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A Novel Approach for Solving Burger's Equation

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Abstract

The paper presents a new analytical method called Variational Homotopy Perturbation Method (VHPM), which is a combination of the well-known Variational Iteration method (VIM) and the Homotopy Perturbation method (HPM) for solving the one-dimensional Burger's equation. Two test problems are presented to demontrate the efficiency and the accuracy of the proposed method.The numerical solutions obtained are compared with the exact solution. Furthermore, this method does not require spatial discretization or restrictive assumptions and is free from round-off errors and therefore reduce the numerical computation significanly. The results reveal that the Variational Homotopy Perturbation Method is very effective and convenient to solve nonlinear partial differential equations.

Keywords: Burger's equation; Variational Homotopy Perturbation Method (VHPM); Variational Iteration Method (VIM); Homotopy Perturbation Method (HPM)

MSC 2010 No.: 76T99, 76505

1. Introduction

In one dimension, Burger's equation is given by

$$\frac{\partial}{\partial t}u(x,t)+u(x,t)\frac{\partial}{\partial x}u(x,t)=\varepsilon\frac{\partial^2}{\partial x^2}u(x,t),\ a\le x\le b,\ t>0, \tag{1}$$

where $\varepsilon > 0$ is kinematic viscosity. The initial condition is given by

$$u(x,0) = u_0(x).$$

Burger used the equation (1) in a mathematical modelling of turbulence (Burger 1939; Burger 1948). This equation arises in many situations such as the theory of shock waves, turbulence problem and continuous stochastic processes. It has been treated as a topic of central interest by many authors for the conceptual understanding of a class of physical flows and for testing various numerical methods. It is well known that the exact solution of Burger's equation can be solved only for restricted values of kinematic viscosity. The mathematical properties of Burger's Equation have been studied by Cole (1951). Ozis (1996) applied a direct variational method to generate a limited form of the solution of Burger's equation. Ozis et al. (2003) applied a simple finite-element approach with linear elements to Burger's equation reduced by the Hopf-Cole transformation. Aksan and Ozdes (2004) have reduced Burger's equation to the system of non-linear ordinary differential equations by discretization in time and solved the system by the Galerkin method in each time step. Aksan et al. (2006) applied the least square method to the solution of equation (1). Abu and Soliman (2005) applied the Variational Iteration Method to obtain the solution of equation (1). Noorzad (2008) applied Homotopy perturbation Method and Variational Iteration Method and made comparison between both the methods. Recently, Deepti Mishra (2012) applied the Homotopy Perturbation Transform Method to solve the non-linear Burger's equation.

In the present paper, Burger's equation is solved by the Variational Homotopy Perturbation Method for two test problems. Numerical and graphical values are also presented for $\varepsilon = 1$ contrary to the conservative methods which require the initial and boundary conditions, the VHPM provide an analytical solution by using only the initial conditions.

2. The Variational Iteration and Homotopy Perturbation Method

To illustrate the basic concepts of the VIM and HPM, we first consider the following nonlinear differential equation

$$Lu + Nu = g(x), \tag{2}$$

where

L = A linear operator,
N = A nonlinear operator,
$g(x)$ = An inhomogeneous term.

According to the VIM (He, 1999; Batiha, 2007) we way construct a correction functional as follows:

$$u_{n+1}(x) = u_{n(x)} + \int_0^t \lambda(\tau)\Big(Lu_n(\tau) + Nu_n(\tau) - g(\tau)\Big)d\tau, \tag{3}$$

where $\lambda(\tau)$ is a general Lagrange multiplier, which can be identified optimally via variational theory. The second term on the right hand side is called the correction and is considered as a restricted variation, i.e., $\delta u_n = 0$. By this method, first it is required to determine the Lagrange multiplier $\lambda(\tau)$ that will be identified optimally. The successive approximations $u_{n+1}(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the determined Lagrange multiplier and any selective function $u_0(x,t)$. Consequently, the solution is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{4}$$

The essential idea of HPM is to introduce a Homotopy parameter (say *p)*, which takes values from 0 to 1. When $p = 0$, the system of equations is in sufficiently simplified form which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of "deformation", the solution of each is "close" to that at the previous stage of "deformation". Eventually at $p = 1$, the system takes the original form of equation and the final stage of "deformation" gives the desired solution. To illustrate the basic concept of Homotopy Perturbation Method, we consider the following nonlinear system of differential equations

$$A(\mathrm{u}) = f(r),\ r \in \Omega, \tag{5}$$

with the boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0\ ,\ r \in \Gamma, \tag{6}$$

where

A = a differential operator,

B = a boundary operator,

$f(r)$ = a known analytic function,

Γ = the boundary of the domain Ω.

Generally, the operator A can be divided into two parts L and N, where L is a linear operator and N is a nonlinear operator. Therefore equation (6) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{7}$$

We construct a Homotopy

$$v(r, p): \Omega \times [0,1] \to R,$$

which satisfies

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \; p \in [0,1], \; r \in \Omega \tag{8}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{9}$$

where u_0 is initial approximation of equation (6). In this method, using the Homotopy parameter p, we can express in terms of power series as

$$v = v_0 + pv_1 + p^2 v_2 + \dots . \tag{10}$$

Setting $p = 1$ yields the approximate solution of equation (10) as below

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots . \tag{11}$$

The convergence of series (11) is discussed by Biazar and Ghazvini (2009).

3. The Variational Homotopy Perturbation Method

In the Homotopy Perturbation Method, the basic assumption is that the solutions can be written in terms of power series in p as

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \dots . \tag{12}$$

To illustrate the concept of the Variational Homotopy Perturbation Method, we consider the general differential equation (5). We construct the correction functional equation (6) and apply the Homotopy Perturbation Method to obtain,

$$\sum_{i=0}^{\infty} p^i u_i = u_0(x,t) + p\int_0^t \lambda(\tau)\left(N\sum_{n=0}^{\infty} p^{(i)} u_i\left(x,\tau\right) - g\left(x,\tau\right)\right) d\tau \tag{13}$$

Thus the procedure is formulated by the coupling of Variational Iteration Method and Homotopy Perturbation Method. A comparison of like powers of p gives solutions of various orders.

4. Statement of Problem

4.1. Problem 1

$$\frac{\partial u}{\partial t}+u\left(\frac{\partial u}{\partial x}\right)-\varepsilon\frac{\partial^2 u}{\partial x^2}=0, \tag{14}$$

$$u(x,0)=x(1-x^2). \tag{15}$$

Equation (14) along with the initial condition (15) has the exact solution (Cole, 1951)

$$u(x,t)=\frac{2\pi\nu\sum_{n=1}^{\infty}A_n\exp(-n^2\pi^2\nu t)\sin n\pi x}{A_0\sum_{n=1}^{\infty}A_n\exp(-n^2\pi^2\nu t)\cos n\pi x}$$

with Fourier coefficients

$$A_0=\int_0^1\exp\left[-(8\nu)^{-1}(2x^2-x^4)\right]dx,$$
$$A_n=\int_0^1\exp\left[-(8\nu)^{-1}(2x^2-x^4)\right]\cos n\pi x\,dx.$$

4.2. Problem 2

Consider the Burger's equation (14) with the initial condition and homogeneous boundary condition

$$u(x,0)=\sin(\pi x)\,,\,0<x<1,$$
$$u(0,t)=u(1,t)=0\,,\,t>0. \tag{16}$$

The exact solution (Cole, 1951) of the Burger's equation (14) with initial and boundary conditions (16) is obtained as

$$u(x,t) = 2\pi\varepsilon \frac{\sum_{n=1}^{\infty} a_n e^{-n^2\pi^2\varepsilon t} n\sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2\varepsilon t} n\cos(n\pi x)},$$

where the Fourier coefficients a_0 and a_n (n=1,2,..) are defined by the following equations, respectively.

$$a_0 = \int_0^1 \exp\left\{-(2\pi\varepsilon)^{-1}\left[1-\cos(\pi x)\right]\right\}dx,$$
$$a_n = 2\int_0^1 \exp\left\{-(2\pi\varepsilon)^{-1}\left[1-\cos(\pi x)\right]\right\}\cos(n\pi x)dx, \quad (n=1,2,3,\cdots).$$

5. Method of Solution

5.1. Solution of Problem 1

According to the Variational Homotopy Perturbation Method, we construct the correction functional for equation (14) as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^{\tau}\left[\lambda(\tau)\left(\frac{\partial u_n}{\partial t} + u_n\frac{\partial u_n}{\partial x} - \varepsilon\frac{\partial^2 u_n}{\partial x^2}\right)\right]d\tau, \tag{17}$$

which yields the stationary conditions,

$$\lambda^{'}(\tau) = 0,$$
$$1+\lambda(\tau) = 0.$$

Therefore, the general Lagrange multiplier can be readily identified as $\lambda = -1$, which yields the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^{\tau}\left(\frac{\partial u_n}{\partial t} + u_n\frac{\partial u_n}{\partial x} - \varepsilon\frac{\partial^2 u_n}{\partial x^2}\right)d\tau \tag{18}$$

Applying the variational Homotopy perturbation method, we get

$$u_0 + pu_1 + p^2u_2 + \cdots = f(x) - p\int_0^t \left[\left(u_0 + pu_1 + p^2u_2 + \cdots\right) \frac{\partial}{\partial x}\left(u_0 + pu_1 + p^2u_2 + \cdots\right) \right] d\tau$$
$$+ p\int_0^t \varepsilon \frac{\partial^2}{\partial x^2}\left(u_0 + pu_1 + p^2u_2 + \cdots\right) d\tau$$

Comparing the coefficient of like powers of p, we have

$$p^0 : u_0(x,t) = x(1-x^2)\,,$$

$$p^1 :\ u_1(x,t) = -\int_0^t u_0 \frac{\partial}{\partial x} u_0 d\tau + \int_0^t \varepsilon \frac{\partial^2}{\partial x^2} u_0 d\tau = -tx(-1+x^2)(-1+3x^2) - 6tx,$$

$$p^1 : u_1(x,t) = tx(-7+4x^2-3x^4)\,,$$

$$p^2 : u_2(x,t) = -\int_0^t u_1 \frac{\partial}{\partial x} u_0 d\tau - \int_0^t u_0 \frac{\partial}{\partial x} u_1 d\tau + \int_0^t \varepsilon \frac{\partial^2}{\partial x^2} u_1 d\tau$$
$$= \frac{1}{2}t^2x(-7+25x^2-15x^4+9x^6) + \frac{1}{2}t^2x(7-19x^2+27x^4-15x^6) + 6t^2x(2-5x^2),$$

$$p^2 : u_2(x,t) = -3t^2x(-4+9x^2-2x^4+x^6)\,.$$

Similarly, further approximations can be obtained up to desired accuracy. The solution becomes

$$u(x,t) = x(1-x^2) + tx(-7+4x^2-3x^4) - 3t^2x(-4+9x^2-2x^4+x^6) + \cdots. \tag{19}$$

5.2. Solution of Problem 2

According to Variational Homotopy Perturbation Method, we construct the correction functional for equation, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula.

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^\tau \left(\frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial x} - \varepsilon \frac{\partial^2 u_n}{\partial x^2} \right) d\tau\,. \tag{20}$$

Applying the Variational Homotopy Perturbation Method, we get

$$u_0 + pu_1 + p^2u_2 + \cdots = f(x) - p\int_0^t \left[\left(u_0 + pu_1 + p^2u_2 + \cdots\right) \frac{\partial}{\partial x}\left(u_0 + pu_1 + p^2u_2 + \cdots\right) \right] d\tau$$
$$+ p\int_0^t \varepsilon \frac{\partial^2}{\partial x^2}\left(u_0 + pu_1 + p^2u_2 + \cdots\right).$$

Comparing the coefficient of like powers of p, we have

$$p^0 : u_0(x,t) = \sin \pi x \,,$$

$$p^1 : u_1(x,t) = -\int_0^t u_0 \frac{\partial}{\partial x} u_0 d\tau + \int_0^t \varepsilon \frac{\partial^2}{\partial x^2} u_0 d\tau = -\int_0^t \pi \sin \pi x \cos \pi x d\tau + \int_0^t \varepsilon \pi^2 \sin \pi x d\tau \,,$$

$$p^1 : u_1(x,t) = -(\pi \sin \pi x \cos \pi x)t + \varepsilon \pi^2 \sin(\pi x)t \,,$$

$$p^2 : u_2(x,t) = -\int_0^t u_1 \frac{\partial}{\partial x} u_1 d\tau - \int_0^t u_0 \frac{\partial}{\partial x} u_1 d\tau + \int_0^t \varepsilon \frac{\partial^2}{\partial x^2} u_1 d\tau$$

$$= -\int_0^t \left[\begin{array}{l} -(\pi \sin \pi x \cos \pi x)t \\ \qquad + \varepsilon \pi^2 \sin(\pi x)t \dfrac{\partial}{\partial x}\left(-(\pi \sin \pi x \cos \pi x)t + \varepsilon \pi^2 \sin(\pi x)t\right) \end{array} \right] d\tau$$

$$- \int_0^t \sin \pi x \left[\frac{\partial}{\partial x}\left(-(\pi \sin \pi x \cos \pi x)t + \varepsilon \pi^2 \sin(\pi x)t\right) \right] d\tau$$

$$+ \int_0^t \varepsilon \left[\frac{\partial^2}{\partial x^2}\left(-(\pi \sin \pi x \cos \pi x)t + \varepsilon \pi^2 \sin(\pi x)t\right) \right] d\tau,$$

$$\begin{aligned} p^2 : \; u_2(x,t) = \pi^2 t(\pi\varepsilon \cos(\pi x) + \pi\varepsilon(\pi\varepsilon + 4\cos(\pi x)) \\ + \cos(2\pi x) - \pi(\pi\varepsilon + \cos(\pi x))(\pi\varepsilon \cos(\pi x) \\ + \cos(2\pi x))) \sin(\pi x). \end{aligned}$$

Similarly, further approximations can be obtained up to desired accuracy. The solution becomes

$$\begin{aligned} u(x,t) = \sin \pi x - (\pi \sin \pi x \cos \pi x)t + \varepsilon \pi^2 \sin(\pi x)t \\ + \pi^2 t(\pi\varepsilon \cos(\pi x) + \pi\varepsilon(\pi\varepsilon + 4\cos(\pi x)) \\ + \cos(2\pi x) - \pi(\pi\varepsilon + \cos(\pi x))(\pi\varepsilon \cos(\pi x) \\ + \cos(2\pi x))) \sin(\pi x) + \cdots. \end{aligned} \tag{21}$$

6. Results and Discussion

Here an approximate solution is obtained for two problems and are compared to the exact solution with put emphasis on the accuracy of the present method where the viscosity value is one. The tabular comparison between the VHPM solutions and the exact solutions at different times for specific value of x are summarized in Table 1 for problem 1. It shows that the solutions are in good harmony with those of the exact solution. The solutions obtained for problem 1 and problem 2 are compared with the exact solution at particular times shown in Table 2 and Table 3 respectively. The plots of the numerical solutions obtained for various values of time and space, considering $\varepsilon = 1$ are shown in Figures 1-3.

Table1. Comparison of VHPM solutions of (Problem 1) with Exact solutions at ε =1 at different times.

x	t	VHPM	Exact
0.25	0.001	0.232687	0.23269
	0.005	0.225987	0.22602
	0.010	0.217729	0.21784
	0.050	0.156313	0.15757
	0.100	0.09117	0.09945
0.50	0.001	0.371909	0.37191
	0.005	0.359601	0.35962
	0.010	0.344341	0.34438
	0.050	0.227285	0.23547
	0.100	0.0935156	0.14389
0.75	0.001	0.323849	0.32385
	0.005	0.306719	0.30671
	0.010	0.285244	0.28623
	0.050	0.110986	0.17647
	0.100	0.112989	0.10414

Table 2. Comparison of VHPM solutionsof (Problem-1) obtained with Exact solutions at t = 0.01 and different values of x

x	Exact	VHPM
0.1	0.09221	0.092157
0.2	0.17863	0.178529
0.3	0.25341	0.253296
0.4	0.31066	0.310566
0.5	0.34438	0.344341
0.6	0.34845	0.348482
0.7	0.31690	0.316668
0.8	0.24531	0.242361
0.9	0.13519	0.118768

Table 3. Comparison of VHPM solution of (Problem 2) with exact solution for ε = 1 and at different time levels

t=0.01			t=0.001	
x	Exact	VHPM	Exact	VHPM
0.1	0.27324	0.26221	0.30509	0.304973
0.2	0.5256	0.507587	0.58057	0.580418
0.3	0.72185	0.714853	0.77762	0.799545
0.4	0.85459	0.859373	0.94082	0.940861
0.5	0.90571	0.919799	0.99018	0.990315
0.6	0.86833	0.885239	0.94261	0.942781
0.7	0.74410	0.75818	0.80252	0.802667
0.8	0.54382	0.552736	0.58347	0.583558
0.9	0.28700	0.291025	0.30688	0.306923

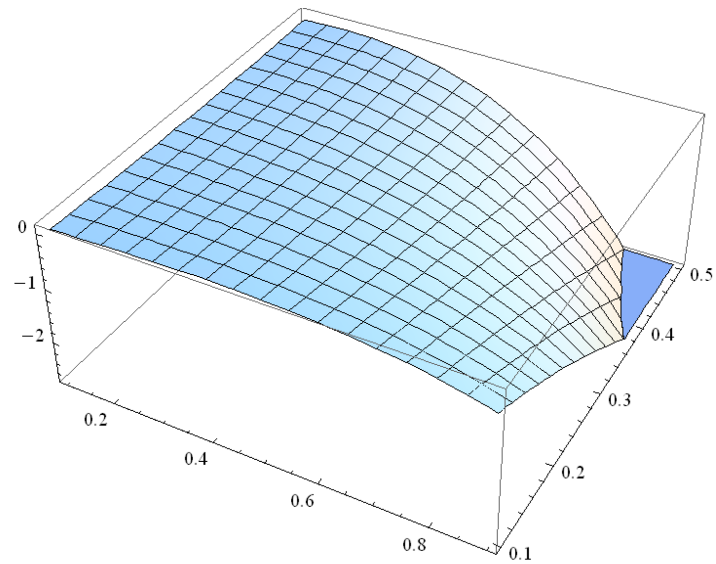

Figure 1. The three dimensional graph of problem 1 for ε =1

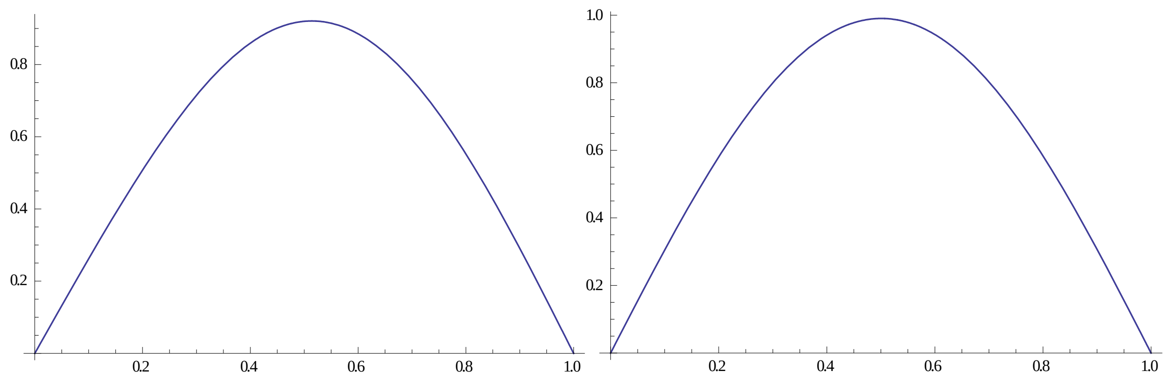

Figure 2. Graph of the solution of problem 2 at time t=0.001 and t=0.01 for ε=1

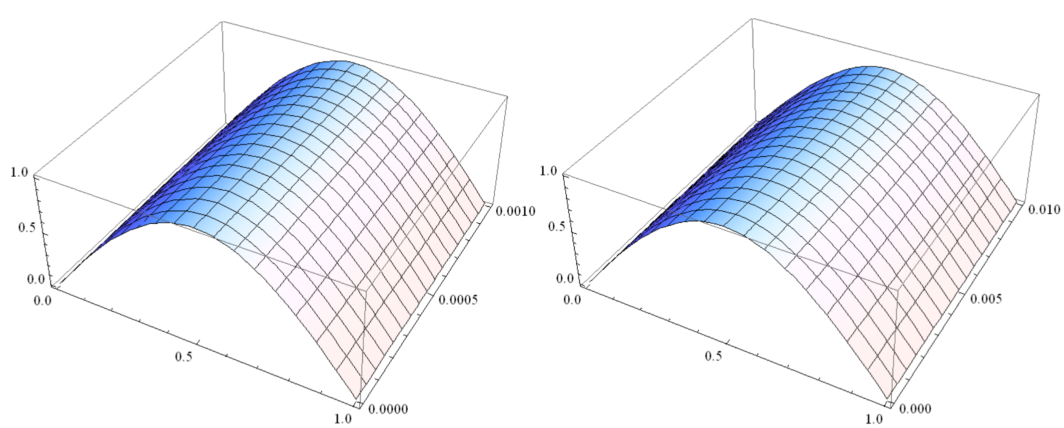

Figure 3. Graph of $u(x,t)$ Vs x of problem 2 at time t=0.001 and t = 0.01 for ε = 1

7. Conclusion

In this paper, solution of Burgers equation is obtained by applying Variational Homotopy Perturbation Method with specific initial conditions. The Variational Homotopy Perturbation Method is proved to be an effective approach for solving the Burger's equation due to the excellent agreement between the obtained numerical solution and the exact solution. A comparison is made to show that method has small size of computation in comparison with the computational size required in other numerical methods and its rapid convergence shows that method is reliable and introduces a significant improvement in solving partial differential equation.

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