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Independent Monopoly Size In Graphs

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Abstract

In a graph $G = (V, E)$, a set $D \subseteq V(G)$ is said to be a monopoly set of G if every vertex $v \in V - D$ has at least $\frac{d(v)}{2}$ neighbors in D. The monopoly size of G, denoted $mo(G)$, is the minimum cardinality of a monopoly set among all monopoly sets in G. The set $D \subseteq V(G)$ is an independent monopoly set in G if it is both a monopoly set and an independent set in G. The number of vertices in a minimum independent monopoly set in a graph G is the independent monopoly size of G and is denoted by $imo(G)$. In this paper, we study the existence of independent monopoly set in graphs, bounds for $imo(G)$, and some exact values for some standard graphs are obtained. Finally we characterize all graphs of order n with $imo(G) = 1, n - 1$ and n.

Keywords: Vertex degree; monopoly set of a graph; independent monopoly set of a graph; independent monopoly size of a graph

MSC 2010 No.: 05C07; 05C69

1. Introduction

We consider here a simple graph $G = (V, E)$ that is finite, having no loops, and no multiple and directed edges. We denote by $n = |V|$ and $m = |E|$ the number of vertices and edges of G, respectively. For a vertex $v \in V$, the open neighborhood of v in G, denoted $N(v)$, is the set of all vertices that are adjacent to v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of vertex v in G, denoted by $d(v)$, is the number of its neighbors in G. By an isolated vertex we mean the vertex of degree 0 and an end-vertex is a vertex of degree 1. We denote by Δ and δ the maximum and minimum degree among the vertices of G, respectively. For a non-empty subset $S \subseteq V$ and a vertex $v \in V$ we denote by $d_S(v)$ the degree of v in S, i.e. $d_S(v) = |N(v) \cap S|$. A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We denote by $\langle S \rangle$ the induced subgraph of G that is induced by vertex set $S \subseteq V$ such that $E(\langle S \rangle) = \{uv \in E(G) | u, v \in S\}$. A complete graph on n vertices, denoted by K_n, is a graph that contains exactly one edge between each pair of distinct vertices. A complete bipartite graph, denoted by $K_{n,m}$, is a graph that has its vertex set partitioned into two subsets V_1 of size n and V_2 of size m such that there is an edge from every vertex in V_1 to every vertex in V_2. A star graph is $K_{1,n}$ complete bipartite graph. The graph $W_n = K_1 + C_{n-1}$, for $n \geq 4$, is called the wheel graph of order n. A k-regular graph is a graph where each vertex has k degree. If G does not contain a graph F as an induced subgraph, then we say that G is F-free. In particular, we say that a graph is claw-free if it is $K_{1,3}$-free. We denote by $\overline{G}$ the complement graph of G, rG is the r disjoint copy of G, $\lceil x \rceil$ is the smallest integer number greater than or equal to x, and $\lfloor x \rfloor$ is the greatest integer number smaller than or equal to x.

Definition. In a graph G, the clique C is a subset of vertices such that $\langle C \rangle$ is complete. The clique number of G, denoted $\omega(G)$, is the number of vertices in the largest clique in G.

For more terminologies and notations in graph theory not defined here, we refer the reader to books by Bondy et al. (2008) and Harary (1969). Independent (or stable) sets of vertices in graphs is one of the most commonly studied concepts in graph theory.

Definition. A set $I \subseteq V$ is independent if no two vertices in I are adjacent. The independent sets of maximum cardinality are called maximum independent sets. The number of vertices in a maximum independent set in a graph G is the independence number (or vertex independence number) of G and is denoted by $\alpha(G)$.

The maximum independent sets are the independent sets that have received the most attention. There are also certain independent sets of minimum cardinality that are of interest. Ordinarily, a graph contains many independent sets.

Definition. An independent set of vertices that is not properly contained in any other independent set of vertices is a maximal independent set of vertices. The minimum number of vertices in a maximal independent set is denoted by $i(G)$.

The parameter $i(G)$ is also called the independence domination number as it is a smallest cardinality of an independent set of vertices that dominate all vertices of G. Some graphs contain independent set D such that every vertex $v \in V - D$ has at least $\frac{d(v)}{2}$ neighbors in D. The goal of this paper is to study the existence of such independent sets in graphs and, when they exist, to investigate the minimum cardinality of such a set.

Definition. A subset $S \subseteq V$ is called a dominating set of G if every vertex in $V - S$ is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G.

For more details in domination theory we refer to Haynes et al. (1998).

Definition. A subset $D \subseteq V$ is called a monopoly set in G if every vertex $v \in V - D$ has at least $\frac{d(v)}{2}$ neighbors in D. The monopoly size, $mo(G)$, of G is the minimum cardinality of a monopoly set among all monopoly sets in G.

The concept of monopoly in graphs was introduced by Khoshkhak et al. (see Khoshkhak et al. (2013)). In particular, monopolies are a dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process. Dynamos were first introduced by Peleg (2002). For more details in dynamos in graphs we refer to Berger (2001), Bermond et al. (2003), Flocchini et al. (2003), Mishra et al. (2006), and Zaker (2012) and references therein.

Definition. A subset $D \subseteq V$ is called an independent monopoly set (shortly imo-set) of a graph G if D is both monopoly and independent set. The independence monopoly size, denoted by $imo(G)$, is the cardinality of a minimum independent monopoly set in G.

To illustrate this concept, consider the graph G in Figure 1. The set $\{v_1, v_2, v_4, v_6, v_8\}$ is a maximum independent set in G and so $\alpha(G) = 5$. The set $D_1 = \{v_3, v_5, v_7\}$ is a minimum monopoly set in G and so $mo(G) = 3$, however, D_1 is not an independent monopoly set in G. The set $D_2 = \{v_3, v_4, v_6, v_8\}$ is a minimum independent monopoly set in G. Hence, $imo(G) = 4$.

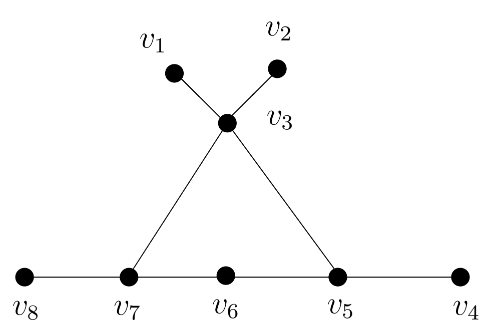

Figure 1: The graph G with $\alpha(G) = 5$, $mo(G) = 3$ and $imo(G) = 4$.

The following are some fundamental results which will be required for many of our arguments in this paper:

Theorem 1. (Khoshkhak et al. (2013))

Let G be a graph on n vertices with m edges whose maximum degree is $\Delta(G)$. Then

$$\frac{2m}{3\Delta(G)} \leq mo(G) \leq \frac{n}{2}.$$

Theorem 2. (Khoshkhak et al. (2013))

Let G be any regular graph on n vertices. Then

$$mo(G) \geq \frac{n}{3}.$$

Theorem 3. (Naji et al. (2015))

Let G be a graph of order n and minimum degree $\delta \geq 1$. Then

$$\frac{\delta}{2} \leq mo(G) \leq n - \frac{\delta + 2}{2}.$$

Observation 1: (Goddard et al. (2013)) If G is a regular graph on n vertices with no isolated vertex, then

$$i(G) \leq \alpha(G) \leq \frac{n}{2}.$$

Theorem 4. (Li et al. (1990))

For any claw-free graph on n vertices

$$\alpha(G) \leq \frac{2n}{\Delta + 2}.$$

Theorem 5. (Bondy et al. (2008))

Any graph G contains a spanning bipartite subgraph H such that $d_H(v) \geq \frac{d(v)}{2}$.

2. The independent monopoly size of a graph

Not all graphs have an independent monopoly set, however, and so $imo(G)$ is not defined for all graphs G. For example, the independent sets of a complete graph K_n consist of only one vertex while a monopoly set of K_n contains at least $\lfloor \frac{n}{2} \rfloor$ vertices. Thus $imo(K_n)$ is not defined for every $n \geq 4$. Figure 2 shows two 3-regular graphs with order 6. A subset $\{v_1, v_3, v_6\}$ is a minimum monopoly set of G_1, while the subset $\{v_1, v_6\}$ is the maximum independent set, i.e. $mo(G_1) = 3$ and $\alpha(G_1) = 2$. Then $imo(G_1)$ is not defined in G_1. For a graph $G_2 = K_{3,3}$, every partite set in G_2 is a monopoly and also an independent set in same time. Hence, $imo(G_2)$ is defined. Furthermore, $imo(G_2) = 3$.

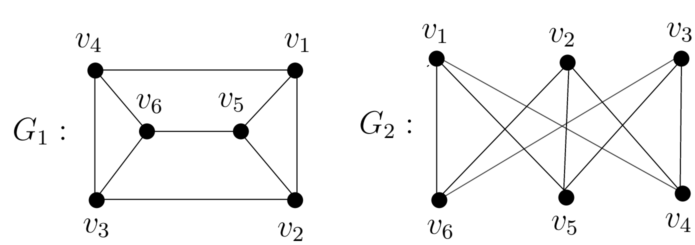

Figure 2: Two 3-regular graphs with 6 vertices.

Since every independent monopoly set in G is a monopoly set, we have the following result.

Observation 2: Let G be a graph of order n, for which $imo(G)$ is defined. Then

$$1 \leq mo(G) \leq imo(G) \leq \alpha(G) \leq n.$$

Since every monopoly set in a graph G is a dominating set (Naji et al. (2015)), then any independent monopoly set in a graph G is an independent dominating set.

Observation 3: Let G be a graph, for which $imo(G)$ is defined. Then

$$\gamma(G) \leq i(G) \leq imo(G).$$

Lemma 1.

For any graph G, if $G = G_1 \cup G_2 \cup ... \cup G_r$, where $2 \leq r \leq n$, then

$$imo(G) = imo(G_1) + imo(G_2) + ... + imo(G_r).$$

Lemma 2.

For any graph G. If every independent set in G is not a monopoly set, then the imo-set of G does not exist.

Observation 4: For any graph G, if $\alpha(G) < mo(G)$, then $imo(G)$ is not defined.

In general, the inverse of the observation 4 is not true. For example, let $\{v_1, v_2, v_3, v_4\}$ be the vertex set of a graph K_4 and let G be a graph obtained from K_4 by join $r \geq 2$ end-vertices (say $u_1, u_2, ..., u_r$) to a vertex v_1 in K_4. The set $I = \{v_2, u_1, ..., u_r\}$ is a maximum independent set in G and so $\alpha(G) = r + 1 \geq 3$. While the set $S = \{v_1, v_3\}$ is a minimum monopoly set in G. However, $\alpha(G) > mo(G)$ but $imo(G)$ is not defined.

Proposition 1.

Let G be a graph and let $D \subseteq V$ be a monopoly set in G. If $\alpha(\langle N(v)\rangle) < \frac{d(v)}{2}$, for any $v \in V - D$, then D is not independent.

Proof:

Let D be a monopoly set in a graph G and let $\alpha(\langle N(v)\rangle) < \frac{d(v)}{2}$ for $v \in V - D$. Since D is a monopoly set and $v \notin D$ it follows that

$$|N(v) \cap D| \geq \frac{d(v)}{2}. \tag{1}$$

On the other hand, since $(N(v) \cap D) \subseteq N(v)$, then

$$\alpha(\langle N(v) \cap D\rangle) \leq \alpha(\langle N(v)\rangle) < \frac{d(v)}{2}. \tag{2}$$

From Equations (2.1) and (2.2) we get $\alpha(\langle N(v) \cap D\rangle) < |N(v) \cap D|$. Hence the subset $N(v) \cap D$ is not independent set in G. Accordingly, D is not an independent set. □

Proposition 2.

If G is a graph of order $n \leq 3$, then imo-set of G exists. Furthermore, $imo(G) = 1$.

Theorem 6.

For any graph G with at least four vertices, if G contains two adjacent vertices v_1 and v_2 with $\alpha(\langle N(v_i)\rangle) < \frac{d(v_i)}{2}$, for every $i = 1, 2$, then $imo(G)$ is not defined.

Proof:

Let G be a graph of order $n \geq 4$ and let G contains two adjacent vertices v_1 and v_2 with $\alpha(\langle N(v_i)\rangle) < \frac{d(v_i)}{2}$, for every $i = 1, 2$. Then any independent set in G does not contain v_1 and v_2 in same time. Let I be an arbitrary independent set in G. Then either $v_1 \notin I$ or $v_2 \notin I$ or both not in I. Assume that $v_1 \notin I$. Since I is an independent set, then

$$|N(v_1) \cap I| = \alpha(\langle N(v_1) \cap I\rangle) \leq \alpha(\langle N(v_1)\rangle) < \frac{d(v_1)}{2}.$$

Hence, I is not a monopoly set in G. Since I is an arbitrary independent set in G, it follows that every independent set in G is not a monopoly set. Therefore, by Lemma 2 $imo(G)$ is not defined. The proof is similar if $v_2 \notin I$ or both v_1 and v_2 not in I. $\square$

Theorem 7.

Let G be a graph and let C be a clique in G with $|C| \geq 4$. If C contains two vertices v_1 and v_2 with $d_C(v_i) \geq \frac{d(v_i)}{2} + 2$, for every $i = 1, 2$, then $imo(G)$ is not defined.

Proof:

Let C be a clique with $|C| \geq 4$ in a graph G and let C contains two vertices v_1 and v_2 with $d_C(v_i) \geq \frac{d(v_i)}{2} + 2$, for every $i = 1, 2$. Since $v_1, v_2 \in C$, it follows that v_1 and v_2 are adjacent vertices in G and $\alpha(\langle N(v_i) \cap C\rangle) = 1$, for every $i = 1, 2$. Then, since $|N(v_i) \cap (V - C)| = d_{V-C}(v_i) = d(v_i) - d_C(v_i)$, it follows that$|N(v_i) \cap (V - C)| \leq \frac{d(v_i)}{2} - 2$, for every $i = 1, 2$. Hence,

$$\begin{aligned}\alpha(\langle N(v_i)\rangle) &= \alpha(\langle N(v_i) \cap C\rangle) + \alpha(\langle N(v_i) \cap (V - C)\rangle) \\ &\leq 1 + |N(v_i) \cap (V - C)| \\ &\leq 1 + \frac{d(v_i)}{2} - 2 \\ &= \frac{d(v_i)}{2} - 1\end{aligned}$$

for every $i = 1, 2$. Accordingly, the vertices v_1 and v_2 are adjacent in G with $\alpha(\langle N(v_i)\rangle) < \frac{d(v_i)}{2}$ for $i = 1, 2$. By Theorem 6, $imo(G)$ is not defined. $\square$

Corollary 1.

Let G be a graph and let C be a clique in G with $|C| \geq 4$. If C contains two vertices v_1 and v_2 with $d(v_i) \leq 2|C| - 6$, for every $i = 1, 2$, then $imo(G)$ is not defined.

Theorem 8.

Every graph G of order n contains a spanning subgraph H with $imo(H) \leq \frac{n}{2}$.

Proof:

By Theorem 5, any graph G contains a spanning bipartite subgraph H such that $d_H(v) \geq \frac{d(v)}{2}$. At least one of the partite sets of H has at most $\frac{n}{2}$ vertices. Hence, each partite set forms an imo-set in G. By taken the partite set which has a smallest cardinality then the bound is held. □

Theorem 9.

Let G be a connected graph with order $n \geq 3$ and a clique number $\omega(G) \geq 2$, for which $imo(G)$ is defined. Then

$$\omega(G) - 2 \leq imo(G) \leq n - \omega(G) + 1.$$

Proof:

Let C_m be a maximum clique in a graph G (i.e. $|C_m| = \omega(G)$) and let D be an imo-set in G. Then $|D \cap C_m| \leq 1$ (by the independence of D) and since $|C_m| \geq 2$, then $C_m - D \neq \phi$. Also, $|N(v) \cap D| \geq \frac{d(v)}{2}$ for every $v \in (C_m - D) \subseteq V - D$ (by the monopoly of D). On the other hand, for every $v \in C_m$, since

$$N(v) = (N(v) \cap C_m) \cup (N(v) \cap (V - C_m))$$

then

$$D \cap N(v) = [D \cap (N(v) \cap C_m)] \cup [D \cap (N(v) \cap (V - C_m))].$$

Thus, $\frac{d(v)}{2} \leq |N(v) \cap D| \leq |(N(v) \cap C_m) \cap D| + |(N(v) \cap (V - C_m) \cap D|$, for every $v \in C_m - D$. Then, $\frac{d(v)}{2} \leq 1 + d(v) - (\omega - 1)$. Sequentially, $\frac{d(v)}{2} \geq \omega - 2$. Therefore, $|D| \geq |N(v) \cap D| \geq \frac{d(v)}{2} \geq \omega - 2$, for every $v \in C_m - D$. For upper bound, since $n - |D| = |V - D| \geq |(V - D) \cap C_m| \geq \omega(G) - 1$, it follows that $imo(G) = |D| \leq n - \omega(G) + 1$. □

The bounds in Theorem 9 are sharp. K_2 achieves the upper bound and K_3 achieves both lower and upper bounds. Also, there is a graph G with order $n = 5$ and $\omega = 4$ which achieves both lower and upper bounds, where G is a graph obtained from K_4 by joining three vertices from $V(K_4)$ by a common vertex not in K_4.

Theorem 10.

Let G be a graph of order $n \geq 3$ and minimum degree $\delta \geq 1$, for which $imo(G)$ is defined. Then

$$\frac{\delta}{2} \leq imo(G) \leq n - \delta.$$

Proof:

Let G be a graph of order $n \geq 3$ and $\delta \geq 1$, for which $imo(G)$ is defined. By Observation 2, $mo(G) \leq imo(G) \leq \alpha(G)$ and by using the well-known result $\alpha(G) \leq n - \delta$ and Theorem 3, we get

$$\frac{\delta}{2} \leq imo(G) \leq n - \delta.$$

□

The bounds in Theorem 10 are sharp. K_3 achieves the lower and the upper bounds. C_4 and $K_{\frac{n}{2},\frac{n}{2}}$ achieve the upper bound.

Corollary 2.

Let G be a graph of order $n \geq 3$ and $\delta \geq \frac{2n}{3}$, for which $imo(G)$ is defined. Then

$$imo(G) = \frac{n}{3}.$$

Since every bipartite graph without isolated vertex is the union of two independent sets, each of which is a monopoly set, we have the following result on the imo-set of a bipartite graph.

Theorem 11.

For a bipartite graph G of order n without isolated vertex,

$$imo(G) \leq \frac{n}{2}.$$

The bound is sharp, and $K_{\frac{n}{2},\frac{n}{2}}$ achieves it.

For some classes of k-regular graph we have the following results.

Theorem 12.

Let G be a k-regular graph of order n without isolated vertex and for which $imo(G)$ is defined, then

$$\left\lceil \frac{n}{3} \right\rceil \leq imo(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof:

Let G be a k-regular graph of order n for which $imo(G)$ is defined. Since,

$$mo(G) \leq imo(G) \leq \alpha(G),$$

it follows by Theorem 2 and Observation 1 that

$$\left\lceil \frac{n}{3} \right\rceil \leq imo(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

□

The bounds in Theorem 12 are sharp. The cycle graph C_n for $n \geq 3$ achieves the lower bound and $K_{\frac{n}{2},\frac{n}{2}}$ achieves the upper bound.

Theorem 13.

For a claw-free k-regular graph G of order n, if $k \geq 5$, then $imo(G)$ is not defined.

Proof:

Let G be a claw-free k-regular graph of order n with $k \geq 5$. By Theorem 4,

$$\alpha(G) \leq \frac{2n}{\Delta+2} = \frac{2n}{k+2} \leq \frac{2n}{5+2} < \frac{2n}{6} = \frac{n}{3}.$$

But Theorem 2 states that $mo(G) \geq \frac{n}{3}$. Hence, $mo(G) < \alpha(G)$. This means that every independent set in G is not a monopoly set. By Observation 4, $imo(G)$ is not defined. $\square$

Theorem 14.

Let G be a cubic graph of order $n \geq 6$, for which $imo(G)$ is defined. Then

$$\frac{2n}{5} \leq imo(G) \leq \frac{n}{2}.$$

Proof:

Let G be the cubic graph of order $n \geq 6$. Since the cubic graph is 3-regular graph without isolated vertex, then by Theorem 12 the upper bound is held. Then

$$\begin{aligned} 3n = \sum_{v\in V} d(v) &= \sum_{v\in V} d_D(v) + \sum_{v\in V} d_{V-D}(v) \\ &= \sum_{v\in D} d_D(v) + \sum_{v\in D} d_{V-D}(v) + \sum_{v\in V-D} d_D(v) + \sum_{v\in V-D} d_{V-D}(v), \end{aligned}$$

for every $v \in V(G)$ and for any $D \subseteq V(G)$.

Now let D be an imo-set in G. Then $\sum_{v\in D} d_D(v) = 0$ and $d_{V-D}(v) \leq 1$ for every $v \in V - D$. Since $\sum_{v\in D} d_{V-D}(v) = \sum_{v\in V-D} d_D(v)$, it follows that

$$\begin{aligned} 3n &= 2\sum_{v\in D} d_{V-D}(v) + \sum_{v\in V-D} d_{V-D}(v) \\ &= 2\sum_{v\in D} d(v) + \sum_{v\in V-D} d_{V-D}(v) \\ &\leq 2\sum_{v\in D} 3 + \sum_{v\in V-D} 1 \\ &= 6|D| + |V-D| \\ &= 6|D| + n - |D| \\ &= 5|D| + n. \end{aligned}$$

Therefore, $|D| \leq \frac{2n}{5}$. $\square$

3. The exact values of independent monopoly size in some standard graphs

In this section, we obtain the exact values of independent monopoly size, $imo(G)$, in some standard graphs.

Theorem 15.

Let G be a graph of order n.

(a) If $G = K_n$, then $imo(G)$ is defined if and only if $n \leq 3$. Furthermore, $imo(G) = 1$.
(b) If $G = P_n$, then $imo(G) = \lceil \frac{n}{3} \rceil$.
(c) If $G = C_n$, then $imo(G) = \lceil \frac{n}{3} \rceil$.
(d) If $G = K_{1,n}$, then $imo(G) = 1$.
(e) If $G = K_{r,s}$ for $2 \leq r, s \leq n$, then $imo(G) = min\{r, s\}$.
(f) If $G = W_n$ for $n \geq 4$, then

$$imo(G) = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd;} \\ \text{Not defined}, & \text{if } n \text{ is even.} \end{cases}$$

Proof:

Parts (a)-(e) in Theorem 15 are consequences of results from section 3. Thus, we will only verify part (f). In $W_n = K_1 + C_{n-1}$, let v_0 be the central vertex and let $\{v_1, v_2, ..., v_{n-1}\}$ be the vertex set of C_{n-1}. There are two cases:

Case 1If n is odd, then $\alpha(W_n) = \frac{n-1}{2}$. Clearly that the subset $D = \{v_1, v_3, ..., v_{n-2}\}$ is an independent set in W_n. Since $|N(v_0) \cap D| = |D| = n-1 = \frac{d(v_0)}{2}$ and $|N(v_i) \cap D| = 2 > \frac{d(v_i)}{2}$ for every $i = 2, 4, ..., n-1$, then D is also a monopoly set in W_n. Furthermore, D is a minimum imo-set in W_n. Therefore $imo(W_n) = \frac{n-1}{2}$ for n odd.

Case 2If n is even, then $\alpha(W_n) = \frac{n-2}{2}$. Since $d(v_0) = n-1$, it follows that v_0 does not belong to any maximal independent set in W_n. Thus for a maximum independent set I in W_n, $|N(v_0) \cap I| = |I| = \frac{n-2}{2} < \frac{d(v_0)}{2}$. Hence any independent set in W_n is not a monopoly set. On other hand, $mo(W_n) = \frac{n-1}{2}$. Hence, any monopoly set in W_n is not independent. Therefore, $imo(W_n)$ is not defined for n even.

□

If G is a non-travail graph of order n, for which $imo(G)$ is defined, then

$$1 \leq imo(G) \leq n.$$

In the following results, we characterize all non-travail graphs G of order n for which $imo(G) \in \{1, n-1, n\}$.

Theorem 16.

Let G be a graph of order $n \geq 2$ for which $imo(G)$ is defined. Then

(1) $imo(G) = 1$ if and only if G has a vertex of degree $n-1$ and the degrees of all remaining vertices is at most 2.
(2) $imo(G) = n$ if and only if $G = \overline{K_n}$.
(3) $imo(G) = n-1$ if and only if $G = (n-2)K_1 \cup K_2$.

Proof:

(1) Let G be a graph of order $n \geq 2$ and let $v_0 \in V$ with $d(v_0) = n - 1$ and $d(v) \leq 2$ for every $v \in V - \{v_0\}$. Clearly that G is connected and $v_0 \in N(v)$ for every $v \neq v_0$. thus, $|N(v) \cap \{v_0\}| = 1 \geq \frac{d(v)}{2}$ for every $v \notin \{v_0\}$. Hence, $\{v_0\}$ is a minimum imo-set in G. Conversely, let the subset $\{w\} \subset V$ be an imo-set in G. Then since $n \geq 2$, it follows that $V - \{w\} \neq \phi$. Furthermore, $|V - \{w\}| = n - 1$. By the definition of the monopoly set in a graph G, $1 \geq |N(v) \cap \{w\}| \geq \frac{d(v)}{2}$, for every $v \notin \{w\}$. Hence, $d(v) \leq 2$ for every $v \in V - \{w\}$. Also, $vw \in E(G)$ for every $v \neq w$. Hence, $d(w) = n - 1$, because if there is a vertex $u \neq w$ such that $wu \notin E(G)$, then $|N(u) \cap \{w\}| = 0$, a contradiction.

(2) The proof is clear from the definition of monopoly.

(3) Let $G = (n-2)K_1 \cup K_2$. Then by Lemma 1 and Theorem 15, we get

$$imo(G) = imo((n-2)K_1) + imo(K_2) = n - 2 + 1 = n - 1.$$

Conversely, let $imo(G) = n - 1$ and let D be an imo-set of G. Then $|V - D| = 1$ and $d(v) \leq 1$ for every $v \in D$. Let $V - D = \{w\}$. On the contrary, assume that $d(w) \neq 1$. Then we consider the following cases

Case 1: $d(w) = 0$, then $G = \overline{K_n}$. Thus, by part (2), $imo(G) = n$, a contradiction.

Case 2: $d(w) = a \geq 2$, then $G = (n-a+1)K_1 \cup K_{1,a}$. Let $S = \{v_1, ..., v_a\} = V(k_{1,a}) - \{w\}$. Then the subset $H = D \cup \{w\} - S$ is an imo-set of G with $|H| = n - a \leq n - 2 < n - 1 = |D|$, a contradiction.

Therefore, $d(w) = 1$, and $G = (n-2)K_1 \cup K_2$. □

Theorem 17.

For a graph G of order n and $\Delta \leq 2$, then

$$imo(G) = i(G).$$

Proof:

The proof is similar to the proof of Theorem 2.3 in Naji et al. (2015). □

4. Conclusion

In this paper, we introduced the concept of independence monopoly size in graphs. We found that not all graphs have independent monopoly set. We studied the existence of independent monopoly set in graphs and we established bounds for independence monopoly size, $imo(G)$, of a graph G, for which imo-set exists, in terms clique number, number of vertices, independence number, and dominating number. We obtained the exact values of independence monopoly size $imo(G)$ of some standard graphs. We characterized all non-trivial graphs of order n with $imo(G)$ equal to $1, n-1$ and n. Investigation of the independent monopoly of graphs is an open area of research. We conclude this paper with the following open problems:

Open Problem. Characterize all graphs G of order n with

(1) $imo(G) = mo(G)$.
(2) $imo(G) = \alpha(G)$.

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REFERENCES

Berger, E. (2001). Dynamic monopolies of constant size, Journal of Combinatorial Theory, Series B, Vol. 83, pp. 191-200.

Bermond, J., Bond, J., Peleg, D., and Perennes, S. (2003). The power of small coalitions in graphs, Discrete Applied Mathematics, Vol. 127, pp. 399-414.

Bondy, J. A. and Murty, U. S. R. (2008). *Graph Theory*, Springer, Berlin.

Goddard, W. and Henning, M. A. (2013). Independent domination in graphs: A survey and recent results, Discrete Mathematics, Vol. 313, pp. 839-854.

Flocchini, P., Kralovic, R., Roncato, A., Ruzicka, P., and Santoro, N. (2003). On time versus size for monotone dynamic monopolies in regular topologies. Journal of Discrete Algorithms, Vol. 1, pp. 129-150.

Harary, F. (1969). *Graph Theory*, Addison Wesley, Massachusetts.

Haynes, T. W., Hedetniemi, S. T., and Slater, P. J. (1998). *Fundamentals of Domination in Graphs*, Marcel Dekker, New York.

Khoshkhak, K., Nemati, M., Soltani, H., and Zaker M. (2013). A study of monopoly in graphs, Graph and Combinatorial Mathematics, Vol. 29, pp. 1417-1427.

Li, H. and Virlouvet, C. (1990). Neighborhhod conditions for claw-free Hamiltonian graph, Ars Combinatoria, Vol. 29A, pp. 109-116.

Mishra, A. and Rao, S. B. (2006). Minimum monopoly in regular and tree graphs, Discrete Mathematics, Vol. 306, No. 14, pp. 1586-1594.

Naji, A. M. and Soner, N. D. (2015). On the monopoly of graphs, Proceeding of the Jangjeon Mathematical Society, Vol. 2, No. 18, pp. 201-210.

Peleg, D. (2002). Local majorities; coalitions and monopolies in graphs; a review, Theoretical Computer Science, Vol. 282, pp. 231-257.

Zaker, M. (2012). On dynamic monopolies of graphs with general thresholds, Discrete Mathematics, Vol. 312, pp. 1136-1143.