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A New Hermite Collocation Method for Solving Differential Difference Equations

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Abstract

The purpose of this study is to give a Hermite polynomial approximation for the solution of m^{th} order linear differential-difference equations with variable coefficients under mixed conditions. For this purpose, a new Hermite collocation method is introduced. This method is based on the truncated Hermite expansion of the function in the differential-difference equations. Hence, the resulting matrix equation can be solved and the unknown Hermite coefficients can be found approximately. In addition, examples that illustrate the pertinent features of the method are presented and the results of the study discussed.

Keywords: Hermite polynomials, Hermite polynomial solutions, differential-difference equations, approximation method

MSC 2010 No.: 41A58; 41A10; 34K28; 39A10

1. Introduction

In recent years, the differential-difference equations, treated as models of some physical phenomena, have been receiving considerable attention. When a mathematical model is developed for a physical system, it is usually assumed that all of the independent variables, such as space and time are continuous. Usually, this assumption leads to a realistic and justified approximation of the real variables of the system. However, there are some physical systems for which, this continuous variable assumption cannot be made. Since then, differential difference equations have played an important role in modeling problems that appear in various branches of science, e.g., mechanical engineering [Funaro (1990), Guo (1999a, 2000a)], condensed matter [Spanier (1987), Tang (2000), Xiong (2007), Zhou (2006)], biophysics and control theory [Zhang (2006), Ocalan (2009, Zhu (2008)].

Differential–difference equations occur wherever discrete phenomena are studied or differential equations discretized. The problems considered in this paper possess positive shifts (termed delays). However, there are other problems where one can have both positive as well as negative shifts (termed delay and advance, respectively). Recently, a number of different methods associated with orthogonal systems for solving higher-order differential-difference equations; the inverse scattering method [Ablowitz (1976), Fox (1971)], Hirota's bilinear form [Hu (2002)], tanh-method [Fan (2001)], Jacobian elliptic function method [Dai (2006)], numerical techniques [Emler (2001,2002)], differential transformation method [Arikoglu (2006), Karakoc (2009)] and Taylor polynomial method [Gulsu (2005, 2005a)] have been given.

In this paper, the basic ideas of the above studies are developed and applied to the m^{th}-order linear differential-difference equation with variable coefficients [Saaty (1981)]:

$$\sum_{k=0}^{m} P_k(t) y^{(k)}(t) + \sum_{j=1}^{n} P_j^*(t) y(t+j) = f(t), \quad k, j \geq 0, k, j \in N \tag{1}$$

with the conditions

$$\sum_{k=0}^{m-1} \left[a_{ik} y^{(k)}(a) + b_{ik} y^{(k)}(b) + c_{ik} y^{(k)}(c) = \mu_i \right], \ i = 0,1,\ldots,m-1 \tag{2}$$

and the solution is expressed in the form

$$y(t) = \sum_{n=0}^{N} a_n H_n(t), \ 0 \leq t \leq 1, \tag{3}$$

where $H_n(t)$ denotes the Hermite polynomials, a_n $(0 \leq n \leq N)$ are unknown Hermite coefficients, and N is any chosen positive integer such that $N \geq m$. Here, $P_k(t)$, $P_j^*(t)$ and $f(t)$ are functions defined on a≤t≤b, the real coefficients a_{ik}, b_{ik}, c_{ik} and μ_i are appropriate constants.

The rest of this paper is organized as follows. We describe the formulation of Hermite functions required for our subsequent development in section 2. Higher-order linear differential-difference equation with variable coefficients and fundamental relations are presented in Section 3. The Hermite collocation method is used. The method of finding approximate solution is described in

Section 4. To support our findings, we present the results of numerical experiments in Section 5. Section 6 concludes this paper with a brief summary. Finally some references are supplied at the end.

2. Properties of the Hermite Polynomials

The general form of the Hermite polynomials of nth degree is given as

$$H_n(t) = n!\sum_{j=0}^{N}(-1)^j \frac{2^{n-2j}}{j!(n-2j)!}t^{n-2j} \tag{4}$$

where $N = n/2$ if n is even and $N = (n-1)/2$ if n is odd. Note that this can also be written, when n = 2, 3, …, as

$$H_n(t) = \sum_{j=0}^{N}\frac{(-1)^j}{j!}n(n-1)...(n-2j+1)(2t)^{n-2j} \quad . \tag{5}$$

Explicit expressions for the first few Hermite polynomials are

$$H_0(t) = 1, \qquad H_1(t) = 2t,$$
$$H_2(t) = 4t^2 - 2, \qquad H_3(t) = 8t^3 - 12t,$$
$$H_4(t) = 16t^4 - 48t^2 + 12, \text{ and } \quad H_5(t) = 32t^5 - 160t^3 + 120t.$$

We finally mention that the Hermite polynomials $H_n(x)$ satisfy the Hermite differential equation

$$H_n''(t) - 2tH_n'(t) + 2nH_n(t) = 0 .$$

In the present application, an approximate solution in terms of linear combination of Hermite polynomials of the following form is assumed:

$$y(t) = \sum_{i=0}^{N} a_i H_i(t), \quad 0 \le i \le N .$$

3. Fundamental Relations

Let us consider the m^{th}-order linear differential-difference equation with variable coefficients (1) and find the matrix form of each term in the equation. We also convert the solution $y(t)$ defined by a truncated Hermite series (3) and its derivative $y^{(k)}(t)$ to m

$$y(t) = \mathbf{H}(t)\mathbf{A}, \quad y^{(k)}(t) = \mathbf{H}^{(k)}(t)\mathbf{A}, \tag{6}$$

where

$$\mathbf{H}(t) = [H_0(t)\ H_0(t)\ ...\ H_N(t)]$$

and

$$\mathbf{A} = [a_0\ a_1 ...\ a_N]^T.$$

On the one hand, it is well known [Sansone (1991)] that the relation between the powers t^n and the Hermite polynomials $H_n(t)$ is

$$t^{2n} = \frac{(2n)!}{2^{2n}} \sum_{n=0}^{r} \frac{H_{2n}(t)}{(r-n)!2n!}, \quad 0 \le t \le 1 \tag{7}$$

and

$$t^{2n+1} = \frac{(2n+1)!}{2^{2n+1}} \sum_{n=0}^{r} \frac{H_{2n+1}(t)}{(r-n)!2n+1!}, \ 0 \le t \le 1. \tag{8}$$

By using the expression (7) and (8) and taking n = 0, 1,…, N, we find the corresponding matrix relation as follows

$$(\mathbf{X}(t))^T = \mathbf{M}(\mathbf{H}(t))^T \quad and \quad \mathbf{X}(t) = \mathbf{H}(t)\mathbf{M}^T, \tag{9}$$

where

$$\mathbf{X}(t) = [1\ t\ \dots\ t^N],$$

for odd N,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \dots & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{8} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{N!}{2^N(\frac{N-1}{2})!1!} 0 & \frac{N!}{2^N(\frac{N-1}{2}-1)!3!} & 0 & \dots & \frac{N!}{2^N 0! N!} \end{bmatrix}$$

and for even N,

$$\mathbf{M}=\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \dots & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{8} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{N!}{2^N(\frac{N}{2})!0!} & 0 & \frac{N!}{2^N(\frac{N}{2}-1)!2!} & 0 & \dots & \frac{N!}{2^N 0!N!} \end{bmatrix}. \tag{10}$$

Then, by taking into account (6) we obtain

$$\mathbf{H}(t)=\mathbf{X}(t)(\mathbf{M}^{-1})^T \tag{11}$$

and

$$(\mathbf{H}(t))^{(k)}=\mathbf{X}^{(k)}(t)(\mathbf{M}^{-1})^T \ , \quad k=0,1,2,\cdots.$$

To obtain the matrix $\mathbf{X}^{(k)}(t)$ in terms of the matrix $\mathbf{X}(t)$, we can use the following relation:

$$\mathbf{X}^{(1)}(t)=\mathbf{X}(t)\mathbf{B}^T$$

$$\mathbf{X}^{(2)}(t)=\mathbf{X}^{(1)}(t)\mathbf{B}^T=\mathbf{X}(t)(\mathbf{B}^T)^2$$

$$\mathbf{X}^{(3)}(t)=\mathbf{X}^{(2)}(t)\mathbf{B}^T=\mathbf{X}(t)(\mathbf{B}^T)^3$$

$$\mathbf{X}^{(k)}(t)=\mathbf{X}^{(k-1)}(t)\mathbf{B}^T=\mathbf{X}(t)(\mathbf{B}^T)^k \ , \tag{12}$$

where

$$\mathbf{B}=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & N & 0 \end{bmatrix}. \tag{13}$$

Similarly, by substituting the matrix forms (11) and (12) into (6) we have the matrix relation

$$y^{(k)}=\mathbf{X}(t)\mathbf{B}^k(\mathbf{M}^T)^{-1}\mathbf{A}\,. \tag{14}$$

To obtain the matrix $\mathbf{X}(t+k)$ in terms of the matrix $\mathbf{X}(t)$, we can use the following relations:

$$\mathbf{X}(t) = [1\; t\; t^2 \ldots t^N]\,,\;\; \mathbf{X}(t+k) = [1\; (t+k)\; (t+k)^2 \ldots (t+k)^N]$$

$$\mathbf{X}(t+k) = \mathbf{X}(t)\mathbf{B}_k\,, \tag{15}$$

where

$$\mathbf{B}_k = \begin{bmatrix} \binom{0}{0}k^0 & \binom{1}{0}k^1 & \binom{2}{0}k^2 & \cdots & \binom{N}{0}k^N \\ 0 & \binom{1}{1}k^0 & \binom{2}{1}k^1 & \cdots & \binom{N}{1}k^{N-1} \\ 0 & 0 & \binom{2}{2}k^0 & \cdots & \binom{N}{2}k^{N-2} \\ \cdots & \cdot\cdot & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}k^0 \end{bmatrix}. \tag{16}$$

Consequently, by substituting the matrix forms (11) and (15) into (6), we have the matrix relation of solution

$$y(t+k) = \mathbf{H}(t+k)\mathbf{A} = \mathbf{X}(t+k)(\mathbf{M}^T)^{-1}\mathbf{A} \tag{17}$$

and by means of (6), (11) and (15), the matrix relation is

$$y(t+k) = \mathbf{X}(t)\mathbf{B}_k(\mathbf{M}^T)^{-1}\mathbf{A}\,. \tag{18}$$

4. Method of Solution

In this section, we consider a high order linear differential-difference equation in (1) and approximate to solution by means of finite Hermite series defined in (3). The aim is to find Hermite coefficients, that is, the matrix **A**. For this purpose, substituting the matrix relations (11) and (15) into equation (1) and then simplifying, we obtain the fundamental matrix equation

$$\sum_{k=0}^{m}\mathbf{P}_k(t)\mathbf{X}(t)\mathbf{B}^k(\mathbf{M}^T)^{-1}\mathbf{A} + \sum_{j=1}^{n}\mathbf{P}_j^*(t)\mathbf{X}(t)\mathbf{B}_k(\mathbf{M}^T)^{-1}\mathbf{A} = f(t)\,. \tag{19}$$

By using in equation (19) collocation points t_i defined by

$$t_i = \frac{i}{N}, \; i = 0,1,...,N \tag{20}$$

we get the system of matrix equations

$$\sum_{k=0}^{m} \mathbf{P}_k(t_i)\mathbf{X}(t_i)\mathbf{B}^k(\mathbf{M}^T)^{-1}\mathbf{A} + \sum_{j=1}^{n} \mathbf{P}_j^*(t_i)\mathbf{X}(t_i)\mathbf{B}_k(\mathbf{M}^T)^{-1}\mathbf{A} = f(t), \; i = 0,1,...N \tag{21}$$

or briefly the fundamental matrix equation

$$\sum_{k=0}^{m} \mathbf{P}_k \mathbf{X}\mathbf{B}^k(\mathbf{M}^T)^{-1}\mathbf{A} + \sum_{j=1}^{n} \mathbf{P}_j^* \mathbf{X}\mathbf{B}_k(\mathbf{M}^T)^{-1}\mathbf{A} = \mathbf{F}, \tag{22}$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k(t_0) & 0 & \dots & \dots & 0 \\ 0 & P_k(t_1) & \dots & \dots & 0 \\ \dots & \dots & & & \dots \\ \dots & \dots & & & \dots \\ 0 & 0 & \dots & \dots & P_k(t_N) \end{bmatrix}, \qquad \mathbf{P}_\mathbf{j}^* = \begin{bmatrix} P_j^*(t_0) & 0 & \dots & \dots & 0 \\ 0 & P_j^*(t_1) & \dots & \dots & 0 \\ \dots & \dots & & & \dots \\ \dots & \dots & & & \dots \\ 0 & 0 & \dots & \dots & P_j^*(t_N) \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} f(t_0) \\ f(t_1) \\ . \\ . \\ . \\ f(t_N) \end{bmatrix}, \qquad \text{and} \qquad \mathbf{X} = \begin{bmatrix} X(t_0) \\ X(t_1) \\ . \\ . \\ . \\ X(t_N) \end{bmatrix} = \begin{bmatrix} 1 & t_0 & t_0{}^2 & . & . & . & t_0{}^N \\ 1 & t_1 & t_1{}^2 & . & . & . & t_1{}^N \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 1 & t_N & t_N{}^2 & . & . & . & t_N{}^N \end{bmatrix}.$$

Hence, the fundamental matrix equation (22) corresponding to equation (1) can be written in the form

$$\mathbf{WA} = \mathbf{F} \text{ or } [\mathbf{W};\mathbf{F}], \quad \mathbf{W} = [w_{i,j}], \; i, j = 0,1,...,N, \tag{23}$$

where

$$\mathbf{W} = \sum_{k=0}^{m} \mathbf{P}_k \mathbf{X}\mathbf{B}^k(\mathbf{M}^T)^{-1} + \sum_{j=1}^{n} \mathbf{P}_j^* \mathbf{X}\mathbf{B}_k(\mathbf{M}^T)^{-1}. \tag{24}$$

Here, Equation (23) corresponds to a system of $(N+1)$ linear algebraic equations with unknown Hermite coefficients $a_0, a_1, ..., a_N$. We can obtain the corresponding matrix forms for the conditions (2), by means of the relation (14),

$$\sum_{k=1}^{m-1}[a_{ik}+b_{ik}+c_{ik}]\mathbf{X}(t)\mathbf{B}_k(\mathbf{M}^T)^{-1}\mathbf{A}=\mu_i\ ,\ i=0,1,2,...,m-1.$$

On the other hand, the matrix form for conditions can be written as

$$\mathbf{U}_i\mathbf{A}=[\mu_i]\ \text{or}\ [\mathbf{U}_i;\mu_i],\ i=0,1,2,...,m-1, \tag{25}$$

where

$$\mathbf{U}_i=\sum_{k=1}^{m-1}[a_{ik}+b_{ik}+c_{ik}]\mathbf{X}(t)\mathbf{B}_k(\mathbf{M}^T)^{-1} \tag{26}$$

and

$$\mathbf{U}_i=[u_{i0}\ u_{i1}\ u_{i2}\ldots u_{iN}],\ \ i=0,1,2,...m-1. \tag{27}$$

To obtain the solution of equation (1) under conditions (2), we replace the row matrices (27) by the last m rows of the matrix (25), giving the new augmented matrix,

$$[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]=\begin{bmatrix} w_{00} & w_{01} & . & . & . & w_{0N} & ; & f(t_0)\\ w_{10} & w_{11} & . & . & . & w_{1N} & ; & f(t_1)\\ \ldots & \ldots & & & & \ldots & ; & \ldots\\ w_{N-m,0} & w_{N-m,1} & . & . & . & w_{N-m,N} & ; & f(t_{N-m})\\ u_{00} & u_{01} & . & . & . & u_{0N} & ; & \mu_0\\ u_{10} & u_{11} & . & . & . & u_{1N} & ; & \mu_1\\ \ldots & \ldots & & & & \ldots & ; & \ldots\\ u_{m-1,0} & u_{m-1,1} & . & . & . & u_{m-1,N} & ; & \mu_{m-1} \end{bmatrix}. \tag{28}$$

If $rank\tilde{\mathbf{W}}=rank[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]=N+1$, then we can write

$$\mathrm{A}=(\tilde{\mathrm{W}})^{-1}\tilde{\mathrm{F}}. \tag{29}$$

Thus, the matrix $\mathbf{A}$ (and thereby the coefficients $a_0,a_1,\ldots,a_N$) is uniquely determined. Also the Eq. (1) with conditions (2) has a unique solution. This solution is given by truncated Hermite series (3).

We can easily check the accuracy of the method. Since the truncated Hermite series (3) is an approximate solution of equation (1), when the solution $y_N(t)$ and its derivatives are substituted in equation (1), the resulting equation must be satisfied approximately; that is, for $t = t_q \in [0,1],\ q = 0,1,2,...$

$$E(t_q) = \left| \sum_{k=0}^{m} P_k(t_q) y^{(k)}(t_q) + \sum_{j=1}^{n} P_j^*(t_q) y(t_q + j) - f(t_q) \right| \cong 0 \qquad (30)$$

and $E(t_q) \le 10^{-k_q}$ (k_q positive integer). If max $10^{-k_q} = 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E(t_q)$ at each of the points becomes smaller than the prescribed 10^{-k}. On the other hand, the error can be estimated by the function

$$E_N(t) = \sum_{k=0}^{m} P_k(t) y^{(k)}(t) + \sum_{j=1}^{n} P_j^*(t) y(t+j) - f(t). \qquad (31)$$

If $E_N(t) \to 0$, when N is sufficiently large enough, then the error decreases.

5. Illustrative Examples

In this section, several numerical examples all of which were performed on the computer using Maple 9 are given to demonstrate the accuracy and effectiveness of the method. The absolute errors in Tables are the values of $|y(t) - y_N(t)|$ at selected points.

Example 1.

We first consider the second order linear differential-difference equation with variable coefficients

$$y''(t) - y'(t) + e^{-t} y(t) + y(t+1) + y(t+2) = 1 + e^{t+1} + e^{t+2}$$

with

$$y(0) = 1, y'(0) = 1$$

and seek the solution $y(t)$ as a truncated Hermite series

$$y(t) = \sum_{n=0}^{N} a_n H_n(t).$$

So that $P_0(t) = e^{-t}$, $P_1(t) = -1$, $P_2(t) = 1$, $P_1^*(t) = 1$, $P_2^*(t) = 1$, $f(t) = 1 + e^{t+1} + e^{t+2}$. Then, for N = 5, the collocation points are

t_0=0, t_1=1/4, t_2=1/2, t_3=3/4, t_4=1 and the fundamental matrix equation of the problem is defined by

$$\{\mathrm{P_0X}(\mathrm{M}^T)^{-1}+\mathrm{P_1XB}(\mathrm{M}^T)^{-1}+\mathrm{P_2XB}^2(\mathrm{M}^T)^{-1}+\mathrm{P}_1^*\mathrm{XB}_1(\mathrm{M}^T)^{-1}+\mathrm{P}_2^*\mathrm{XB}_2(\mathrm{M}^T)^{-1}\}\mathrm{A}=\mathrm{F},$$

where $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_1^*$, $\mathbf{P}_2^*$, $\mathbf{X}$ are matrices of order (6×6) defined by

$$\mathbf{X}=\begin{bmatrix}1&0&0&0&0&0\\1&\frac{1}{5}&\frac{1}{25}&\frac{1}{125}&\frac{1}{625}&\frac{1}{3125}\\1&\frac{2}{5}&\frac{4}{25}&\frac{8}{125}&\frac{16}{625}&\frac{32}{3125}\\1&\frac{3}{5}&\frac{9}{25}&\frac{27}{125}&\frac{81}{625}&\frac{243}{3125}\\1&\frac{4}{5}&\frac{16}{25}&\frac{64}{125}&\frac{256}{625}&\frac{1024}{3124}\\1&1&1&1&1&1\end{bmatrix},\qquad \mathbf{P}_0=\begin{bmatrix}1&0&0&0&0&0\\0&e^{-1/5}&0&0&0&0\\0&0&e^{-2/5}&0&0&0\\0&0&0&e^{-3/5}&0&0\\0&0&0&0&e^{-4/5}&0\\0&0&0&0&0&e^{-1}\end{bmatrix}$$

$$\mathbf{P}_1=\begin{bmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\end{bmatrix},\quad \mathbf{P}_2=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\quad \mathrm{B}=\begin{bmatrix}0&1&0&0&0&0\\0&0&2&0&0&0\\0&0&0&3&0&0\\0&0&0&0&4&0\\0&0&0&0&0&5\\0&0&0&0&0&0\end{bmatrix},$$

$$\mathrm{B}_1=\begin{bmatrix}1&1&1&1&1&1\\0&1&2&3&4&5\\0&0&1&3&6&10\\0&0&0&1&4&10\\0&0&0&0&1&5\\0&0&0&0&0&1\end{bmatrix},\quad \mathrm{B}_2=\begin{bmatrix}1&2&4&8&16&32\\0&1&4&12&32&80\\0&0&1&6&24&80\\0&0&0&1&8&40\\0&0&0&0&1&10\\0&0&0&0&0&1\end{bmatrix},$$

$$\mathbf{Q}_1=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\quad \mathrm{Q}_2=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}.$$

If these matrices are substituted in (22), we obtain the linear algebraic system. This system yields the approximate solution of the problem. The result with N = 6(1)8 using the Hermite collocation method discussed in Section 3 and also the exact values of $y = \exp(t)$ are shown in Table1.

Table1. Error analysis of Example 1 for the t value

t	Exact	Present Method					
	Solution	N=6	N_e=6	N=7	N_e=7	N=8	N_e=8
0.0	1.000000	1.000000	0.000000	0.999999	0.00000	0.999999	0.300E-9
0.1	1.105171	1.105193	0.227E-4	1.105174	0.391E-5	1.105169	0.127E-5
0.2	1.221403	1.221493	0.907E-4	1.221421	0.185E-4	1.221398	0.439E-5
0.3	1.349859	1.350057	0.199E-3	1.349906	0.478E-4	1.349508	0.797E-5
0.4	1.491825	1.492161	0.336E-3	1.491919	0.951E-4	1.491814	0.102E-4
0.5	1.648721	1.649206	0.484E-3	1.648883	0.162E-3	1.648712	0.903E-5
0.6	1.822119	1.822738	0.619E-3	1.822368	0.249E-3	1.822116	0.225E-5
0.7	2.013753	2.014469	0.711E-3	2.014106	0.353E-3	2.013764	0.122E-4
0.8	2.225541	2.226271	0.730E-3	2.226009	0.468E-3	2.225576	0.360E-4
0.9	2.459603	2.460248	0.645E-3	2.460189	0.586E-3	2.459673	0.702E-4
1.0	2.718282	2.718711	0.429E-3	2.718975	0.693E-3	2.718396	0.114E-3

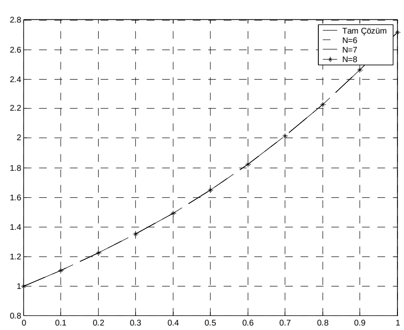

Figure 1. Numerical and exact solution of the Example1 for N=6,7,8

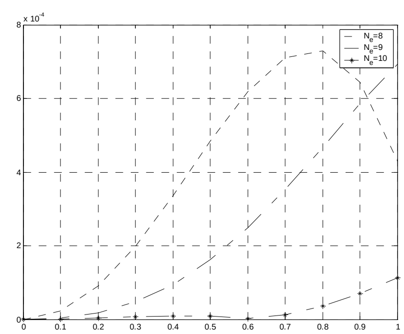

Figure 2. Error function of Example1 for various N

Figure 1 shows the resulting graph of solution of Example1 for N = 6,7,8 and it is compared with exact solution. In Figure 2, we plot error function for Example 1.

Example 2.

Let us find the Hermite series solution of the following second order linear differential-difference equation

$$y''(t) - y'(t) + y(t) - y(t+1) + y(t+2) = -\cos(t) - \sin(t+1) + \sin(t+2)$$

with $y(0) = 1, y'(0) = 0$. The exact solution of this problem is $y(t) = \sin t$. Using the procedure in Section 3 and taking N=8, 9 and 10 the matrices in equation (22) are computed. Hence linear algebraic system is gained. This system is approximately solved using the Maple 9. We display a plot of absolute difference exact and approximate solutions in Figure 3 and error functions for various N is shown in Figure 4. The solution of the linear differential difference equation is obtained for N = 8, 9, 10.

Table2. Error analysis of Example 2 for the t value

t	Exact Solution	Present Method N=8	N_e=8	N=9	N_e=9	N=10	N_e=10
0.0	0.000000	0.000000	0.000000	0.30E-13	0.300E-13	0.68E-13	0.68E-13
0.1	0.099833	0.099833	0.388E-6	0.099833	0.622E-8	0.099833	0.971E-7
0.2	0.198669	0.198669	0.129E-5	0.198669	0.151E-6	0.198668	0.414E-6
0.3	0.295520	0.295517	0.221E-5	0.295520	0.767E-6	0.295519	0.976E-6
0.4	0.389418	0.389415	0.255E-5	0.389420	0.214E-5	0.389416	0.178E-5
0.5	0.479426	0.479423	0.164E-5	0.479430	0.457E-5	0.479422	0.283E-5
0.6	0.564642	0.564643	0.117E-5	0.564650	0.827E-5	0.564638	0.404E-5
0.7	0.644218	0.644224	0.651E-5	0.644231	0.134E-4	0.644212	0.536E-5
0.8	0.717356	0.717370	0.148E-4	0.717376	0.199E-4	0.717349	0.666E-5
0.9	0.783327	0.783353	0.263E-4	0.783354	0.278E-4	0.783319	0.783E-5
1.0	0.841470	0.841511	0.409E-4	0.841507	0.368E-4	0.841462	0.873E-5

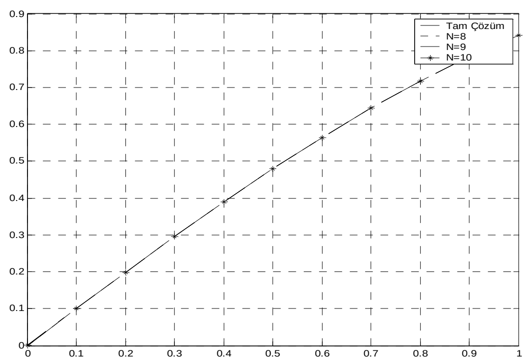

Figure 3. Numerical and exact solution of the Example 2 for N=8,9,10

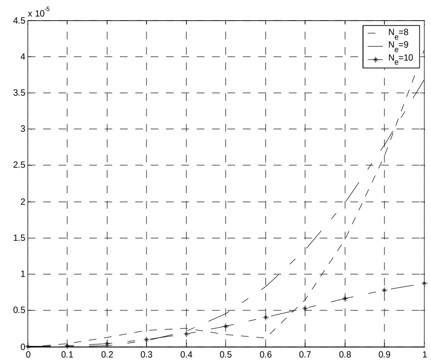

Figure 4. Error function of Example 2 for various N

Example 3.

Let us find the Hermite series solution of the first order linear differential difference equation

$$y'(t) + y(t) + y(t+1) - y(t+2) = t^2 - t - 3$$

with condition

$$y(0) = 0$$

and the exact solution is $y = t^2 - t$. Using the procedure in Section 3, we find the approximate solution of this equation which is the same as the exact solution.

6. Conclusion

In recent years, the study of high order linear differential-difference equation has attracted the attention of many mathematicians and physicists. The Hermite collocation methods are used to solve the high order linear differential-difference equation numerically. A considerable advantage of the method is that the Hermite polynomial coefficients of the solution are found very easily using computer programs. Shorter computation time and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy. Illustrative examples are included to demonstrate the validity and applicability of the technique which can be performed on the computer using a program written in Maple 9. To get the best approximating solution of the equation, we take more forms from the Hermite expansion of the functions, that is, the truncation limit N must be chosen large enough. In addition, an interesting feature of this method is to find the analytical solutions if the equation has an exact solution that is a polynomial function.

Illustrative examples with satisfactory results are used to demonstrate the application of this method. Suggested approximations make this method very attractive and contribute to the good agreement between approximate and exact values in the numerical example.

As a result, the power of the employed method is confirmed. We are assured of the correctness of the obtained solutions by putting them back into the original equation with the aid of Maple. This provides an extra measure of confidence in the results. The Hermite expansion method is a promising method for investigating exact analytic solutions to linear differential-difference equations with some modifications this method can also be extended to the systems of linear differential-difference equations with variable coefficients.

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