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A Boundedness and Stability Results for a Kind of Third Order Delay Differential Equations

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Abstract

The objective of this study was to get some sufficient conditions which guarantee the asymptotic stability and uniform boundedness of the null solution of some differential equations of third order with the variable delay. The most efficient tool for the study of the stability and boundedness of solutions of a given nonlinear differential equation is provided by Lyapunov theory. However the construction of such functions which are positive definite with corresponding negative definite derivatives is in general a difficult task, especially for higher-order differential equations with delay. Such functions and their time derivatives along the system under consideration must satisfy some fundamental inequalities. Here the Lyapunov second method or direct method is used as a basic tool. By defining an appropriate Lyapunov functional, we prove two new theorems on the asymptotic stability and uniform boundedness of the null solution of the considered equation. Our results obtained in this work improve and extend some existing well-known related results in the relevant literature which were obtained for nonlinear differential equations of third order with a constant delay. We also give an example to illustrate the importance of the theoretical analysis in this work and to test the effectiveness of the method employed.

Keywords: Stability, Boundedness, Lyapunov functional, Third-order Delay differential equations

MSC 2010 No.: 34C11, 34D20, 34K20

1. Introduction

The qualitative behavior of solutions of various differential equations of third order with and without delay have been extensively studied. In the relevant literature, a good deal of work has been done and many interesting results have been obtained. We refer readers to the papers of Hara (1971), Zhu (1992), Omeike (2009) and Tunç (2009), to mention a few as well as the references cited therein for some works on the subject, where the Lyapunov function or functional approach have been the most effective method to determine the stability and boundedness of solutions.

Omeike (2009) considered the following nonlinear differential equation of third order, with a constant deviating argument r,

$$x'''(t) + a(t)x''(t) + b(t)f(x'(t)) + c(t)h(x(t-r)) = e(t).$$

He studied the stability and boundedness of solutions of this equation when $e(t) = 0$ and $e(t) \neq 0$. In Tunç (2009), the author discussed sufficient conditions which ensure the boundedness of the delay differential equation of the form

$$\begin{aligned} x'''(t) + a(t)\psi(x'(t))x''(t) + b(t)f(x'(t)) + c(t)h(x(t-r)) \\ = e(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)). \end{aligned}$$

Our objective in this paper is to extend the results verified by Tunç (2009) to obtain sufficient conditions for the stability and boundedness of solutions of the following delay differential equation:

$$\begin{aligned} [\phi(x(t))x'(t)]'' + a(t)\psi(x'(t))x''(t) + b(t)f(x'(t)) + c(t)h(x(t-r(t))) \\ = e(t, x(t), x(t-r(t)), x'(t), x'(t-r(t)), x''(t)), \end{aligned} \tag{1}$$

where ϕ is a twice differentiable function and ρ is a positive constant with $0 \leq r(t) \leq \rho$, which will be determined later. The functions $a(t),\ b(t),\ c(t),\ f(x'),\ h(x),\ \psi(x')$, and $e(t, x(t), x(t-r(t)), x'(t), x'(t-r(t)), x''(t))$ are continuous in their respective arguments; the derivatives $a'(t),\ b'(t),\ c'(t),\ h'(x),\ f'(x')$ are continuous for all $x,\ y$ with $h(0) = f(0) = 0$. In addition, it is also assumed that the functions $e(t, x, x', x(t-r(t)), x'(t-r(t)), x'')$, $f(x'(t))$ and $h(x(t-r(t)))$ satisfy a Lipschitz condition in $x,\ x'$, x'', $x(t-r(t))$, and $x'(t-r(t))$.

Remark 1.1.

Clearly the equation discussed in Tunç (2009) is a special case of equation (1) when $\phi(x) = 1$ and $r(t) = r$. Moreover, if $\psi(x') = 1$ and $f(x') = x'$, then (1) reduces to the case studied by Remili and Oudjedi (2014).

2. Preliminaries

First we will give some basic definitions and important stability criteria for the general non-autonomous delay differential system. We consider

$$x' = f(t, x_t), \quad x_t(\theta) = x(t+\theta)\ , \quad -r \le \theta \le 0, \quad t \ge 0, \tag{2}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping,

$$f(t,0) = 0,\ C_H := \{\phi \in (C[-r,0],\ \mathbb{R}^n) : \|\phi\| \le H\},$$

and for $H_1 < H$, there exist $L(H_1) > 0$, with $|f(t,\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 2.1. Burton (2005)

An element $\psi \in C$ is in the $\omega -$ limit set of ϕ, say $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,+\infty)$ and there is a sequence $\{t_n\}, t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)$ for $-r \le \theta \le 0$.

Definition 2.2 Burton (2005)

A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t,0,\phi)$, is defined on $[0,\infty)$ and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.

Lemma 2.3. Burton (1985)

If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2.1) with $x_0(\phi) = \phi$ is defined on $[0,\infty)$ and $\|x_t(\phi)\| \le H_1 < H$ for $t \in [0,\infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and

$$\mathrm{dist}(x_t(\phi), \Omega(\phi)) \to 0 \quad \text{as} \quad t \to \infty.$$

Lemma 2.4. Krasovskii (1963)

If there is a continuous functional $V(t,\phi) : [0,+\infty] \times C_H \to [0,+\infty]$ locally Lipschitz in ϕ and wedges W_i such that:

(i) If $W_1(\|\phi\|) \le V(t,\phi),\ V(t,0) = 0$ and $V'_{(2,1)}(t,\phi) \le 0$

then the zero solution of (2.1) is stable. If in addition $V(t,\phi) \le W_2(\|\phi\|)$ then the zero solution of (2.1) is uniformly stable.

(ii) If $W_1(\|\phi\|) \le V(t,\phi) \le W_2(\|\phi\|)$ and $V'_{(2,1)}(t,\phi) \le -W_3(\|\phi\|)$,

then the zero solution of (2.1) is uniformly asymptotically stable.

3. Assumptions and main results

We assume that there are positive constants $a,\ b,\ c,\ \delta,\ \delta_0,\ \delta_1,\ \delta_2,\ \psi_1,\ \phi_0,\ \phi_1,\ \rho$, and σ, such that the following conditions hold

i) $0 < a \le a(t) \le A,\ 0 < b \le b(t) \le B,\ 0 < c \le c(t) \le C,$

ii) $1 \le \psi(y) \le \psi_1,\ 0 < \phi_0 \le \phi(x) \le \phi_1,$

iii) $\dfrac{h(x)}{x} \geq \delta > 0 \ (x \neq 0)$, and $|h'(x)| \leq \delta_1$ for all x,

iv) $\delta_2 \geq \dfrac{f(y)}{y} \geq \delta_0 > 0 \ (y \neq 0)$,

v) $\displaystyle\int_{-\infty}^{+\infty} |\phi'(u)| \, du < \infty$,

vi) $\displaystyle\int_0^{\infty} |c'(s)| \, ds \leq N_1 < \infty$ and $c'(t) \to 0$ as $t \to \infty$,

vii) for some $\rho \geq 0$ and $0 < \sigma < 1, \ 0 \leq r(t) \leq \rho, \ r'(t) \leq \sigma$.

Before stating theorems, we introduce the following notation $\Delta = \mu b \delta_0 - C \delta_1 \phi_1$.

For the case $e \equiv 0$, the following result is introduced.

Theorem 3.1.

In addition to the assumptions (i)-(vii), assume that the following conditions are satisfied

H1) $\dfrac{\phi_1 \delta_1 C}{b \delta_0} < \mu < a$,

H2) $\dfrac{\mu \psi_1}{\phi_0^2}|a'(t)| + \dfrac{\delta_2}{\phi_0}|b'(t)| - \dfrac{\delta_1}{\mu}c'(t) < \dfrac{\Delta}{\phi_1^2}$.

Then the zero solution of (1) is uniformly asymptotically stable, provided that

$$\rho < \min\left\{ \frac{2(a-\mu)}{\phi_1 C \delta_1}, \frac{\phi_0^3 \Delta (1-\sigma)}{C \delta_1 \phi_1^2 (\mu + \phi_0 + \mu \phi_0^2 (1-\sigma))} \right\}.$$

Proof:

Equation (1) can be expressed as the following system

$$\begin{aligned} x' &= \frac{1}{\phi(x)} y \\ y' &= z \\ z' &= -\frac{a(t)}{\phi(x)} \psi(\frac{y}{\phi(x)}) z + \frac{a(t)\phi'(x)}{\phi^3(x)} \psi(\frac{y}{\phi(x)}) y^2 - b(t) f\left(\frac{y}{\phi(x)}\right) \\ &\quad -c(t)h(x) + c(t) \int_{t-r(t)}^{t} \frac{y(s)}{\phi(x(s))} h'(x(s)) ds. \end{aligned} \tag{3}$$

We define the following Lyapunov functional $W = W(t, x_t, y_t, z_t)$:

$$W(t, x_t, y_t, z_t) = e^{-\beta(t)} V(t, x_t, y_t, z_t) = e^{-\beta(t)} V, \tag{4}$$

where

$$\begin{aligned} V &= \mu c(t)H(x) + c(t)h(x)y + b(t)\phi(x)F(\frac{y}{\phi(x)}) + \frac{1}{2}z^2 + \frac{\mu}{\phi(x)}yz \\ &\quad + \mu a(t)\int_0^{\frac{y}{\phi(x)}} \psi(u)udu + \lambda \int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds, \end{aligned} \tag{5}$$

and

$$\beta(t) = \int_0^t [\frac{|\alpha(s)|}{\omega} + \frac{|c'(s)|}{c}]ds,$$

such that

$$H(x) = \int_0^x h(u)du,\ F(y) = \int_0^y f(u)du,\ \text{and } \alpha(t) = \frac{\phi'(x(t))}{\phi^2(x(t))}x'(t).$$

ω and λ are positive constants which will be specified later in the proof. The above Lyapounov functional V can be rewritten as the following

$$\begin{aligned} V &= \mu c(t)\left[H(x) + \frac{1}{\mu}h(x)y + \frac{\delta_1}{2\mu^2}y^2\right] + \mu a(t)\int_0^{\frac{y}{\phi(x)}} [\psi(u) - \frac{\mu}{a(t)}]udu \\ &\quad + \frac{1}{2}(z + \frac{\mu}{\phi(x)}y)^2 + b(t)\phi(x)F(\frac{y}{\phi(x)}) - \frac{\delta_1 c(t)}{2\mu}y^2 \\ &\quad + \lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds. \end{aligned}$$

Letting

$$G(x,y) = H(x) + \frac{1}{\mu}yh(x) + \frac{\delta_1}{2\mu^2}y^2,$$

assumption (iii) implies that

$$\begin{aligned} G(x,y) &= H(x) + \frac{\delta_1}{2\mu^2}(y + \frac{\mu}{\delta_1}h(x))^2 - \frac{1}{2\delta_1}h^2(x) \\ &\geq \int_0^x \left(1 - \frac{h'(u)}{\delta_1}\right)h(u)du \geq 0. \end{aligned} \tag{6}$$

It is clear from (iv) that

$$F\left(\frac{y}{\phi(x)}\right) = \int_0^{\frac{y}{\phi(x)}} f(u)du \geq \frac{\delta_0}{2}\frac{y^2}{\phi^2(x)}.$$

Hence, using the above estimate, the assumption (ii), (3.4), and the fact that the integral $\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds$ is positive, we deduce that

$$\begin{aligned} V &\geq \mu c(t)G(x,y) + \frac{1}{2}(z + \frac{\mu}{\phi(x)}y)^2 \\ &\quad + \frac{1}{2}\left[\frac{\mu a}{\phi^2(x)}\left(1 - \frac{\mu}{a}\right) + \left(\frac{\delta_0 b}{\phi_1} - \frac{C\delta_1}{\mu}\right)\right]y^2. \end{aligned}$$

Condition (H1) implies that $1-\dfrac{\mu}{a}>0$ and $\dfrac{\delta_0 b}{\phi_1}-\dfrac{C\delta_1}{\mu}>0$. Thus there exist a positive constant k small enough such that

$$V \geq \mu c G(x,y) + k(y^2+z^2). \tag{7}$$

From (ii), (v), and (vi), we get

$$\begin{aligned}
\beta(t) &= \int_0^t \left[\frac{|\alpha(s)|}{\omega}+\frac{|c'(s)|}{c}\right] ds,\\
&\leq \frac{1}{\omega}\int_{\theta_1(t)}^{\theta_2(t)} \frac{|\phi'(u)|}{\phi^2(u)}du+\frac{N_1}{c}\\
&\leq \frac{1}{\omega\phi_0^2}\int_{-\infty}^{+\infty} |\phi'(u)|\, du+\frac{N_1}{c} \leq \frac{N_2}{\omega\phi_0^2}+\frac{N_1}{c}=N<\infty,
\end{aligned}$$

where $\theta_1(t)=\min\{x(0),x(t)\}$ and $\theta_2(t)=\max\{x(0),x(t)\}$. Therefore we can find a continuous function $W_1(\|X\|)$, where $X=(x,y,z)$, such that

$$W_1(\|X\|)\geq 0 \quad \text{and} \quad W_1(\|X\|)\leq W.$$

The existence of a continuous function $W_2(\|X\|)$ which satisfies the inequality $W\leq W_2(\|X\|)$ is easily verified.

Let $V'_{(3)}(t,x_t,y_t,z_t)=V'_{(3)}$ denote the time derivative of the Lyapunov functional $V(t,x_t,y_t,z_t)$ along the trajectories of the system (3). An easy computation shows that

$$\begin{aligned}
V'_{(3)} &= \mu c'(t)H(x)+c'(t)yh(x)+b'(t)\phi(x)F(\frac{y}{\phi(x)})\\
&+\frac{\mu}{\phi(x)}z^2-\frac{a(t)}{\phi(x)}\Psi(\frac{y}{\phi(x)})z^2-\mu b(t)\frac{y}{\phi(x)}f(\frac{y}{\phi(x)})\\
&+\mu a'(t)\int_0^{\frac{y}{\phi(x)}}\Psi(u)udu+\frac{c(t)h'(x)}{\phi(x)}y^2+\lambda r(t)y^2\\
&+\alpha(t)\left[b(t)\phi^2(x)F(\frac{y}{\phi(x)})-b(t)y\phi(x)f(\frac{y}{\phi(x)})+(a(t)\Psi(\frac{y}{\phi(x)})-\mu)zy\right]\\
&+c(t)(\frac{\mu}{\phi(x)}y+z)\int_{t-r(t)}^t y(s)\frac{h'(x(s))}{\phi(x(s))}ds-\lambda(1-r'(t))\int_{t-r(t)}^t y^2(\xi)d\xi.
\end{aligned}$$

When we apply the hypotheses of the theorem we obtain

$$\begin{aligned}
V'_{(3.1)} &\leq \mu c'(t)G(x,y)+\left[\frac{c(t)h'(x)}{\phi(x)}y^2-\mu b(t)\frac{y}{\phi(x)}f(\frac{y}{\phi(x)})+\lambda\rho y^2\right]\\
&+\left[\mu a'(t)\int_0^{\frac{y}{\phi(x)}}\Psi(u)udu+b'(t)\phi(x)F(\frac{y}{\phi(x)})-\frac{\delta_1}{2\mu}c'(t)y^2\right]\\
&+\left[\frac{\mu-a(t)}{\phi(x)}\right]z^2+|\alpha(t)|\left[\frac{3}{2}\delta_2 By^2+(A\psi_1-\mu)|zy|\right]\\
&+c(t)(\frac{\mu}{\phi(x)}y+z)\int_{t-r(t)}^t y(s)\frac{h'(x(s))}{\phi(x(s))}ds-\lambda(1-\sigma)\int_{t-r(t)}^t y^2(\xi)d\xi.
\end{aligned}$$

Using the Schwartz inequality $|uv| \le \frac{1}{2}(u^2+v^2)$, we have

$$\begin{aligned}|\alpha(t)|\left[\frac{3\delta_2}{2}By^2+(A\psi_1-\mu)|zy|\right] &\le \frac{1}{2}|\alpha(t)|\left[3\delta_2By^2+(A\psi_1-\mu)(y^2+z^2)\right]\\ &\le k_1|\alpha(t)|(y^2+z^2),\end{aligned}$$

where $k_1=\dfrac{1}{2}(A\psi_1-\mu+3\delta_2B)$. Since $|h'(x)|\le\delta_1$, we get the following inequalities

$$\frac{\mu c(t)}{\phi(x)}y\int_{t-r(t)}^{t}\frac{y(s)}{\phi(x)}h'(x(s))ds \le \frac{C\delta_1\mu\rho}{2\phi_0}y^2+\frac{C\mu\delta_1}{2\phi_0^3}\int_{t-r(t)}^{t}y^2(\xi)d\xi,$$

and

$$c(t)z\int_{t-r(t)}^{t}\frac{y(s)}{\phi(x)}h'(x(s))ds \le \frac{C\delta_1\rho}{2}z^2+\frac{C\delta_1}{2\phi_0^2}\int_{t-r(t)}^{t}y^2(\xi)d\xi.$$

It can be easily deduced from the above estimates that

$$\begin{aligned}V'_{(3)} \le\ & \mu c'(t)G(x,y)+\frac{1}{2}\left[\frac{\mu\psi_1}{\phi_0^2}|a'(t)|+\frac{\delta_2}{\phi_0}|b'(t)|-\frac{\delta_1}{\mu}c'(t)\right]y^2\\ &-\left[\frac{\Delta}{\phi_1^2}-(\lambda+\frac{\mu C\delta_1}{2\phi_0})\rho\right]y^2-\left[\frac{a-\mu}{\phi_1}-\frac{C\delta_1\rho}{2}\right]z^2\\ &+\left[\frac{C\delta_1}{2\phi_0^2}(1+\frac{\mu}{\phi_0})-\lambda(1-\sigma)\right]\int_{t-r(t)}^{t}y^2(\xi)d\xi+k_1|\alpha(t)|(y^2+z^2).\end{aligned}$$

If we take $\dfrac{C\delta_1(\phi_0+\mu)}{2\phi_0^3(1-\sigma)}=\lambda$, by using (H2) the last inequality becomes

$$\begin{aligned}V'_{(3)} \le\ & \mu c'(t)G(x,y)-\left[\frac{\mu b\delta_0-C\delta_1\phi_1}{2\phi_1^2}-\frac{C\delta_1}{2\phi_0}(\frac{\phi_0+\mu}{\phi_0^2(1-\sigma)}+\mu)\rho\right]y^2\\ &-\left[\frac{a-\mu}{\phi_1}-\frac{C\delta_1\rho}{2}\right]z^2+k_1|\alpha(t)|(y^2+z^2).\end{aligned}\tag{8}$$

Taking $\omega=\dfrac{k}{k_1}$ and using the inequalities (3.6), (3.5), and (3.2) we obtain

$$\begin{aligned}W'_{(3)} =\ & e^{-\beta(t)}\left[V'_{(3)}-(\frac{k_1|\alpha(t)|}{k}+\frac{|c'(t)|}{c_0})V\right]\\ \le\ & e^{-\beta(t)}\Bigg[\mu c'(t)G(x,y)-\left(\frac{a-\mu}{\phi_1}-\frac{C\delta_1\rho}{2}\right)z^2\\ &-\left(\frac{\Delta}{2\phi_1^2}-\frac{C\delta_1}{2\phi_0}\left(\frac{\phi_0+\mu}{\phi_0^2(1-\sigma)}+\mu\right)\rho\right)y^2+k_1|\alpha(t)|(y^2+z^2)\\ &-\left(\frac{k_1|\alpha(t)|}{k}+\frac{|c'(t)|}{c}\right)\left(\mu cG(x,y)+k(y^2+z^2)\right)\Bigg].\end{aligned}$$

Since $c'(t) - |c'(t)| \leq 0$ we have

$$\begin{aligned} W'_{(3)} \;\; &\leq \;\; e^{-\beta(t)} \Bigg[- \left(\frac{a-\mu}{\phi_1} - \frac{C\delta_1\rho}{2} \right) z^2 \\ & - \left(\frac{\Delta}{2\phi_1^2} - \frac{C\delta_1}{2\phi_0} \left(\frac{\phi_0+\mu}{\phi_0^2(1-\sigma)} + \mu \right) \rho \right) y^2 \Bigg]. \end{aligned}$$

Therefore if we choose

$$\rho < \min \left\{ \frac{2(a-\mu)}{\phi_1 C\delta_1}, \frac{\phi_0^3\Delta(1-\sigma)}{C\delta_1\phi_1^2(\mu+\phi_0+\mu\phi_0^2(1-\sigma))} \right\},$$

then

$$W'_{(3)}(t, x_t, y_t, z_t) \leq -\gamma(y^2+z^2), \quad \text{for some} \quad \gamma > 0,$$

From (3), $W_3(\|X\|) = \gamma(y^2+z^2)$ is positive definite function. Hence, Lemma 2.4 guarantees that the trivial solution of Equation (1) is uniformly asymptotically stable and completes the proof of the theorem. □

We will now state our main results for the case $e \neq 0$.

Theorem 3.2.

If all the assumptions of Theorem 3.1 are satisfied then all solutions of equation (1) are bounded provided

$$\begin{aligned} & |e(t, x, x(t-r(t)), y, y(t-r(t)), z)| \leq |q(t)|, \quad \text{and} \\ & \int_0^t |q(s)|ds < \infty, \text{for all} \quad t \geq 0. \end{aligned}$$

Proof:

The proof of this theorem is similar to that of the proof of the theorem in Tunç (2009) and hence it is omitted. □

4. Example

We consider the following fourth order non-autonomous differential equation

$$\begin{aligned} & \left(\left(\frac{\sin^2(x)}{1+x^2} + 3 \right) x' \right)'' + \left(\frac{1}{4}e^{-t} + 3 \right) \left(\frac{\cos^2(y)}{1+y^2} + 8 \right) x'' \\ & + \left(\frac{1}{3}\cos t + \frac{193}{3} \right) \left(x' + \frac{2x'}{(1+x'^2)} \right) \\ & + \left(\frac{1}{4}\sin t + \frac{15}{4} \right) \left(x(t-r(t)) + \frac{x(t-r(t))}{1+x^2(t-r(t))} \right) \\ & = \frac{1}{1+t^2+x^2+x^2(t-r(t))+y^2+y^2(t-r(t))+z^2}. \end{aligned} \tag{9}$$

Take

$$\phi(x)=\frac{\sin^2(x)}{1+x^2}+3\ ,\ f(y)=y+\frac{2y}{(1+y^2)}\ ,\ \psi(y)=\frac{\cos^2(y)}{1+y^2}+8\ ,\ h(x)=\frac{x}{x^2+1}+x\ ,$$

$$a(t)=\frac{1}{4}e^{-t}+3\ ,\ b(t)=\frac{1}{3}\cos t+\frac{193}{3}\ ,\ c(t)=\frac{1}{4}\sin t+\frac{15}{4}$$

and

$$e(t,x,x(t-r(t)),y,y(t-r(t)),z)=\frac{1}{1+t^2+x^2+x^2(t-r(t))+y^2+y^2(t-r(t))+z^2}.$$

It can be seen that

$$\begin{aligned}
&3=a\le a(t)=\frac{1}{4}e^{-t}+3\le\frac{13}{4},\ \ |a'(t)|=|-\frac{1}{4}e^{-t}|\le\frac{1}{4},\ t\ge 0,\\
&64=b\le b(t)=\frac{1}{3}\cos t+\frac{193}{3}\le\frac{194}{3},\ 0\le|b'(t|=|\frac{1}{3}\sin t|\le\frac{1}{3},\ t\ge 0,\\
&\frac{7}{2}=c\le c(t)=\frac{1}{4}\sin t+\frac{15}{4}\le 4=C,\ -\frac{1}{4}\le c'(t)=\frac{1}{4}\cos t\le\frac{1}{4},\ t\ge 0,\\
&1=\delta\le\frac{h(x)}{x}=1+\frac{1}{1+x^2}\text{ with } x\neq 0,\ |h'(x)|\le\delta_1=2\text{ and }\mu=1,
\end{aligned}$$

$$\begin{aligned}
&1=\delta_0\le\frac{f(y)}{y}=1+\frac{2}{(1+y^2)}\le\delta_2=3\\
&1<8=\psi_0\le\psi(y)=\frac{\cos^2(y)}{1+y^2}+8\le\psi_1=9,\\
&3=\phi_0\le\phi(x)=\frac{\sin^2(x)}{1+x^2}+3\le\phi_1=4.
\end{aligned}$$

Easy computations show that conditions (H1) and (H2) are satisfied.
Indeed, $\dfrac{\phi_1\delta_1C}{b\delta_0}=\dfrac{1}{2}<\mu<a=3$. We have also

$$\frac{\mu\psi_1}{\phi_0^2}|a'(t)|+\frac{\delta_2}{\phi_0}|b'(t)|-\frac{\delta_1}{\mu}c'(t)\le\frac{\mu}{4}+\frac{1}{3}+\frac{1}{2\mu}=\frac{13}{12}<\frac{\mu b\delta_0-\phi_1C\delta_1}{\phi_1^2}=2.$$

It is straightforward to verify that

$$\begin{aligned}
\int_{-\infty}^{+\infty}|\phi'(u)|\,du\ &\le\ \int_{-\infty}^{+\infty}\left[\left|\frac{2\cos u\sin u}{1+u^2}\right|+\left|\frac{2u\sin^2 u}{(1+u^2)^2}\right|\right]du\\
&\le\ 2\pi+2.
\end{aligned}$$

We have

$$
\begin{aligned}
e(t,x,x(t-r(t)),y,y(t-r(t)),z) &= \frac{1}{1+t^2+x^2+x^2(t-r(t))+y^2+y^2(t-r(t))+z^2} \\
&\leq \frac{1}{1+t^2},
\end{aligned}
$$

and

$$\int_0^{+\infty} \frac{1}{1+t^2} dt < \infty.$$

All assumptions of Theorems 3.1 and 3.2 hold true, thus their conclusions also follow.

5. Conclusion

The problem of the stability and boundedness of solutions of differential equations is very important in the theory and applications of differential equations. In the present work, conditions were obtained for the stability and boundedness for certain third order non-linear non-autonomous differential equations with the variable delay. Using Lyapunov second or direct method, a Lyapunov functional was defined and used to obtain our results.

REFERENCES

Ademola, A.T. and Arawomo, P.O. (2013). Uniform stability and boundedness of solutions of nonlinear delay differential equations of third order, Math. J. Okayama Univ., Vol. 55, pp. 157-166.

Ademola, A.T., Arawomo, P.O., Ogunlaran, O.M. and Oyekan, E.A. (2013). Uniform stability, boundedness and asymptotic behavior of solutions of some third order nonlinear delay differential equations, Differential Equations and Control Processes, N4.

Bai, Y. and Guo, C. (2013). New results on stability and boundedness of third order nonlinear delay differential equations, Dynam. Systems Appl., Vol. 22 , No. 1, pp. 95-104.

Burton, T.A. (1985). Stability and periodic solutions of ordinary and functional differential equations, Mathematics in Science and Engineering, Vol. 178. Academic Press, Inc., Orlando,FL.

Burton, T.A. (2005). *Volterra Integral and Differential Equations*, Mathematics in Science and Engineering V (202), second edition.

Greaf, J.R. and Remili, M. (2012). Some properties of monotonic solutions of $x''' + p(t)x' + q(t)f(x) = 0$, Pan. American Mathematical Journal, Vol. 22, No. 2, pp. 31-39.

Hara, T. (1971). On the asymptotic behavior of solutions of certain of certain third order ordinary differential equations, Proc. Japan Acad., Vol. 47, pp. 03-908.

Krasovskii, N.N. (1963). *Stability of Motion*. Stanford Univ. Press.

Omeike, M.O. (2010). New results on the stability of solution of some non-autonomous delay differential equations of the third order, Differential Equations and Control Processes, Vol. 2010, No. 1, pp. 18–29.

Omeike, M.O. (2009). Stability and boundedness of solutions of some non-autonomous delay differential equation of the third order, An. Stiint. Univ. Al. I. Cuza Iasi. Mat.(N.S.), Vol. 55, No. 1, pp. 49-58.

Remili, M. and Oudjedi, D. L. (2014). stability and boundedness of the solutions of non autonomous third order differential equations with delay, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica, Vol. 53, No. 2, pp. 137-149.

Remili, M. and Beldjerd, D. (2014). On the asymptotic behavior of the solutions of third order delay differential equations, Rend. Circ. Mat. Palermo, Vol. 63, No. 3 , pp. 447-455.

Sadek, A. I. (2005). On the Stability of Solutions of Some Non-Autonomous Delay Differential Equations of the Third Order, Asymptot. Anal., Vol. 43, No. (1-2), pp. 1-7.

Sadek, A. I. (2003). Stability and Boundedness of a Kind of Third-Order Delay Differential System, Applied Mathematics Letters, Vol. 16, No. 5, pp. 657-662.

Swick, K. (1969). On the boundedness and the stability of solutions of some non-autonomous differential equations of the third order, J. London Math. Soc., Vol. 44 , pp. 347–359.

Tunç, C.(2007). Stability and boundedness of solutions of nonlinear differential equations of third order with delay, Journal Differential Equations and Control Processes, No. 3, pp. 1-13.

Tunç, C.(2009). Boundedness in third order nonlinear differential equations with bounded delay. Bol. Mat. Vol. 16, No. 1, pp. 1-10.

Tunç, C. (2010). On some qualitative behaviors of solutions to a kind of third order nonlinear delay differential equations, Electron. J. Qual. Theory Differ. Equ., No. 12, pp. 1-9.

Tunç, C. (2010). Some stability and boundedness conditions for non-autonomous differential equations with deviating arguments, E. J. Qualitative Theory of Diff. Equ., No. 1., pp. 1–12.

Tunç, C. (2010). On the stability and boundedness of solutions of nonlinear third order differential equations with delay, Filomat, Vol. 24, No. 3, pp. 1-10.

Tunç, C. (2013). Stability and boundedness of the nonlinear differential equations of third order with multiple deviating arguments, Afr. Mat., Vol. 24, No. 3, pp. 381-390.

Yoshizawa, T. (1966). *Stability theory by Liapunov's second method*, The Mathematical Society of Japan, Tokyo.

Zhu, Y.F. (1992). On stability, boundedness and existence of periodic solution of a kind of third order nonlinear delay differential system, Ann. Differential Equations, Vol. 8, No. 2, pp. 249-259.