N	Min	Max	Mean	std	K-Smirnov	P-value
data	400	0.69	4.05	1.3178	0.88613	6.021	0.0000101
data	600	0.69	4.05	1.3104	0.87522	7.37	0.0000071
data	800	0.69	4.05	1.2392	0.77338	8.47	0.000048
error	400	0.4016	0.49	0.06	0.1511	1.128	0.157
error	600	-0.44	0.75	0.0592	0.1754	1.353	0.0865
error	800	-0.73	0.84	0.06	0.1911	1.421	0.0721

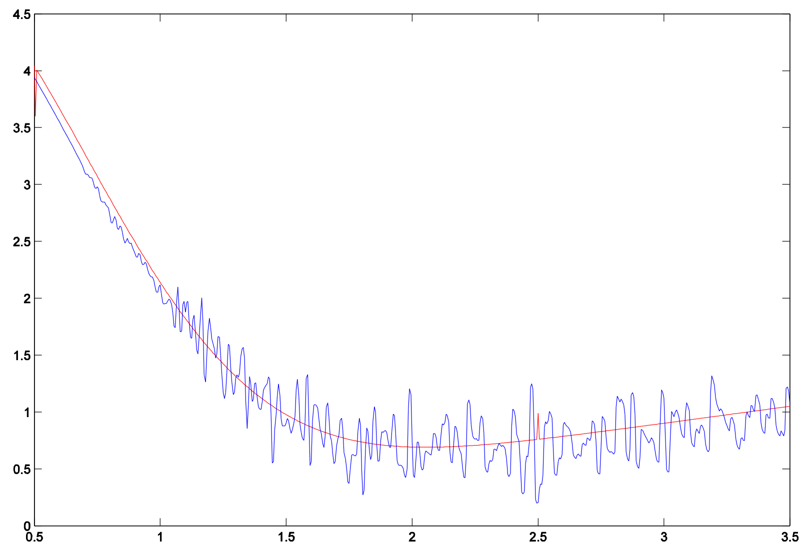

Figure 1. $f(x) = 5e^{-x^2}$, n=600, RMSE=0.0725.

Table 2. Descriptive statisticsfor the simulation data and errors of the model $f(x) = 5e^{-3x} + 0.1cos(x)$

var	N	Min	Max	Mean	std	K-Smirnov	P-value
data	400	0.11	1.05	0.4093	0.2525	3.061	0.000146
data	600	0.11	1.05	0.4089	0.2523	3.748	0.000105
data	800	0.11	1.05	0.4087	0.2519	4.32	0.000091
error	400	-0.13	0.12	-0.004	0.04715	1.1755	0.126
error	600	-0.15	0.152	-0.002	0.03412	0.888	0.41
error	800	-0.121	0.17	-0.003	0.04431	1.238	0.092

Table 3. RMSE for estimating the model $f(x) = 5e^{-x^2}$

n	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\sigma}$	RMSE
400	4.769153	0.523312	0.09	0.0806
600	5.2131	0.495932	0.112	0.0725
800	5.1771	0.472340	0.135	0.0643

Table 4. RMSE for estimating the model $f(x) = 5e^{-3x} + 0.1cos(x)$

n	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\beta}$	$\hat{\sigma}$	RMSE
400	7.86	4.2137	0.3754	0.13219	0.0265
600	7.85	4.1943	0.3901	0.10538	0.0249
800	6.6987	4.1131	0.1077	0.1325	0.0142

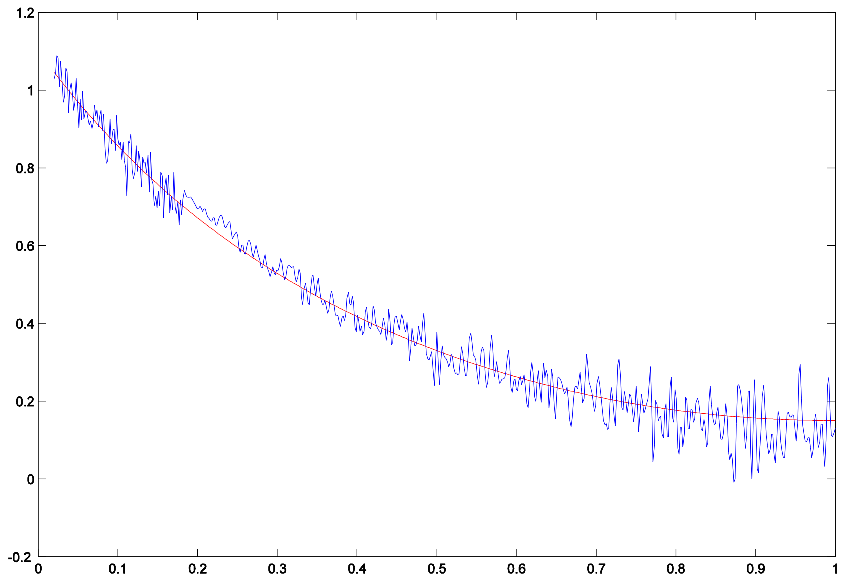

Figure 2. $f(x) = 5e^{-3x} + 0.1cos(x)$, n=600, RMSE=0.0249

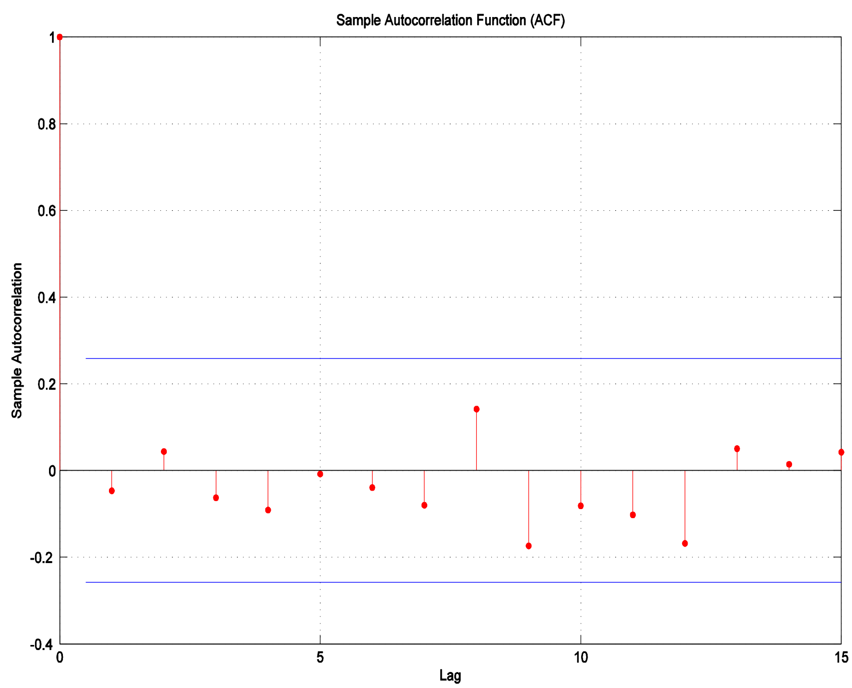

Figure 3. ACF of errors of the model $f(x) = 5e^{-x^2}$, n=600

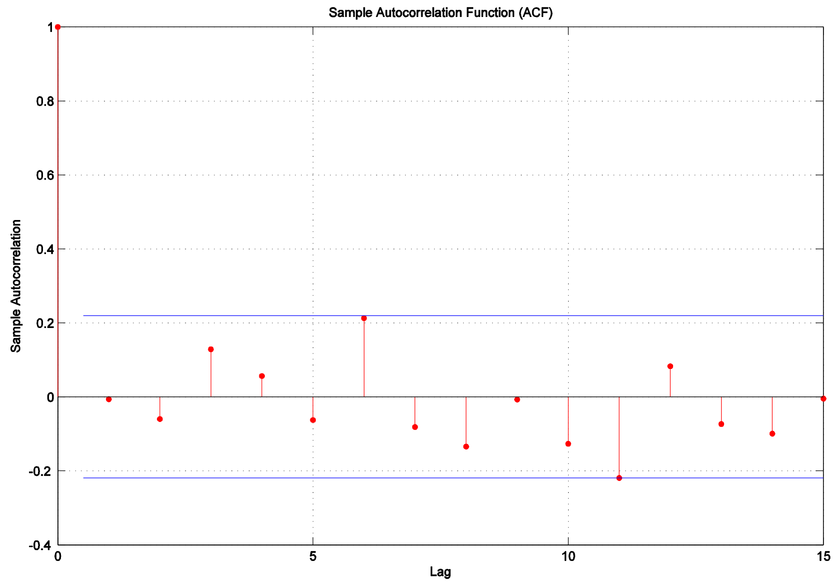

Figure 4. ACF of errors of the model $f(x) = 5e^{-3x} + 0.1\,cos(x)$, n=600

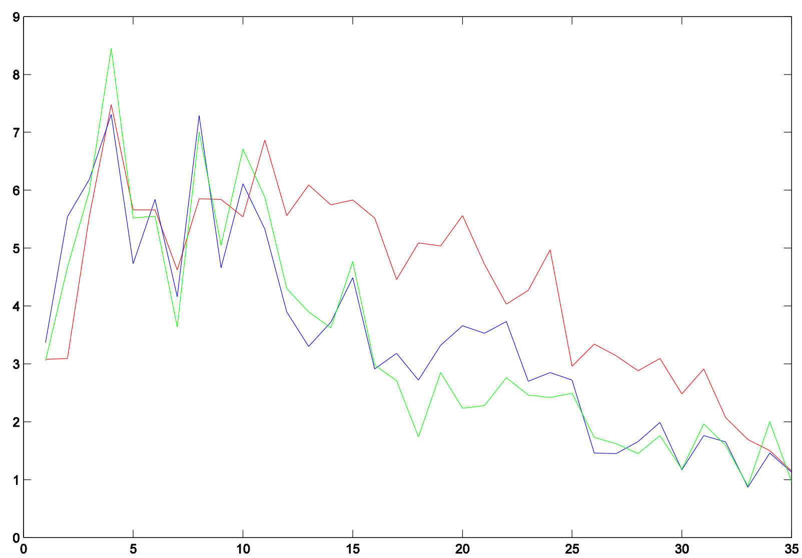

Figure 5. Exact values of ARW for three species (Pinus Eldarica) of Kelardasht site (The north of Iran)

5. Empirical Application

In this research, three normal Pinus eldarica trees were randomly selected from a plantation at Garagpas-Kelardasht site, which is located in the western part of the Mazandaran province in the north of Iran. These trees have grown for over 35 years in this site. The Pinus eldarica Medw is mixed with some Pinus sylvestris, Pinus nigra and Picea abies at the Garagpas-Kelardasht site. The Pinus eldarica trees were cut for this study in January 2009. To illustrate the suitability of our methodology to the ARW data, we use an additive functional autoregressive model to forecast the ARW of kelardasht site in the north of Iran, from 1974 to 2008. The following functional-autoregressive model is proposed as empirical model

$$Y_t = \mu + \beta Y_{t-1} + f(Y_{t-2}) + \varepsilon_t, \quad t = 2, \dots, n,$$

where $\{\varepsilon_t\}$ is a sequence of i.i.d random variables with mean zero and variance σ^2. Also, ε_t and Y_{t-1} are independent for each t and μ is constant. Using the presented semiparametric method, we estimate the regression function.

Table 5, respectively, shows the observed-estimated value for the ARW data of kelardasht site in the north of Iran, from 1974 to 2008. The descriptive statistics of the ARW of pine wood are shown in Table 6. There are significant differences between the growth and ring width (radius) of a tree in a year. The ARW values were increasing by increasing age of tree (radial axis). The mean of the ARW of three trees was 3.74 mm.

The RMSE and ASE values of the regression function for the functional autoregressive models are shown in Table 7. Figure 5 shows the curves of the observations (the ARW data in three trees of Pinus Elderica) from 1974 to 2008. Figure 6 shows the curves of the observations (the mean of ARW data) and its semiparametric estimator with selected bandwidth.

The red and blue lines are the curves of the observations and the semiparametric predictor, respectively. Figure 7 shows the ACF errors of the proposed empirical model. The errors of the model are almost uncorrelated. We see that the presented semiparametric method for a functional autoregressive model is more efficient.

Table 5. The observed - forcasted mean -values for the ARW data of kelardasht site in the north of Iran, from 1974 to 2008

Time	Exact-value	Forecasted-value	Time	Exact-value	Forecasted-value
1974	3.17	3.21	1992	3.74	4.21
1975	4.43	4.39	1993	3.82	3.13
1976	5.91	6.30	1994	3.51	4.07
1977	7.75	7.73	1995	3.51	3.62
1978	5.30	3.52	1996	3.14	3.34
1979	5.68	3.90	1997	3.41	3.74
1980	4.14	4.91	1998	2.72	2.47
1981	6.71	5.39	1999	2.18	2.71
1982	5.18	5.83	2000	2.07	2.97
1983	6.12	8.34	2001	2.00	2.57
1984	6.03	6.66	2002	2.28	2.96
1985	4.59	4.87	2003	1.61	2.33
1986	4.43	4.47	2004	2.21	2.53
1987	4.36	6.33	2005	1.77	2.02
1988	5.03	4.47	2006	1.15	1.56
1989	3.80	6.64	2007	1.65	1.67
1990	3.45	3.51	2008	1.08	1.12
1991	3.18	3.02	−	−	−

Table 6. The centeral tendency and dispersion of mean -values for the ARW data of kelardasht site in the north of Iran, from 1974 to 2008

n	mean	std	min	max	sum
35	3.7463	1.66	1.08	7.75	131.12

Table 7. RMSE for estimating regression function for the ARW data of kelardasht site in the north of Iran, from 1974 to 2008

n	ASE	$\hat{\sigma}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\mu}$	RMSE
35	1.0113	1.23	0.8011	2.79	2.021	1.0056

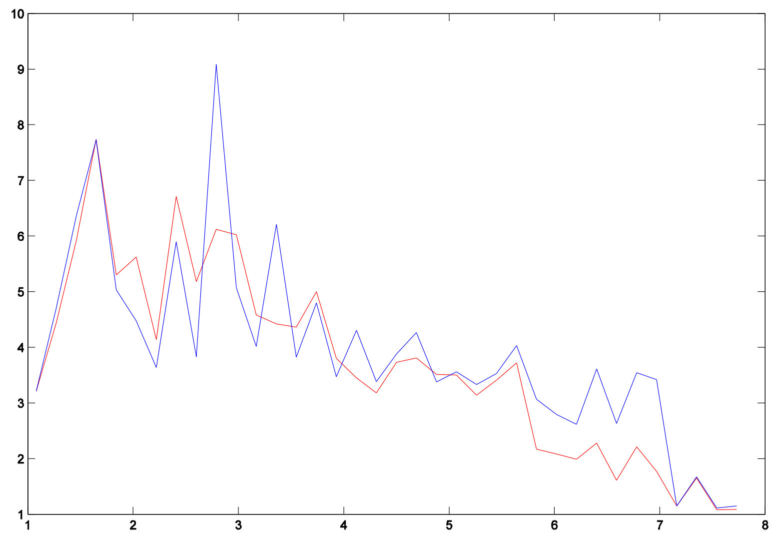

Figure 6. Exact and estimated mean values of ARW of Kelardasht site (The north of Iran)

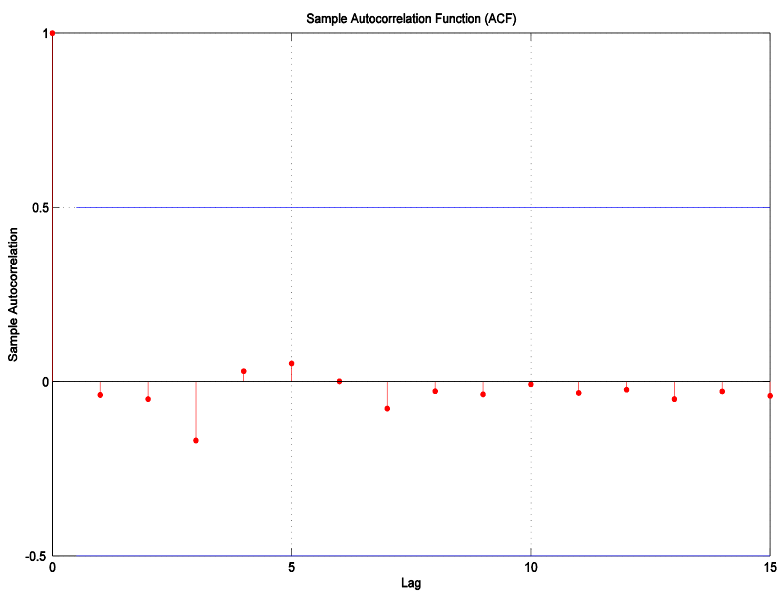

Figure 7. ACF errors of the empirical model

6. Summary

The partially linear autoregressive model is currently used in a variety of fields, including econometric studies, finance, wood industry science, biometrics, engineering, genetics, ecology and biology. This paper proposed a combination of parametric forms and nonlinear functions, in which the errors are independent. The errors and observations are also independent for each *t*. Since the parametric methods are not very efficient to estimate the regression functions, semiparametric methods are used.

At first, we suppose that the regression function *f*(.) has a parametric framework, that can be formed as $g(x,\gamma)$, where $g(x,\gamma)$ is a function of x with $\gamma \epsilon \Theta$ as an unknown parameter of the model. Therefore, we suggested a semiparametric form $g(x,\gamma)\xi(x)$ for the unknown autoregression function *f*(.), where $\xi(x)$ is a nonparametric adjustment. The unknown parameters are estimated using the conditional nonlinear least squares method and by defining and minimizing the local L2-fitting criterion with respect to $\xi(x)$, the nonparametric adjustment is also estimated and then, with smooth-kernel method, these estimates are corrected. So, the estimator of *f*(*x*) can be obtained. Because the formula $\hat{\xi}(x)$ contains the unknown function *f*(*x*), and with regard to the fact the errors of the model are small values, we can obtain $\tilde{\xi}(x)$ as an estimator of $\xi(x)$.

In order to investigate the efficiency of the semiparametric method in our model, we consider an empirical application. Hereby, three normal Pinus eldarica trees were randomly selected from a plantation at Garagpas-Kelardasht site, which is located in the western part of the Mazandaran province in the north of Iran. These trees have grown over 35 years in this site. The Pinus eldarica trees were cut for this study in January 2009. The ARW Pinus eldarica was predicted by an additive functional autoregressive model, from 1974 to 2008. The results are shown in Tables (5-7) and Figures (5-7) which indicate that the presented semiparametric method for a functional autoregressive model is more efficient.

Table 5, respectively, shows the observed-estimated value for the ARW data of kelardasht site in the north of Iran. In Table 6, the descriptive statistics of the ARW of pine wood is shown. We see that there are significant differences between the growth and ring width (radius) of a tree in a year, so much so that the ARW values were increasing by increasing age of tree (radial axis). The RMSE and ASE criterions are also applied to verify the efficiency of the suggested model. The RMSE and ASE values of the regression function for the functional autoregressive models are shown in Table 7. As we can see, it supports the great efficiency of the suggested model.

The curves of the ARW data in three trees of Pinus Elderica are shown in Figure 5. The curves of the mean of ARW data and its semiparametric estimator with selected bandwidth are also shown in Figure 6. Also, Figure 7 shows the ACF errors of the proposed empirical model which indicates that the errors of the model are almost uncorrelated. The results of the study show that the semiparametric estiamator of the autoregression function performs well.

7. Conclusion

The simulation results show that the semiparametric estiamator of the autoregression function performs well. Furthermore, the method is applied for annual ring width prediction to show that the partially linear autoregressive model is an efficient model for prediction of annual ring width. The autocorrelation function errors of the proposed empirical model with selected bandwidth and different sample sizes are almost uncorrelated. We see that the presented semiparametric method for a functional autoregressive model has proved to be more efficient.

Acknowledgment

The authors would like to thank the Editor and the referees for careful reading and for their comments which greatly improved the paper.

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