



Finite Thermal Wave Propagation in a Half-Space Due to Variable Thermal Loading

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Received: July 9, 2013 ; Accepted: February 21, 2014

Abstract

The thermoelastic interaction for the dual-phase-lag (DP) heat conduction in a thermoelastic half space is studied in the light of two-temperature generalized thermoelasticity theory (2TT). The medium is assumed to be initially quiescent. Using Laplace transform, the fundamental equations are expressed in the form of a vector-matrix differential equation which is then solved by state-space approach. The obtained general solution is then applied to the mechanical loading and various types of thermal loading (the thermal shock and the ramp-type heating). The numerical inversion of the Laplace transforms are carried out by the method of Fourier series expansion technique. The numerical results are computed for copper like material. Significant dissimilarities between two models (the two-temperature Lord-Shulman (2TLS) and the two-temperature Dual-phase-lag model (2TDP)) are shown graphically. Because of the short duration of the second sound effect, the small-time solutions are analyzed and the discontinuities that occur at the wave fronts are also discussed. The effects of two-temperature and ramping parameters are studied.

Keywords: Two-temperature generalized thermoelasticity; Dual-phase-lag model; State-space approach; Vector matrix differential equation

MSC 2010 No.: 74F05

Nomenclature

- λ, μ Lamé's constant,
 ρ density,
 c_v specific heat at constant strain,
 t time,
 ϕ conductive temperature,
 θ thermodynamic temperature,
 α_t coefficient of linear thermal expansion,
 $\beta = \alpha_t(3\lambda + 2\mu)$,
 σ_{ij} components of stress tensor,
 e_{ij} components of strain tensor,
 u_i components of displacement vector,
 k thermal conductivity,
 τ_q phase lag of heat flux vector,
 τ_T phase-lag of temperature gradient,
 τ_0 relaxation time,
 $c_0 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ (longitudinal wave speed),
 $\eta = \frac{\rho c_v}{k}$ (thermal viscosity),
 a the two temperature parameter,
 $\omega = ac_0^2 \eta^2$ (dimensionless two temperature parameter),
 $\alpha = \frac{\beta \theta_0}{\lambda + 2\mu}$ (dimensionless mechanical coupling constant),
 $\Delta = \varepsilon_{kk}$ (dilatation),
 Q heat source,
 \bar{q} heat flux vector,
 $E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ (Young's modulus),
 $\nu = \frac{\lambda}{2(\lambda + \mu)}$ (Poisson's ratio),
 $\varepsilon = \frac{\beta}{\rho c_v}$ (thermoelastic coupling constant).

1. Introduction

The classical theory of thermoelasticity (CTE) involving infinite speed of propagation of thermal signals, contradicts physical facts. The theory of heat conduction derived from classical Fourier's law presumes heat to propagate with infinite speed. During the last five decades, non-classical theories involving finite speed of heat transportation in elastic solids have been developed to

remove this paradox. In contrast with the conventional coupled thermo-elasticity theory, which involves a parabolic-type heat transport equation, these generalized theories involving a hyperbolic-type heat transport equation are supported by experiments exhibiting the actual occurrence of wave-type heat transport in solids, called second sound effect. The first generalization of thermal relaxation time (single-phase-lag model) is known as extended thermoelasticity (ETE). In order to overcome the paradox of an infinite speed of thermal wave inherent in CTE and CCTE (classical coupled theory of thermoelasticity), efforts were made to modify coupled thermoelasticity, on different grounds to obtain a wave-type heat conduction equation by different researchers. Lord and Shulman (1967) have formulated the generalized thermoelasticity theory introducing one relaxation time in Fourier's law of heat conduction equation and thus transforming the heat conduction equation into a hyperbolic type. Uniqueness of the solution for this theory has been proved under different conditions by Dhaliwal and Sherief (1981) and Sherief (1987). The second generalization of the coupled thermoelasticity theory is due to Green and Lindsay (1972), which is known as temperature rate dependent thermoelasticity (TRDTE). The problem concerning this theory has been solved recently by Ghosh and Kanoria (2009), Kanoria and Ghosh (2010). Experimental study showed that the relaxation times can be of relevance in cases involving a rapidly propagating crack tip, a localized moving heat source with intensity, shock waves propagation, laser technique etc. Because of the experimental evidence in support of finiteness of heat propagation speed, the generalized thermoelasticity theories are considered to be more realistic than the conventional theories in dealing with practical problems involving large heat fluxes at short intervals like those occurring in laser units and energy channels. The third generalization is known as low-temperature thermoelasticity introduced by Hetnarski and Ignaczak (1993, 1994) called H-I theory. Most engineering materials such as metals possess a relatively high rate of thermal damping and thus are not suitable for use in experiments concerning second sound propagation. The fourth generalization is concerned with the thermoelasticity without energy dissipation (TEWED) introduced by Green and Naghdi (1991, 1992, 1993) and provides sufficient basic modifications in the constitutive equations that permit treatment of a much wider class of heat flow problems, labeled as types I, II, III. The natures of these three types of constitutive equations are such that when the respective theories are linearized, type-I is the same as the classical heat equation (based on Fourier's law) whereas the linearized versions of type-II and type-III theories permit propagation of thermal waves at finite speed. The entropy flux vector in type-II and type-III (i.e., thermoelasticity with energy dissipation (TEWED)) models are determined in terms of the potential that also determines stresses. When Fourier conductivity is dominant, then the temperature equation reduces to classical Fourier's law of heat conduction and when the effect of conductivity is negligible, then the equation has undamped thermal wave solutions without energy dissipation. Applying the above theories of generalized thermoelasticity, several problems have been solved by Chandrasekharaiah (1996a, 1996b), Bagri and Islami (2004), Roychoudhury and Dutta (2005), Mallik and Kanoria (2007), Kar and Kanoria (2009), Mallik and Kanoria (2009), Islam and Kanoria (2011) and Banik et al. (2009).

The fifth generalization of the thermoelasticity theory is known as dual-phase-lag thermoelasticity developed by Tzou (1995) and Chandrasekharaiah (1998). Tzou has considered microstructural effects into the delayed response in time in the macroscopic formulation by taking into account that increase of the lattice temperature is delayed due to phonon-electron interactions on the macroscopic level. Tzou introduced two-phase-lags to both the heat flux

vector and the temperature gradient. According to this model, classical Fourier's law $\vec{q} = -k\vec{\nabla}T$ has been replaced by $\vec{q}(P, t + \tau_q) = -k\vec{\nabla}T(P, t + \tau_T)$, where the temperature gradient $\vec{\nabla}T$ at a point P of the material at time $t + \tau_T$ corresponds to the heat flux vector \vec{q} at the same point at time $t + \tau_q$. Here k is the thermal conductivity of the material. The delay time τ_T is interpreted as that caused by the microstructural interactions and is called the phase-lag of the temperature gradient. The other delay time τ_q is interpreted as the relaxation time due to the fast transient effects of thermal inertia and is called the phase-lag of the heat flux. The case when $\tau_q = \tau_T = 0$, correspond to classical Fourier's law. If $\tau_q = \tau$ and $\tau_T = 0$, Tzou refers to the model as single-phase-lag model. Roychoudhury (2007) has studied one dimensional thermo-elastic wave propagation in an elastic half-space in the context of dual-phase-lag model. Recently, several researchers have attempted to solve their problems on the basis of the theory of Dual-phase-lag (DP) model. Quintanilla (2005, 2006, 2009) has solved several problems on the basis of this model. The exponential stability and condition of the delay parameters in the dual-phase-lag theory under this model have been studied by Quintanilla (2002, 2003). Wang and Mingtian (2002) have studied the well-posedness and solution structure of the dual-phase-lag heat conduction equation. Kumar, Prasad and Mukhopadhyay (2010) have studied the propagation of finite thermal wave in the context of dual-phase-lag model.

Theory of heat conduction in a deformable body, which depends on two different temperatures, the conductive temperature ϕ and the thermodynamic temperature θ , has been formulated by Chen and Gurtin (1968) and Chen et al (1968). The key element that sets the two temperature thermoelasticity (2TT) apart from the classical theory of thermoelasticity (CTE) is the material parameter a (≥ 0), called the temperature discrepancy (Chen and Gurtin; 1969). Specifically, if $a = 0$, then $\phi = \theta$ and the field equations of the 2TT reduce to those of one-temperature theory.

The linearized version of two-temperature theory (2TT) has been studied by many authors. Warren and Chen (1973) have investigated the wave propagation in the two-temperature theory of thermoelasticity. Leasn (1970) has established the uniqueness and also the reciprocity theorems for the 2TT. The existence, structural stability and and the spatial behavior of the solution in 2TT has been discussed by Quintanilla (2004). Puri and Jordan (2006) have studied the propagation of plane harmonic waves under the 2TT.

Youssef (2006) has developed the theory of two-temperature generalized thermoelasticity based on the Lord-Shulman (LS) model. Youssef and Al-Harby (2007) have solved a problem of an infinite body with a spherical cavity employing the two-temperature LS model by applying a state space approach. An half-space problem filled with an elastic material has been solved in the context of the two-temperature generalized thermoelasticity theory using the state space approach by Youssef and Al-Lehaibi (2007). Mukhopadhyay and Kumar (2009) have studied the thermoelastic interaction on two-temperature generalized thermoelasticity in an infinite medium with a cylindrical cavity. Variational and reciprocal principles have been studied by Kumar et al. (2010) and the effect of the thermal relaxation time on plane wave propagation under the two-temperature generalized thermoelasticity has been studied by Kumar and Mukhopadhyay (2010). Uniqueness and growth of solutions in two-temperature generalized thermoelastic theories have been studied by Magaïne and Quintanilla (2009). Banik and Kanoria (2011, 2012) have studied

the thermoelastic interactions in an infinite body with spherical cavity under 2TT. Also they have studied the effect of three-phase-lag model under this new theory (Banik, 2012). The thermoelastic interactions in an infinite body under 2TT in the context of fractional heat equation have been studied by Sur and Kanoria (2012, 2014).

A method for solving coupled thermoelastic problems by using the state space approach was developed by Bahar and Hetnerski (1978). The state space formulation for problems not containing heat sources has been made by Sherief and Anwar (1994). Sherief and Hamza (1994) have solved some two-dimensional problems and studied the wave propagation in this theory. El-Maghraby and Youssef (2004) have used the state space approach to solve a thermomechanical shock problem.

In this work we have investigated the thermoelastic stress, strain, displacement, conductive temperature and the thermodynamic temperature in an infinite isotropic elastic half space under thermal shock using the two-temperature generalized thermoelasticity theory in the context of two-temperature Lord-Shulman (2TLS) and two-temperature Dual-phase-lag (2TDP) models. The governing equations of two-temperature generalized thermoelasticity theory are transformed in the Laplace transform domain which are then solved using the state-space approach. The inversion of the transform solution is carried out numerically by applying a method based on a Fourier-series expansion technique (Honig and Hirdes; 1984). A complete and comprehensive analysis of the results has been presented for 2TLS and 2TDP models. These results have also been compared with those of the 2TLS model (Youssef, 2007). The effects of two-temperature and the comparisons between different models (Lord-Shulman (LS) model and Dual-Phase-lag (DP) model) have been studied.

2. Formulation of the Problem

We consider a homogeneous isotropic elastic half space $x \geq 0$ with stress free boundary which is subjected to a thermal shock. We assume that the body be initially at rest and the undisturbed state is maintained at an uniform reference temperature θ_0 . We shall consider one dimensional disturbance of the medium. Then all the thermophysical quantities can be taken as functions of x and t only. It follows, therefore, that the displacement components take the following form

$$u_x = u(x, t), \quad u_y = u_z = 0. \quad (1)$$

The strain component is given by

$$e = e_{xx} = \frac{\partial u}{\partial x}. \quad (2)$$

Stress-strain-temperature relation in the present problem is

$$\sigma = \sigma_{xx} = (\lambda + 2\mu)e - \beta\theta, \quad (3)$$

where λ and μ are Lamé's constants, $\beta = (3\lambda + 2\mu)\alpha_i$; α_i being the coefficient of linear thermal

expansion.

The equation of motion in absence of the body forces is

$$\frac{\partial \sigma}{\partial x} = \rho \ddot{u}, \quad (4)$$

which can be written as

$$\frac{\partial^2 \sigma}{\partial x^2} = \rho \ddot{e}. \quad (5)$$

The displacement equation of motion in absence of body forces is

$$(\lambda + 2\mu) \frac{\partial e}{\partial x} - \beta \frac{\partial \theta}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}. \quad (6)$$

The relation between the conductive temperature and the thermodynamic temperature is given by

$$\phi - \theta = a \nabla^2 \phi, \quad (7)$$

where $a (> 0)$ is the two-temperature parameter.

In the context of two-temperature generalized thermoelasticity based on the Dual-Phase-Lag model, the heat conduction equation is given by (Quintanilla and Jordan, 2009)

$$k \left(1 + \tau_T \frac{\partial}{\partial t} \right) \frac{\partial^2 \phi}{\partial x^2} = \left\{ 1 + \tau_q \frac{\partial}{\partial t} + \frac{1}{2} \tau_q^2 \frac{\partial^2}{\partial t^2} \right\} (\rho c_v \dot{\theta} + \beta \theta_0 \dot{e}), \quad (8)$$

where ρ is the density, k is the coefficient of thermal conductivity, θ_0 is the reference thermodynamic temperature, c_v is the specific heat at constant strain, τ_q is the phase lag of heat flux vector, τ_T is the phase lag of temperature gradient.

Note that for $\tau_T = 0$ and neglecting the term $\tau_q^2 = 0$, we have two-temperature Lord-Shulman (2TLS) model.

We use the following non-dimensional variables

$$x' = c_0 \eta x, \quad t' = c_0^2 \eta t, \quad (\tau_T', \tau_q', \tau_0') = c_0^2 \eta (\tau_T, \tau_q, \tau_0), \quad \sigma' = \frac{\sigma}{\lambda + 2\mu}, \quad \theta' = \frac{\theta}{\theta_0}, \quad \phi' = \frac{\phi}{\phi_0},$$

where

$$c_0^2 = \left(\frac{\lambda + 2\mu}{\rho} \right) \quad \text{and} \quad \eta = \frac{\rho c_v}{k}.$$

Hence, we have

$$\left(1 + \tau_r \frac{\partial}{\partial t}\right) \frac{\partial^2 \phi}{\partial x^2} = \left\{1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2}{\partial t^2}\right\} (\dot{\theta} + \varepsilon \dot{e}), \quad (9)$$

$$\sigma = e - \alpha \theta, \quad (10)$$

$$\frac{\partial^2 \sigma}{\partial x^2} = \ddot{e}, \quad (11)$$

$$\phi - \theta = \omega \nabla^2 \phi, \quad (12)$$

where

$$\omega = ac_0^2 \eta^2, \quad \varepsilon = \frac{\beta}{\rho c_v}, \quad \alpha = \frac{\beta \theta_0}{\lambda + 2\mu}.$$

The initial and the regularity conditions are given by

$$u = \theta = \phi = 0 \quad \text{at } t = 0 \quad \text{for } x \geq 0,$$

$$\frac{\partial u}{\partial t} = \frac{\partial \theta}{\partial t} = \frac{\partial \phi}{\partial t} = 0 \quad \text{at } t = 0,$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 \theta}{\partial t^2} = \frac{\partial^2 \phi}{\partial t^2} = 0 \quad \text{at } t = 0,$$

$$u = \theta = \phi = 0 \quad \text{as } x \rightarrow \infty.$$

The problem is to solve the equations (9)- (12) subject to the following boundary conditions

(i) Thermal boundary condition

The boundary plane $x = 0$ is subjected to a thermal loading as follows

$$\phi(0, t) = F(t). \quad (13)$$

(ii) Mechanical boundary condition

The boundary plane $x = 0$ is free of traction, i.e., we have

$$\sigma(0, t) = \sigma_0 = 0. \quad (14)$$

3. Method of Approach

Applying the Laplace transform defined by the relation

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad \text{Re}(s) > 0, \quad (15)$$

to equations (9)–(12), we obtain

$$\frac{d^2 \bar{\phi}}{dx^2} = a_3(\bar{\theta} + \varepsilon \bar{e}), \quad (16)$$

$$\bar{\sigma} = \bar{e} - \alpha \bar{\theta}, \quad (17)$$

$$\frac{d^2 \bar{\sigma}}{dx^2} = s^2 \bar{e}, \quad (18)$$

$$\bar{\phi} - \bar{\theta} = \omega \frac{d^2 \bar{\phi}}{dx^2}, \quad (19)$$

where

$$a_3 = \frac{s \left(1 + \tau_q s + \frac{\tau_q^2}{2} s^2 \right)}{(1 + \tau_r s)}.$$

Eliminating \bar{e} and $\bar{\theta}$ from (16)–(19), we obtain

$$\frac{d^2 \bar{\phi}}{dx^2} = L_1 \bar{\phi} + L_2 \bar{\sigma}, \quad (20)$$

and

$$\frac{d^2 \bar{\sigma}}{dx^2} = M_1 \bar{\phi} + M_2 \bar{\sigma}, \quad (21)$$

where

$$L_1 = \frac{a_3(1 + \alpha \varepsilon)}{1 + a_3 \omega + a_3 \alpha \varepsilon \omega}, \quad L_2 = \frac{a_3 \varepsilon}{1 + a_3 \omega + a_3 \alpha \varepsilon \omega},$$

$$M_1 = \alpha s^2(1 - \omega L_1) \quad \text{and} \quad M_2 = s^2(1 - \alpha \omega L_2).$$

Equations (20) and (21) can be written in a vector-matrix differential equation as follows

$$\frac{d^2 \bar{V}(x, s)}{dx^2} = A(s) \bar{V}(x, s), \quad (22)$$

where

$$\bar{V}(x, s) = \begin{pmatrix} \bar{\phi}(x, s) \\ \bar{\sigma}(x, s) \end{pmatrix}, \quad (23)$$

and

$$A(s) = \begin{pmatrix} L_1 & L_2 \\ M_1 & M_2 \end{pmatrix}. \quad (24)$$

3.1. State-space approach

As a solution of (22), we may take

$$\bar{V}(x, s) = \exp[-\sqrt{A(s)}x]\bar{V}(0, s), \quad (25)$$

where

$$\bar{V}(0, s) = \begin{pmatrix} \bar{\phi}(0, s) \\ \bar{\sigma}(0, s) \end{pmatrix} = \begin{pmatrix} \bar{\phi}_0 \\ \bar{\sigma}_0 \end{pmatrix}.$$

The characteristic equation of the matrix $A(s)$ takes the form (Youssef; 2007)

$$k^2 - k(L_1 + M_2) + (L_1M_2 - L_2M_1) = 0, \quad (26)$$

where, the roots k_1 and k_2 satisfy

$$k_1 + k_2 = L_1 + M_2, \quad (27)$$

$$k_1k_2 = L_1M_2 - L_2M_1. \quad (28)$$

Now, the spectral decomposition of $A(s)$ is

$$A(s) = k_1E_1 + k_2E_2,$$

where E_1 and E_2 are called the projections of $A(s)$ and satisfy the following conditions

$$E_1 + E_2 = I, \quad I \text{ being the identity matrix,}$$

$$E_1E_2 = \text{zero matrix,}$$

$$E_i^2 = E_i \quad \text{for } i = 1, 2.$$

The Taylor series expansion of the matrix exponential in equation (25) is given by

$$\exp[-\sqrt{A(s)}x] = \sum_{n=0}^{\infty} \frac{[-\sqrt{A(s)}x]^n}{n!}. \quad (29)$$

Then, we have

$$\sqrt{A(s)} = \sqrt{k_1}E_1 + \sqrt{k_2}E_2,$$

where

$$E_1 = \frac{1}{k_1 - k_2} \begin{pmatrix} L_1 - k_2 & L_2 \\ M_1 & M_2 - k_2 \end{pmatrix} \quad (30)$$

and

$$E_2 = \frac{1}{k_1 - k_2} \begin{pmatrix} k_1 - L_1 & -L_2 \\ -M_1 & k_1 - M_2 \end{pmatrix}. \quad (31)$$

Therefore, we get

$$B(s) = \sqrt{A(s)} = \frac{1}{\sqrt{k_1} + \sqrt{k_2}} \begin{pmatrix} L_1 + \sqrt{k_1 k_2} & L_2 \\ M_1 & M_2 + \sqrt{k_1 k_2} \end{pmatrix}. \quad (32)$$

Using Cayley-Hamilton theorem, we can express B^2 and higher powers of the matrix B in terms of I and B , where I is the identity matrix of order two.

Thus the infinite series in equation (29) can be reduced to the following form

$$\exp[-B(s)x] = a_0(x, s)I + a_1(x, s)B(s), \quad (33)$$

where, a_0 and a_1 are coefficients depending on x and s .

To find the matrix $\exp[-B(s)x]$, we now apply Cayley-Hamilton theorem. The characteristic equation of the matrix $B(s)$ can be written as follows

$$m^2 - m(\sqrt{k_1} + \sqrt{k_2}) + \sqrt{k_1 k_2} = 0,$$

where the roots of the equation, taken as m_1, m_2 , are as follows

$$m_1 = \sqrt{k_1} \quad \text{and} \quad m_2 = \sqrt{k_2}.$$

The characteristic roots k_1 and k_2 of the matrix $A(s)$ must satisfy equation (33). Thus, we have

$$\exp(-m_1 x) = a_0 + a_1 m_1, \quad (34)$$

and

$$\exp(-m_2 x) = a_0 + a_1 m_2. \quad (35)$$

On solving these two equations, we obtain

$$a_0 = \frac{m_1 e^{-m_2 x} - m_2 e^{-m_1 x}}{m_1 - m_2}, \quad (36)$$

$$a_1 = \frac{e^{-m_1 x} - e^{-m_2 x}}{m_1 - m_2}. \quad (37)$$

Hence, from equation (33) we have

$$\exp[-B(s)x] = L_{ij}(x, s), \quad i, j = 1, 2, \quad (38)$$

where,

$$L_{11} = \frac{e^{-m_2 x}(m_1^2 - L_1) - e^{-m_1 x}(m_2^2 - L_1)}{m_1^2 - m_2^2}, \quad (39)$$

$$L_{12} = \frac{L_2(e^{-m_1 x} - e^{-m_2 x})}{m_1^2 - m_2^2}, \quad (40)$$

$$L_{21} = \frac{M_1(e^{-m_1 x} - e^{-m_2 x})}{m_1^2 - m_2^2}, \quad (41)$$

$$L_{22} = \frac{e^{-m_1 x}(m_2^2 - M_2) - e^{-m_2 x}(m_1^2 - M_2)}{m_2^2 - m_1^2}. \quad (42)$$

Now, we can write equation (25) as

$$\bar{V}(x, s) = L_{ij}(x, s)\bar{V}(0, s). \quad (43)$$

The solutions $\bar{\phi}$ and $\bar{\sigma}$ can be obtained from equation (43) using boundary conditions (20) and (21) as follows

$$\bar{\phi} = \frac{\bar{F}(s) \left[(m_1^2 - L_1)e^{-m_2 x} - (m_2^2 - L_1)e^{-m_1 x} \right]}{(m_1^2 - m_2^2)}, \quad (44)$$

$$\bar{\sigma} = \frac{\bar{F}(s)M_1 \left[e^{-m_1x} - e^{-m_2x} \right]}{(m_1^2 - m_2^2)}. \quad (45)$$

By using equations (44) in equation (19) , we get

$$\bar{\theta} = \frac{\bar{F}(s) \left[(m_1^2 - L_1)(1 - \omega m_2^2) e^{-m_2x} - (m_2^2 - L_1)(1 - \omega m_1^2) e^{-m_1x} \right]}{(m_1^2 - m_2^2)}, \quad (46)$$

and, from equation (17), using (45) and (46) we obtain

$$\bar{e} = \frac{\bar{F}(s) \left[e^{-m_2x} \left\{ (m_1^2 - L_1)(1 - \omega m_2^2)\alpha - M_1 \right\} - e^{-m_1x} \left\{ (m_2^2 - L_1)(1 - \omega m_1^2)\alpha - M_1 \right\} \right]}{(m_1^2 - m_2^2)}. \quad (47)$$

Further, from equation (18), we obtain the displacement in the following form

$$\bar{u} = \frac{1}{s^2} \frac{\partial \bar{\sigma}}{\partial x}. \quad (48)$$

Substituting from equation (45) into equation (48), we obtain

$$\bar{u} = \frac{-M_1 \bar{F}(s) (m_1 e^{-m_1x} - m_2 e^{-m_2x})}{(m_1^2 - m_2^2)}. \quad (49)$$

Equations (44)-(47) and (49) together constitute the complete solution of the problem in the Laplace transform domain.

3.2. Thermal shock problem

We consider the thermal loading on the bounding plane $x=0$ in the thermal shock form as follows:

$$F(t) = \begin{cases} \phi_0, & t > 0, \\ 0, & t < 0. \end{cases} \quad (50)$$

Thus, taking the Laplace transform, we have

$$\bar{\phi}_0 = \bar{F}(s) = \frac{\phi_0}{s}. \quad (51)$$

3.3. Ramp-Type Heating

We consider the thermal loading of the bounding plane $x=0$ in the ramp-type form as follows:

$$F(t) = \begin{cases} 0, & t \leq 0, \\ \frac{F_0}{t_0} t, & 0 < t \leq t_0, \\ F_0, & t > t_0, \end{cases} \quad (52)$$

where F_0 is a constant, and t_0 is the ramp-type parameter. After making dimensionless and using the Laplace transform, we have

$$\bar{\phi}_0 = \bar{F}(s) = \frac{F_0(1 - e^{-st_0})}{t_0 s^2}. \quad (53)$$

3.4. Mechanical Boundary Condition

Since the boundary plane $x = 0$ is free of traction, we have

$$\bar{\sigma}(0, s) = 0. \quad (54)$$

Thus, we get the complete solution to the ramp-type problem in the Laplace transform domain using (53)-(54) into (44)-(47) and (49).

3.5. Derivation of Small-Time Solutions

From equations (6)-(8), we have the following equation in Laplace transform domain satisfied by \bar{e} and $\bar{\phi}$ as

$$\left[(N + M\omega + \alpha\epsilon M\omega)D^4 - (s^2N + M + \alpha\epsilon M + s^2M\omega)D^2 + s^2M \right] (\bar{\phi}, \bar{e}) = 0, \quad (55)$$

where

$$M = s \left(1 + \tau_q + \frac{1}{2} \tau_q^2 s^2 \right), \quad N = 1 + \tau_T s,$$

where $m_{1,2}$ are the roots with positive real part of the equation

$$(N + M\omega + \alpha\epsilon M\omega)m^4 - (s^2N + M + \alpha\epsilon M + s^2M\omega)m^2 + s^2M = 0. \quad (56)$$

The roots of the biquadratic equation are given by

$$m_{1,2}^2 = \frac{1}{2(N + M\omega + \alpha\varepsilon M\omega)} \left[s^2(N + M\omega) + M(1 + \alpha\varepsilon) \pm \left\{ \left[s^2(N + M\omega) + M(1 + \alpha\varepsilon) \right]^2 - 4Ms^2(N + M\omega + \alpha\varepsilon M\omega) \right\}^{\frac{1}{2}} \right]. \quad (57)$$

Clearly the roots of the given equation (57) are real if s is real. Taking positive sign in (57), we have for large s ,

$$m_1 \cong \frac{s}{v_1} + \frac{1}{2} \frac{\lambda_2}{\lambda_1} \frac{1}{v_1} + \frac{1}{2v_1} \left(\frac{\lambda_3}{\lambda_1} - \frac{1}{4} \frac{\lambda_3^2}{\lambda_1^2} \right) \frac{1}{s}, \quad (58)$$

Taking the negative sign in (57), we have for large s

$$m_2 \cong \frac{s}{v_2} + \frac{1}{2} \frac{\mu_2}{\mu_1} \frac{1}{v_2} + \frac{1}{2v_2} \left(\frac{\mu_3}{\mu_1} - \frac{1}{4} \frac{\mu_3^2}{\mu_1^2} \right) \frac{1}{s}, \quad (59)$$

where

$$\lambda_1 = A + \sqrt{A^2 - 4F}, \quad \lambda_2 = B + \frac{L_1}{2} \sqrt{A^2 - 4F} - \frac{2}{\tau_q} \lambda_1, \quad v_1 = \tau_q \sqrt{\frac{1 + \alpha\varepsilon}{\lambda_1}}, \quad v_2 = \tau_q \sqrt{\frac{1 + \alpha\varepsilon}{\mu_1}},$$

$$\lambda_3 = C + \frac{1}{8} \sqrt{A^2 - 4F} (4L_2 - L_1^2) - \frac{2}{\tau_q} \left(B + \frac{L_1}{2} \sqrt{A^2 - 4F} \right) + \lambda_1 \left\{ \frac{2}{\tau_q^2} - \frac{2\tau_r}{\tau_q^2(1 + \alpha\varepsilon)} \right\},$$

$$L_1 = \frac{2AB - 2AE\tau_q}{A^2 - 4F}, \quad L_2 = \frac{2AC - 8AE}{A^2 - 4F}, \quad A = \frac{1}{2} \omega \tau_q^2, \quad B = \omega \tau_q, \quad C = \frac{1}{2} \tau_q^2 (1 + \alpha\varepsilon) + \omega,$$

$$D = 1 + \tau_r + \tau_q(1 + \alpha\varepsilon), \quad E = 1 + \alpha\varepsilon, \quad F = \frac{1}{4} \tau_q^4 \omega (\alpha\varepsilon - 1),$$

$$\mu_1 = A - \sqrt{A^2 - 4F}, \quad \mu_2 = B - \frac{L_1}{2} \sqrt{A^2 - 4F} - \frac{2}{\tau_q} \lambda_1,$$

$$\mu_3 = C - \frac{1}{8} \sqrt{A^2 - 4F} (4L_2 - L_1^2) - \frac{2}{\tau_q} \left(B - \frac{L_1}{2} \sqrt{A^2 - 4F} \right) + \mu_1 \left\{ \frac{2}{\tau_q^2} - \frac{2\tau_r}{\tau_q^2(1 + \alpha\varepsilon)} \right\}.$$

Clearly, $\lambda_1, \mu_1 > 0$ since $A > \sqrt{A^2 - 4F}$ and $F > 0$. Further $\lambda_1 > \mu_1$ implies $v_2 > v_1$. Now, we have to prove that, under suitable restrictions on material constants, λ_2 and μ_2 are positive.

Now,

$$\lambda_2 > 0 \quad \text{if} \quad B + \frac{L_1}{2} \sqrt{A^2 - 4F} > \frac{2}{\tau_q} (A + \sqrt{A^2 - 4F}),$$

that is, if

$$B > \frac{2A}{\tau_q} + \frac{2}{\tau_q} \sqrt{A^2 - 4F} \left(1 - \frac{L_1 \tau_q}{4} \right).$$

Similarly,

$$\mu_2 > 0 \quad \text{if} \quad B > \frac{2A}{\tau_q} - \frac{2}{\tau_q} \sqrt{A^2 - 4F} \left(1 - \frac{L_1 \tau_q}{4} \right).$$

We impose the restriction on material parameters such that $1 > \frac{L_1 \tau_q}{4}$. This leads to the inequality

$$A^2 - 4F > \frac{1}{2} A \tau_q (B - \tau_q E).$$

Since λ_2 and $\mu_2 > 0$, we have

$$m_1 \cong \frac{s}{v_1} + \frac{1}{2} \frac{\lambda_2}{\lambda_1} \frac{1}{v_1} \quad \text{and} \quad m_2 \cong \frac{s}{v_2} + \frac{1}{2} \frac{\mu_2}{\mu_1} \frac{1}{v_2}.$$

For the thermal shock problem, we have the solutions of the stress and strain component in Laplace transform domain as follows

$$\bar{\sigma}(x, s) = \phi_0 \left\{ \frac{e^{-m_2 x} M_1}{s(m_2^2 - m_1^2)} - \frac{e^{-m_1 x} M_1}{s(m_2^2 - m_1^2)} \right\}, \quad (60)$$

$$\bar{e}(x, s) = \phi_0 \left[\frac{e^{-m_1 x} \{ (m_2 - L_1)(1 - \omega m_1) \alpha - M_1 \}}{s(m_2^2 - m_1^2)} - \frac{e^{-m_2 x} \{ (m_1 - L_1)(1 - \omega m_2) \alpha - M_1 \}}{s(m_2^2 - m_1^2)} \right], \quad (61)$$

For large s , we obtain the following results after simplification

$$M_1 \cong \delta_1 s + \delta_2 s^2,$$

$$\frac{M_1}{s(m_2^2 - m_1^2)} \cong \frac{\delta_2}{L_0} \frac{1}{s} + \frac{1}{s^2} \left(\frac{\delta_1}{L_0} - \frac{\delta_2 M_0}{L_0^2} \right),$$

$$\frac{(m_2^2 - L_1)(1 - \omega m_1^2) \alpha - M_1}{s(m_2^2 - m_1^2)} \cong \frac{\alpha}{L_0 v_2 s^2} - \frac{\omega}{v_1 v_2 s},$$

$$\frac{(m_1^2 - L_1)(1 - \omega m_2^2)\alpha - M_1}{s(m_2^2 - m_1^2)} \cong \frac{\alpha}{L_0 v_1 s^2} - \frac{\omega}{v_1 v_2 s},$$

where

$$L_0 = \frac{1}{v_2^2} - \frac{1}{v_1^2}, \quad M_0 = \frac{\mu_2}{\mu_1 v_2^2} - \frac{\lambda_2}{\lambda_1 v_1^2}, \quad N_0 = \frac{1}{4} \left(\frac{\mu_2^2}{\mu_1^2 v_2^2} - \frac{\lambda_2^2}{\lambda_1^2 v_1^2} \right),$$

$$\delta_1 = \frac{\alpha}{\tau_T + (1 + \alpha \varepsilon) \tau_q}, \quad \delta_2 = \frac{\alpha \tau_T}{\tau_T + (1 + \alpha \varepsilon) \tau_q}.$$

Finally, we obtain the solutions in Laplace transform domain as follows

$$\bar{\sigma}(x, s) = \phi_0 \left[\left\{ \frac{\delta_2}{L_0} \frac{1}{s} + \left(\frac{\delta_1}{L_0} - \frac{\delta_2 M_0}{L_0^2} \right) \frac{1}{s^2} \right\} e^{-\left(\frac{s}{v_2} + \frac{\mu_2}{2\mu_1 v_2} \right) x} - \left\{ \frac{\delta_2}{L_0} \frac{1}{s} + \left(\frac{\delta_1}{L_0} - \frac{\delta_2 M_0}{L_0^2} \right) \frac{1}{s^2} \right\} e^{-\left(\frac{s}{v_1} + \frac{\lambda_2}{2\lambda_1 v_1} \right) x} \right], \quad (62)$$

$$\bar{e}(x, s) = \phi_0 \left[\left(\frac{\alpha}{L_0 v_2 s^2} - \frac{\omega}{v_1 v_2 s} \right) e^{-\left(\frac{s}{v_1} + \frac{\lambda_2}{2\lambda_1 v_1} \right) x} - \left(\frac{\alpha}{L_0 v_1 s^2} - \frac{\omega}{v_1 v_2 s} \right) e^{-\left(\frac{s}{v_2} + \frac{\mu_2}{2\mu_1 v_2} \right) x} \right]. \quad (63)$$

Taking inverse Laplace transform, we have the solutions in space-time domain as follows

$$\sigma(x, t) = \phi_0 \left[e^{-\frac{\mu_2 x}{2\mu_1 v_2}} \left\{ \frac{\delta_2}{L_0} + \left(\frac{\delta_1}{L_0} - \frac{\delta_2 M_0}{L_0^2} \right) \left(t - \frac{x}{v_2} \right) \right\} H \left(t - \frac{x}{v_2} \right) - e^{-\frac{\lambda_2 x}{2\lambda_1 v_1}} \left\{ \frac{\delta_2}{L_0} + \left(\frac{\delta_1}{L_0} - \frac{\delta_2 M_0}{L_0^2} \right) \left(t - \frac{x}{v_1} \right) \right\} H \left(t - \frac{x}{v_1} \right) \right]. \quad (64)$$

$$e(x, t) = \phi_0 \left[e^{-\frac{\lambda_2 x}{2\lambda_1 v_1}} \left\{ \frac{\alpha}{L_0 v_2} \left(t - \frac{x}{v_1} \right) - \frac{\omega}{v_1 v_2} \right\} H \left(t - \frac{x}{v_1} \right) - e^{-\frac{\mu_2 x}{2\mu_1 v_2}} \left\{ \frac{\alpha}{L_0 v_1} \left(t - \frac{x}{v_2} \right) - \frac{\omega}{v_1 v_2} \right\} H \left(t - \frac{x}{v_2} \right) \right]. \quad (65)$$

The short-time solutions for stress and strain components reveal the existence of two waves. Each of σ and e is composed of two parts and the each part corresponds to a wave propagating with a finite speed. In the stress component, the speed of the wave corresponding to the first part is v_2 and that corresponding to the second part is v_1 whereas in the strain component, the speed of the wave corresponding to the first part is v_1 and that corresponding to the second part is v_2 . The faster wave has its speed equal to v_2 and the slower wave has its speed equal to v_1 . Since $v_1 < v_2$, the faster wave is the predominantly modified Tzou wave (T-wave) and the slower is a predominantly modified elastic wave (E-wave). The first term of the solutions represents the contribution of the E-wave near its wave-front $x = v_1 t$ and the second term represents the contribution of the T-wave near its wave front $x = v_2 t$. We observe also that both the waves experience decay exponentially with the distance (attenuation). We further note that σ and e are identically zero for $x > v_2 t$. This implies that at a given instant of time t^* , the points of the solid

$x > 0$ that are beyond the faster wave front $x = v_2 t^*$ do not experience any disturbances. This observation confirms that like other thermoelasticity theories, the two-temperature dual-phase-lag theory is also a wave thermoelasticity theory. Moreover, it is interesting to record that at a given instant, the region $0 < x < v_2 t^*$ is the domain of influence of the disturbance, contrary to the result that this domain extends and the effects occur instantaneously everywhere in the solid as predicted by the classical thermoelasticity theory.

3.6. Numerical Inversion of Laplace Transform

It is difficult to find the analytical inverse Laplace transform of the complicated solutions for the displacement, thermodynamic temperature, conductive temperature and stress in Laplace transform domain. So we have to resort to numerical computations. We now outline the numerical procedure to solve the problem. Let $\bar{f}(x, s)$ be the Laplace transform of a function $f(x, t)$.

Then the inversion formula for Laplace transform can be written as

$$f(x, t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{st} \bar{f}(x, s) ds, \quad (66)$$

where d is an arbitrary real number greater than real parts of all the singularities of $\bar{f}(x, s)$. Taking $s = d + iw$, the preceding integral takes the form

$$f(x, t) = \frac{e^{dt}}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} f(x, d + i\omega) d\omega. \quad (67)$$

Expanding the function $h(x, t) = e^{-dt} f(x, t)$ in a Fourier series in the interval $[0, 2T]$, we obtain the approximate formula (Honig and Hirdes; 1984)

$$f(x, t) = f_{\infty}(x, t) + E_D, \quad (68)$$

where

$$f_{\infty}(x, t) = \frac{1}{2} c_0 + \sum_{k=1}^{\infty} c_k \quad \text{for } 0 \leq t \leq 2T \quad (69)$$

and

$$c_k = \frac{e^{dt}}{T} \left[e^{\frac{ik\pi t}{T}} \bar{f} \left(x, d + \frac{ik\pi t}{T} \right) \right]. \quad (70)$$

The discretization error E_D can be made arbitrarily small by choosing d large enough (Honig and Hirdes; 1984). Since the infinite series in equation (69) can be summed upto a finite number

N of terms, the approximate value of $f(x,t)$ becomes

$$f_N(x,t) = \frac{1}{2}c_0 + \sum_{k=1}^N c_k, \quad \text{for } 0 \leq t \leq 2T. \quad (71)$$

Using the preceding formula to evaluate $f(x,t)$ we introduce a truncation error E_T that must be added to the discretization error to produce total approximation error.

Two methods are used to reduce the total error. First, the ‘Korrektur’ method is used to reduce the discretization error. Next, the ε -algorithm is used to accelerate convergence (Honig and Hirdes; 1984).

The Korrektur method uses the following formula to evaluate the function $f(x,t)$ as follows (Honig; 1984)

$$f(x,t) = f_\infty(x,t) - e^{-2dT} f_\infty(x, 2T+t) + E'_D, \quad (72)$$

where the discretization error $|E'_D| \ll |E_D|$. Thus, the approximate value of $f(x,t)$ becomes

$$f_{NK}(x,t) = f_N(x,t) - e^{-2dT} f_{N'}(x, 2T+t), \quad (73)$$

where N' is an integer such that $N' < N$.

We shall now describe the ε -algorithm that is used to accelerate the convergence of the series in equation (71). Let $N = 2q+1$, where q is a natural number and let $s_m = \sum_{k=1}^m c_k$ be the sequence of partial sums of series in (71).

We define the ε -sequence by

$$\varepsilon_{0,m} = 0, \quad \varepsilon_{1,m} = s_m$$

and

$$\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + \frac{1}{\varepsilon_{p,m+1} - \varepsilon_{p,m}}, \quad p = 1, 2, 3, \dots$$

It can be shown that the sequence

$$\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \dots, \varepsilon_{N,1}$$

converges to $f(x,t) + E_D - \frac{c_0}{2}$ faster than the sequence of partial sums $s_m, m = 1, 2, 3, \dots$

The actual procedure used to invert the Laplace transform consists of using equation (73) together with the ε -algorithm. The values of d and T are chosen according to the criterion outlined in (Honig and Hirdes; 1984).

4. Numerical Results and Discussion

To get the solution for the conductive temperature, thermodynamic temperature, displacement and the thermal stress in the space-time domain, we have to apply numerical inversion of the Laplace transform. This has been done using a method based on the Fourier series expansion technique as mentioned above. The numerical code has been prepared using *Fortran-77* programming language. For the purpose of illustration, here we have used the copper material. The material constants are as follows (Youssef and Lehaibi; 2007)

$$\begin{aligned} k &= 386 \text{N/K s}, & \alpha_T &= 1.78(10)^{-5} \text{K}^{-1}, & c_v &= 383.1 \text{m}^2/\text{K}, & \eta &= 8886.73 \text{m/s}^2, \\ \mu &= 3.86(10)^{10} \text{N/m}^2, & \lambda &= 7.76(10)^{10} \text{N/m}^2, & \rho &= 8954 \text{kg/m}^3, & \tau_T &= 0.015, \\ \tau_q &= 0.02, & \theta_0 &= 293 \text{K}, & \varepsilon &= 0.0168, & \alpha &= 0.0104, & F_0 &= 1. \end{aligned}$$

The temperature, the stress, the displacement and the strain distributions are represented graphically to study the effect of the two-temperature parameter.

The numerical values of the conductive temperature, thermodynamic temperature, stress and strain have been calculated for a fixed time $t = 0.2$ and for x ranging from $x = 0.0$ to $x = 2.2$.

Figures 1-4 are representing the differences between the theory of one-temperature ($\omega = 0.0$) and two-temperature ($\omega = 0.1$) using Lord -Shulman (LS) and Dual-phase-lag (DP) models for $t = 0.2$. In these figures, continuous lines represent the variations corresponding to the one-temperature theory whereas the dotted line corresponds to the two-temperature theory.

Figure 1 exhibits the variation of conductive temperature (ϕ) with distance (x) for both one-temperature ($\omega = 0.0$) and two-temperature ($\omega = 0.1$) theories. It is observed that ϕ corresponding to both $\omega = 0.0$ and $\omega = 0.1$ satisfy our theoretical boundary condition as laid down in equation (13). This fact establishes the correctness of the numerical codes prepared.

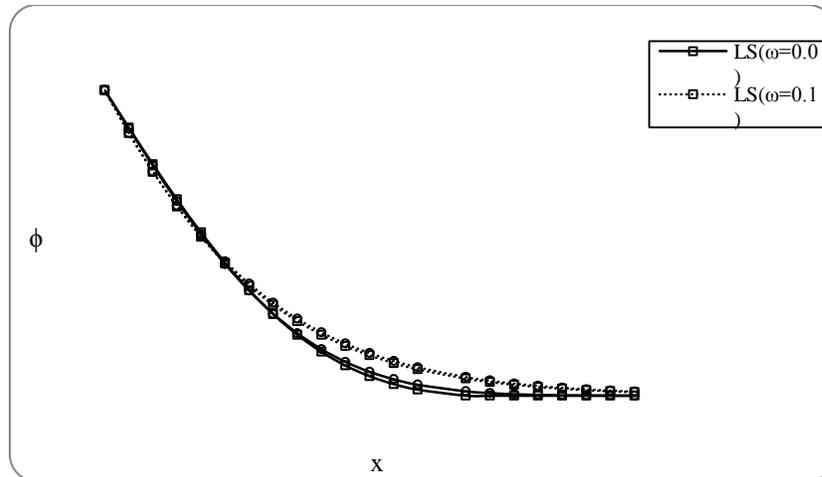


Figure 1. Variation of conductive temperature (ϕ) vs. x and $t = 0.2$ for thermal shock problem

Also, it is observed that ϕ decreases with an increase of the distance and finally vanishes for both the one-temperature and the two-temperature theory. But, the presence of $\omega(=0.1)$ decelerates ϕ to vanish as compared to when $\omega=0.0$. Also, for the Dual-phase-lag model, the decay of ϕ is slower than that of the LS model.

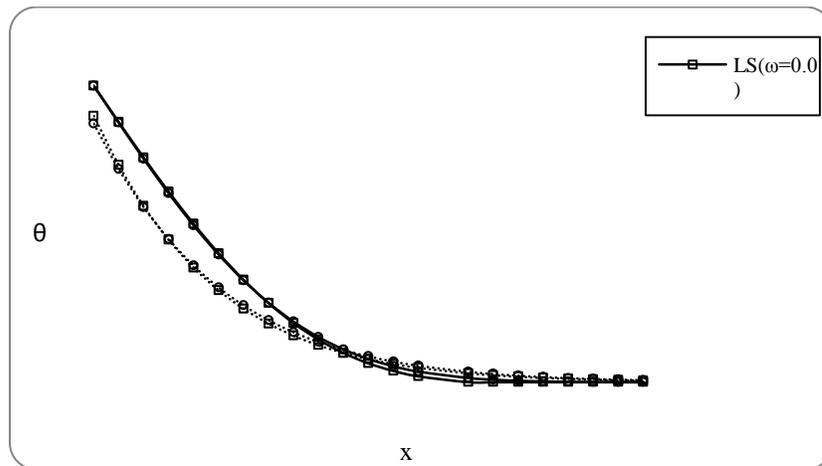


Figure 2. Variation of thermodynamic temperature (θ) vs. x and $t = 0.2$ for thermal shock problem

Figure 2 gives the variation of the thermodynamic temperature (θ) with distance (x) for the one-temperature ($\omega=0.0$) and the two-temperature ($\omega=0.1$) theories. As seen from the figure, the conductive temperature is maximum in magnitude for one-temperature theory for both the models near the bounding plane $x=0$. Here, the rate of decay of θ is slower for the two-temperature theory than for the one-temperature theory.

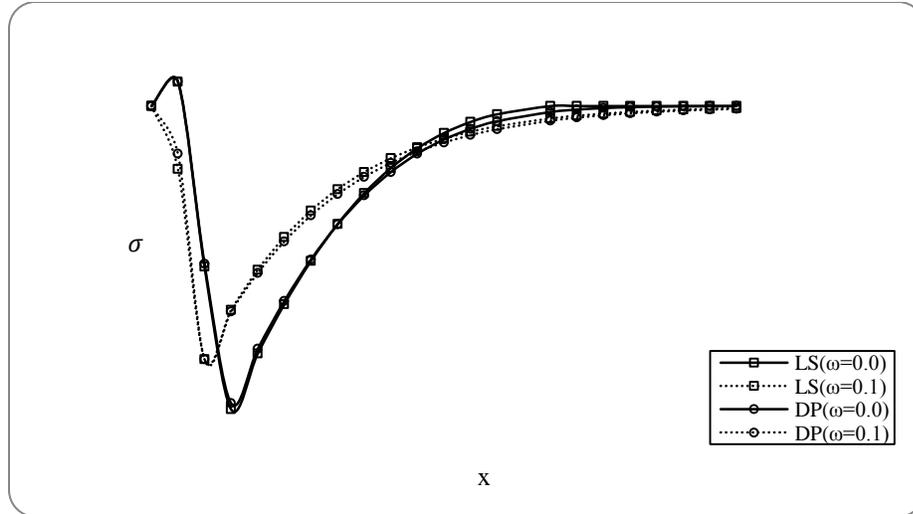


Figure 3. Variation of thermal stress (σ) vs. x and $t = 0.2$ for thermal shock problem

Figure 3 depicts the variation of the stress wave (σ) against x for $t = 0.2$ and $\omega = 0.0, 0.1$. In each case, σ vanishes at $x = 0$ satisfying the theoretical boundary condition. It is observed from the figure, that the presence of the dimensionless two-temperature parameter has a tendency to decrease the stress concentration for both the models. It is seen that the stress wave is expansive in the range $0 < x < 0.12$ for the one-temperature theory for both the models. The steep jump of σ occurs for a particular value of x and this is also seen analytically from equation (64). Here, for both the LS and the DP models, we have fairly close results.

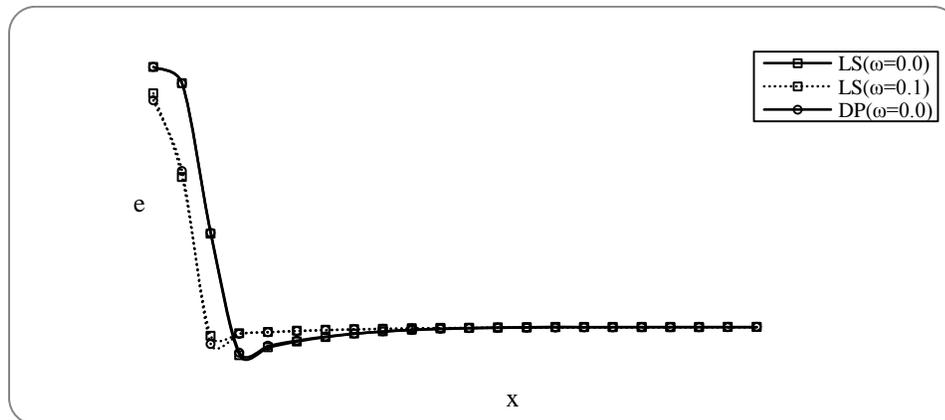


Figure 4. Variation of strain (e) vs. x and $t = 0.2$ for thermal shock problem

Figure 4 depicts the variation of the strain (e) versus the space variable x for the two types of temperature and for the same parameters as in figure 3. It is observed that the magnitude of e is larger for one-temperature theory when $0 \leq x < 0.7$ for both the models. The strain shows its compressive nature in the range $0.27 < x < 0.6$ for the two-temperature theory and $0.29 < x < 0.6$ for the one-temperature theory. Also, it is observed that the shift profile of e is larger for the one-temperature theory than for the two-temperature theory.

Figures 5-8 are representing the variation of the thermophysical quantities versus x at $t = 0.2$ and

$\omega = 0.1$ for the two-temperature Lord-Shulman (2TLS) and the two-temperature Dual-phase-lag (2TDP) models for different ramping parameter $t_0 (= 0.1, 0.2, 0.3)$.

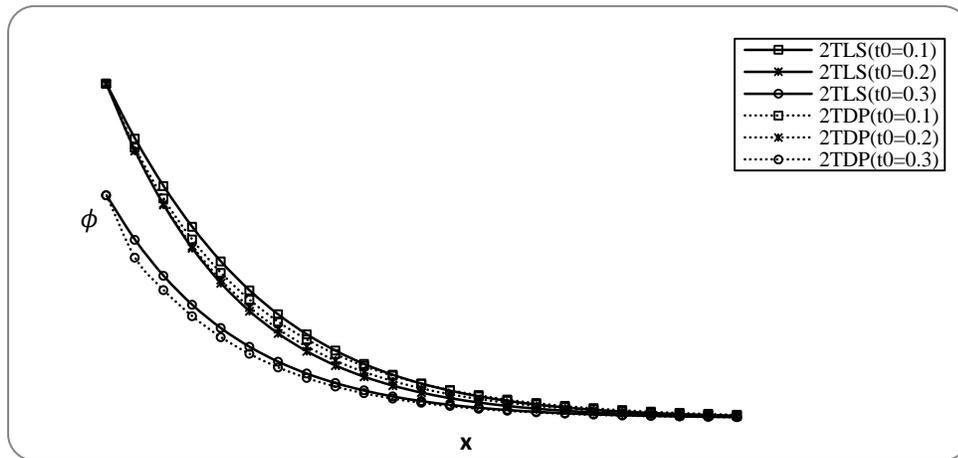


Figure 5. Variation of conductive temperature (ϕ) vs. x for $\omega = 0.1, t = 0.2$ for ramp-type problem

Figure 5 depicts the variation of the conductive temperature (ϕ) with distance (x) for both the models. It is observed that ϕ attains the maximum magnitude near the bounding plane $x = 0$ and finally the magnitude decays gradually as x increases. In the figure, when $t_0 = 0.1$ and 0.2 , ϕ has the value 1 for $x = 0.0$ and when instead $t_0 = 0.3$, it has the value $\phi \simeq 0.667$, thus satisfying our theoretical boundary condition. Also, it is seen that, for both the models, the magnitude of the conductive temperature decreases with the increase of the ramping parameter t_0 and the rate of decay is slower in the case of the DP model than for that of the LS model.

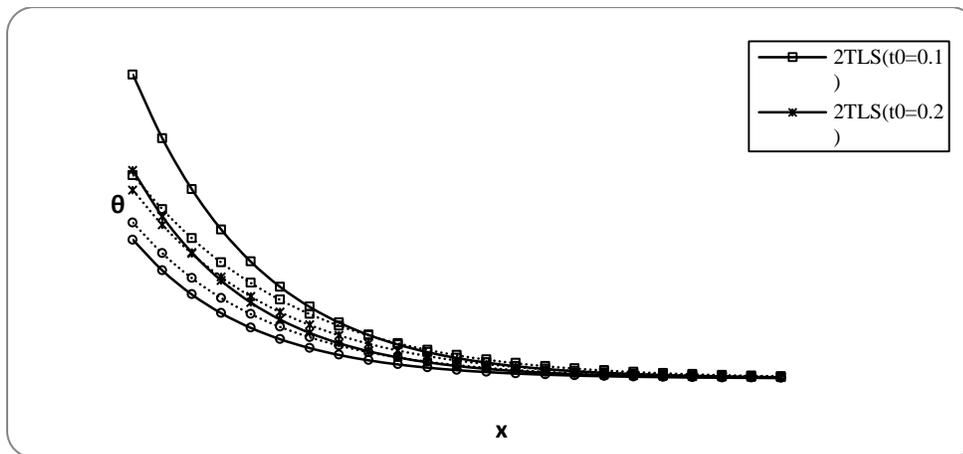


Figure 6. Variation of thermodynamic temperature (θ) vs. x for $\omega = 0.1, t = 0.2$ for ramp-type problem

Figure 6 exhibits the variation of the thermodynamic temperature (θ) against x for both the models and for different ramping parameter t_0 . For Dual-phase-lag model, the rate of decay of θ is slower than that of Lord-Shulman model.

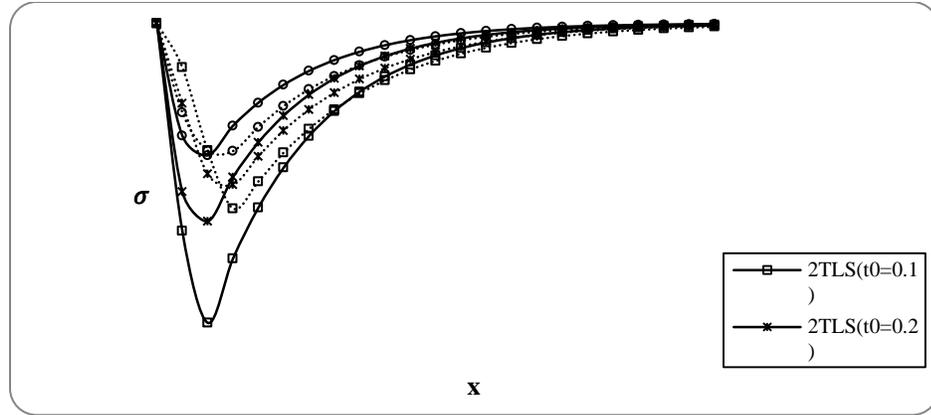


Figure 7. Variation of thermal stress (σ) vs. x for $\omega = 0.1, t = 0.2$ for ramp-type problem

Figure 7 displays the comparison of stress (σ) against x for different ramping parameter t_0 . It is observed that each σ is compressive at the beginning of the thermal loading to the bounding plane and finally reaches zero. It can be concluded that the increase of the ramping parameter t_0 will decrease the stress concentration for both the models. It is observed that the curves corresponding to $t_0 = 0.3$ is smoother than that for $t_0 = 0.2$ which in turn, is smoother than for $t_0 = 0.1$. Thus, the presence of t_0 in these models may have an important role in maintaining the continuity of stress distribution in solids. Also, for the 2TDP model, the rate of decay is slower than that of 2TLS model.

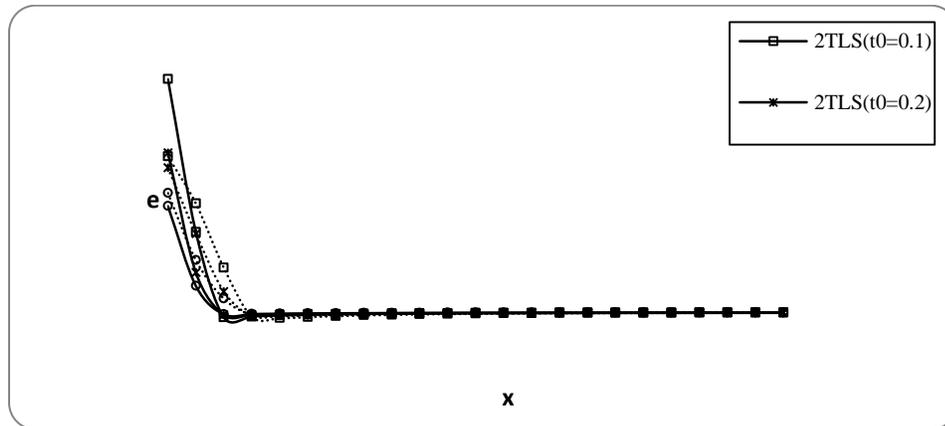


Figure 8. Variation of strain (e) vs. x for $\omega = 0.1, t = 0.2$ for ramp-type problem

Figure 8 gives the variation of the strain (e) versus the distance (x) for the same set of parameters as mentioned above. It is seen that the elongation of the solid is maximum in magnitude near the bounding plane for both models. Also, the elongation almost disappears for $0.6 < x < 2.2$. For 2TDP model, the decay of e is slower than that of 2TLS model.

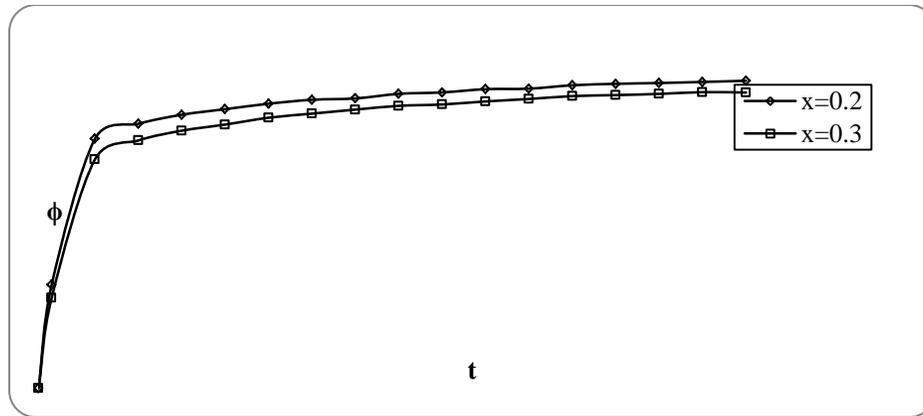


Figure 9. Variation of conductive temperature (ϕ) vs. t for $x=0.2,0.3$, $\omega=0.1$ and $t_0=0.1$

Figures 9 and 10 are representing the variation of the conductive temperature (ϕ) and the thermodynamic temperature (θ) against time (t) at $x=0.2,0.3$ for the two-temperature Dual-phase-lag (2TDP) model for ramping parameter $t_0=0.1$. From these figures, it is observed that as t increases, the magnitude of θ and ϕ will almost attain a steady state which supports the physical fact.

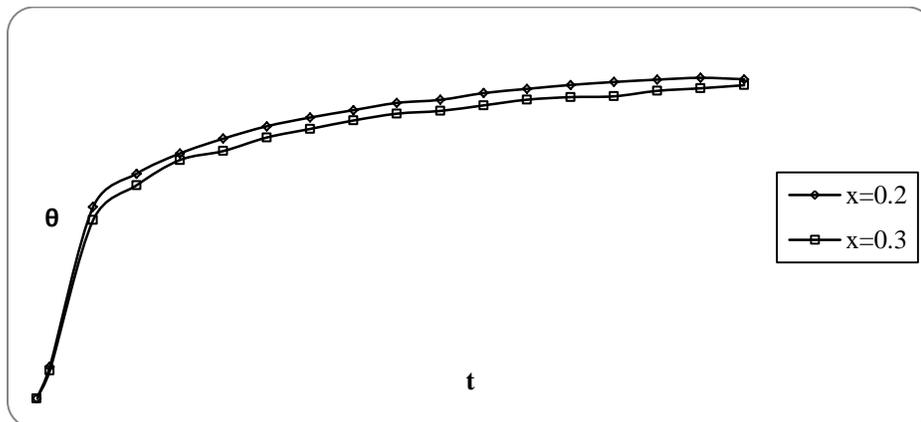


Figure 10. Variation of thermodynamic temperature (θ) vs. t for $x=0.2,0.3$, $\omega=0.1$ and $t_0=0.1$

5. Conclusions

The problem of investigating the thermoelastic stress, strain, displacement, conductive temperature and the thermodynamic temperature in an infinite, homogenous, isotropic elastic half space is studied in the light of two-temperature generalized thermoelasticity with dual-phase-lag effects for different types of thermal loading. The method of Laplace transform is used to write the basic equations in the form of vector-matrix differential equation, which is then solved by state-space approach. The numerical inversion of the Laplace transform is done by using Fourier series expansion technique (Honig and Hirdes; 1984). The analysis of the results permit some concluding remarks.

1. Significant differences in the physical quantities are observed between the one-

temperature dual-phase-lag (DP) model and the two-temperature dual-phase-lag (2TDP) model. The two-temperature theory is more realistic than the one-temperature theory in the case of generalized thermoelasticity.

2. As observed from the figures, an increase of the ramping parameter results in a decrease of the magnitudes of the thermophysical quantities.

3. Here all the results for the two temperature Lord-Shulman model in the case of thermal shock problem agree with the results of Youssef and Lehaibi (2007).

Acknowledgments

We are grateful to Professor S. C. Bose of the Department of Applied mathematics, University of Calcutta, for his kind assistance and guidance in the preparation of the paper. We also express our sincere thanks to the reviewers for their valuable suggestions for the improvement of the paper.

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