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On mathematical modeling, nonlinear properties and stability of secondary flow in a dendrite layer

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Abstract

This paper studies instabilities in the flow of melt within a horizontal dendrite layer with deformed upper boundary and in the presence or absence of rotation during the solidification of a binary alloy. In the presence of rotation, it is assumed that the layer is rotating about a vertical axis at a constant angular velocity. Linear and weakly nonlinear stability analyses provide results about various flow features such as the critical mode of convection, neutral stability curve, preferred flow pattern and the solid fraction distribution within the dendrite layer. The preferred shape of the deformed upper boundary of the layer, which is found to be caused by the temperature variations of the secondary flow, is detected to be the same as that for the stable and preferred horizontal flow pattern within the dendrite layer.

Keywords: convective flow; flow stability; mathematical model; dendrite layer; nonlinear flow

MSC 2000: 76E06, 76E30

1. Introduction

This paper considers the problem of convective flow and the associated instabilities in a dendrite region, which exists adjacent to a solidifying alloy. There have been a number of research studies of convective flow during solidification of binary alloys (Roberts and Loper 1983; Fowler 1985; Worster 1991, 1992; Tait et al. 1992; Amberg and Homsy 1993; Anderson and Worster 1995; Chung and Chen 2000, 2003; Guba 2001; Riahi 2002, 2003, 2005, 2006; Guba and Worster 2006). Understanding such flow and the associated instabilities in the region adjacent to the solidifying interface, which is important both in fundamental research study and in the engineering and geophysical applications (Copley et al.1970; Fowler1985; Tait and Jaupart1992; Bergman1997), has been a main goal of these studies.

Anderson and Worster (1995) studied the problem of steady convection during alloy solidification in a dendrite layer, which is also called a mushy layer in the literature (Worster1991, 1992; Tait and Jaupart1992), and used a single-layer model of the mushy zone developed first by Amberg and Homsy (1993). The single-layer model developed by these later authors was under a near-eutectic approximation in the limit of large far-field temperature, and the mushy layer was assumed to be bounded by two horizontal rigid and flat boundaries. Anderson and Worster (1995) calculated the steady solutions in the form of two-dimensional rolls and hexagons (Busse 1978) and analyzed the stability of these solutions. They found that rolls or hexagons with either up-flow or down-flow at the cells' centers could be stable, depending on the parameter values.

Guba (2001) studied steady convection in a rotating horizontal mushy layer bounded above and below by two rigid and flat boundaries. His model excluded the interactions between the local solid fraction and the convection associated with the Coriolis term in the Darcy-momentum equation (Joseph 1976). He determined the finite-amplitude steady solutions in the form of rolls and hexagons. Riahi (2003) extended this work by taking into account the interactions between the local solid fraction and the flow associated with the Coriolis term and, in addition, carried out a stability analysis of a number of determined steady solutions.

Riahi (2002) studied weakly nonlinear oscillatory modes of two- and three-dimensional convection and their stability in a mushy layer, which was bounded again by two horizontal rigid and flat boundaries. He found, in particular, that depending on the values of the parameters, only supercritical modes of convection in the form traveling or standing rolls can be stable. Riahi (2006) extended this work to the rotating case for both two- and three-dimensional flows. Guba and Worster (2006) considered two-dimensional case and extended the work by Riahi (2002) to a wider range of the parameter values.

Chung and Chen (2003) studied linear flow instabilities in an inclined liquid layer during directional solidification and under inclined rotation. They found, in particular, that morphological mode was slightly stabilized while convective modes in the liquid layer were significantly stabilized. Riahi (2005) extended this work and studied linear problem of the flow instabilities in a horizontal dendrite layer rotating about an inclined axis.

In the present problem we make use of a mathematical model for flow in a dendrite layer adjacent to a horizontal solidifying alloy in the presence or absence of rotation in order to determine qualitative results about various flow features and the deformed upper boundary of the layer. Such qualitative results can, in particular, be beneficial in materials processing and geophysical applications. For example, in alloy solidification processes it is desirable to impose certain external constraints, such as rotation, in an optimized manner upon the solidification system, in order to reduce the induced flow and instabilities. The induced flow and the associated instabilities are undesirable since they can lead to micro-defect density and imperfections, which are called freckles (Copley et al. 1970; Fowler 1985), in the alloy crystal, and, thus, can reduce the quality of the produced crystal. The prediction of the preferred morphology of the upper boundary of the dendrite layer for the nonlinear flow problem, which is another goal of the present study, can be important and of interest, in particular, in the geophysical applications. For example, implications with respect to the observed earth's inner-core anisotropy (Bergman 1997), which is one of the major unsolved problem about the earth's deep interior, and the relationship with respect to the important geodynamo problem. The present mathematical model for the dendrite layer with deformed upper boundary is relatively more realistic than any other model that has been studied mathematically so far for such problem (Fowler 1985; Amberg and Homsy 1993; Anderson and Worster 1995; Chung and Chen 2000, 2003; Guba 2001; Riahi 2002, 2003, 2005, 2006; Guba and Worster 2006). Thus, the present results can be more beneficial at least as far as the above applications are concerned.

2. Mathematical formulation

We consider a horizontal layer of a binary alloy melt of some constant composition C_0 and temperature T_∞, which is cooled from below and is solidified at a constant speed V_0, with the eutectic temperature T_e at the position $\tilde{z}=0$. This position is held fixed in a frame moving with the solidification speed in the vertical $\tilde{z}$-direction. The solidifying system is assumed to be rotating at a constant speed Ω about a unit vector $\mathbf{z}$ in the positive direction of the $\tilde{z}$-axis. Here $\tilde{z}$-axis is anti-parallel to the gravity vector. Within the layer of melt, there is a dendrite layer adjacent to the solidifying surface and of thickness $\tilde{h}(\tilde{x}, \tilde{y}, \tilde{t})$, where the solid dendrites and the liquid melt coexist. Here $\tilde{t}$ is the time variable, and $\tilde{x}$ and $\tilde{y}$ are the horizontal variables along the $\tilde{x}$- and $\tilde{y}$-axes in the horizontal plane $\tilde{z}=0$.

In this paper we focus on the so-called mushy-layer mode (Worster 1992) at the onset of convection for the flow within the dendrite layer only. It is known both theoretically (Worster 1992) and experimentally (Tait and Jaupart 1992) that such mode is mainly responsible for the formation of convection in the so-called chimneys within the dendrite layer, which can lead to undesirable freckle effects in the solidified alloy. The present dendrite-layer model is assumed to be in local thermodynamic equilibrium (Worster 1992) and, thus,

$$\tilde{T}=T_L(C_0)+M(\tilde{C}-C_0). \tag{1}$$

Here $\tilde{T}$ is the temperature, T_L is the liquidus temperature of the alloy, $\tilde{C}$ is the composition and M is the slope of the liquidus (Worster 1992).

Next, we consider the equations for momentum, continuity, heat and solute (Worster 1991; Riahi 2003) for both liquid region ($\tilde{z}>\tilde{h}$), which is assumed to be motionless, and dendrite region ($0<\tilde{z}<\tilde{h}$) in the frame $o\tilde{x}\tilde{y}\tilde{z}$. This frame is assumed to be moving with the solid-mush interface ($\tilde{z}=0$) at speed V_0 and rotating with the solidifying system at the constant angular velocity Ω. The governing system is non-dimensionalized by using V_0, k/V_0, k/V_0^2, $\beta\Delta C\rho_0 gk/V_0$, ΔC and ΔT as scales for velocity, length, time, pressure, solute and temperature, respectively. Here k is the thermal diffusivity, ρ_0 is a reference (constant) density, $\beta=\beta^*-M\alpha^*$, α^* and β^* are the expansion coefficients for the heat and solute respectively and M is assumed to be constant, $\Delta C=C_0-C_e$, $\Delta T=T_L(C_0)-T_e$, g is acceleration due to gravity and C_e is the eutectic composition. Just as in a number of problems in applications where the effect of the centrifugal force is negligible in comparison to the force of gravity, the present investigation takes into account the rotational effects only through the presence of the Coriolis force.

The non-dimensional form of the equations for momentum, temperature and solute concentration in the liquid layer in the motionless state, where the continuity equation is satisfied identically, are

$$\nabla\tilde{P}+R^{*}\tilde{S}\mathbf{z}=0, \tag{2a}$$

$$(\partial/\partial\tilde{t}-\partial/\partial\tilde{z})\tilde{\theta}=\nabla^{2}\tilde{\theta}, \tag{2b}$$

$$(\partial/\partial\tilde{t}-\partial/\partial\tilde{z})\tilde{S}=\lambda\nabla^{2}\tilde{S}. \tag{2c}$$

Here $\tilde{P}$ is the modified pressure, $R^{*}=k^{2}\beta g\Delta C/(V_{0}{}^{2}\nu)$ is the solutal Rayleigh number in the liquid layer (Worster 1992), $\tilde{S}$ is the non -dimensional concentration, $\tilde{\theta}$ is the non-dimensional temperature, $\lambda=k_{s}/k$ is the inverse of a Lewis number, k_s is the solute diffusivity, and $\mathbf{z}$ is a unit vector in the positive direction of the $\tilde{z}$-axis. For simplicity of notation, all the non-dimensional independent variables are designated by their dimensional symbols. The parameter λ is assumed to be small as is the case in the applications (Worster1992).

The non-dimensional form of the equations for momentum, continuity, temperature and solute concentration in the dendrite layer, which is assumed to be a porous layer under Darcy's law (Roberts and Loper 1983; Fowler 1985; Worster 1991), are

$$K(\tilde{\phi})\tilde{\mathbf{u}}=-\nabla\tilde{P}-\tilde{R}\,\tilde{\theta}\,\mathbf{z}+T\,\tilde{\mathbf{u}}\times\mathbf{z}/(1-\tilde{\phi}), \tag{3a}$$

$$\nabla.\tilde{\mathbf{u}}=0, \tag{3b}$$

$$(\partial/\partial\tilde{t}-\partial/\partial\tilde{z})(\tilde{\theta}-S_{t}\tilde{\phi})+\tilde{\mathbf{u}}.\nabla\tilde{\theta}=\nabla^{2}\tilde{\theta}, \tag{3c}$$

$$(\partial/\partial\tilde{t}-\partial/\partial\tilde{z})[(1-\tilde{\phi})\tilde{\theta}+C_{r}\tilde{\phi}]+\tilde{\mathbf{u}}.\nabla\tilde{\theta}=0. \tag{3d}$$

Here $\tilde{\mathbf{u}}=\tilde{u}\mathbf{x}+\tilde{v}\mathbf{y}+\tilde{w}\mathbf{z}=(1-\tilde{\phi})\mathbf{U}$ is the volume flux per unit area (Joseph 1976), $\tilde{\phi}$ is the local solid fraction within the dendrite layer, $\mathbf{U}$ is the velocity vector, $\tilde{u}$ and $\tilde{v}$ are the horizontal components of $\tilde{\mathbf{u}}$ along the $\tilde{x}$- and $\tilde{y}$-directions, respectively, $\mathbf{x}$ and $\mathbf{y}$ are unit vectors along the positive $\tilde{x}$- and $\tilde{y}$-directions, and $\tilde{w}$ is the vertical component of $\tilde{\mathbf{u}}$ along the $\tilde{z}$-direction. The main non-dimensional parameter $\tilde{R}=\beta\Delta Cg\Pi(0)/(V_{0}\nu)$ is the Rayleigh number in the dendrite layer, $\Pi(0)$ is a reference value at $\tilde{\phi}=0$ of the permeability $\Pi(\tilde{\phi})$ of the porous medium, and ν is the kinematic viscosity. In addition, $K(\tilde{\phi})\equiv\Pi(0)/\Pi(\tilde{\phi})$, $S_t=L/(C_l\,\Delta T)$ is the Stefan number, C_l is the specific heat per unit volume, L is the latent heat of solidification per unit volume, $C_r=(C_s-C_0)/\Delta C$ is a concentration ratio, C_s is the composition of the solid phase forming the dendrites and $T=2\Omega\Pi(0)/\nu$ is the Coriolis parameter. The equation (3d) is based on the limit of sufficiently large value of the Lewis number.

The governing equations (2)-(3) are subjected to the following boundary conditions:

$$\tilde{\theta}+1=\tilde{w}=0 \quad \text{at} \quad \tilde{z}=0, \tag{4a}$$

$$[\tilde{\theta}]=[\mathbf{n}.\nabla\tilde{\theta}]=[\mathbf{n}.\tilde{\mathbf{u}}]=\tilde{\phi}=0 \quad \text{at} \quad \tilde{z}=h, \tag{4b}$$

$$\tilde{\theta}\to\theta_\infty,\ \tilde{S}\to 0 \text{ as } \quad z\to\infty \tag{4c}$$

Here the square brackets denote the jump in the enclosed quantity across the mush-liquid interface, $\mathbf{n}$ is a unit vector normal to the interface, $\tilde{h}=Vh_0/k$ is the dimensionless depth of the dendrite layer and θ_∞ is the non-dimensional form of T_∞.

Following Amberg and Homsy (1993) and Anderson and Worster (1995) in reducing the model asymptotically, we assume certain rescaling to be given below in the limit of sufficiently small δ, where δ is the assumed constant value of h in the absence of motion.

$$C_r=C/\delta,\ S_t=S/\delta,\ \ \varepsilon<<\delta<<1, \tag{5a}$$

$$(\tilde{x},\tilde{y},\tilde{z})=\delta(x, y, z), \quad \tilde{t}=\delta^2 t, \quad R^2=\delta\tilde{R}, \tag{5b}$$

$$\tilde{\mathbf{u}}=\varepsilon R\mathbf{u}/\delta, \quad \tilde{P}=RP. \tag{5c}$$

Here C is an order one quantity, and S is either an order one or an order less than one quantity as $\delta\to 0$. As discussed in Anderson and Worster (1995), the assumption of thin dendrite layer ($\delta<<1$) is associated with the large non-dimensional far-field temperature, which can occur when the initial $\tilde{C}$ is close to C_e. The assumption of order one quantity C corresponds to the near-eutectic approximation (Fowler 1985), which allows one to describe the dendrite layer as a porous layer of constant permeability to the leading order.

The rescaling (5a)-(5c) are then used in the governing system (2)-(4). The resulting system admits a motionless basic state, which is steady and horizontally uniform. The basic state solution, denoted by the subscript 'B', which is also referred to as the primary solution, is given below to leading orders in terms of the asymptotic expansions for $\delta<<1$.

$$\theta_B=\theta_\infty+(\theta_i-\theta_\infty)\exp[\delta(1-z)],\ \ \theta_i=\lambda\theta_\infty/(\lambda-1), \quad z>1, \tag{6a}$$

$$S_B=\exp[\delta(1-z)/\lambda], \quad z>1, \tag{6b}$$

$$P_B=P_0+R/2-R\delta\theta_i\exp[\delta(1-z)\lambda], \quad z>1, \tag{6c}$$

$$\theta_B=(z-1)-\delta G(z^2-z)/2+\ldots,\ 0<z<1, \tag{6d}$$

$$\phi_B=-\delta(z-1)/C+\delta^2[-(z-1)^2/C^2+G(z^2-z)/(2C)]+\ldots,0<z<1, \tag{6e}$$

$$P_B=P_0-R[(z^2/2-z)-\delta G(z^3/2-z^2/2)/2+\ldots], \quad 0<z<1, \tag{6f}$$

where P_0 is a constant, $G\equiv S/C+1$ and the above solutions are assumed to be valid in the limits of

$$\lambda << \delta,\ \lambda^2 << 1/R. \tag{7}$$

Since $\tilde{\phi}$ is expected to be small, according to the results (6e), the following expansion for $K(\tilde{\phi})$ will be implemented later in the governing system:

$$K(\tilde{\phi})=1+K_1\tilde{\phi}+K_2\tilde{\phi}^{\,2}+\ldots, \tag{8}$$

where the coefficients K_1 and K_2 are constants.

For the analysis to be presented in the next section, it was found convenient to use the general representation

$$\mathbf{u}=\mathbf{\Omega}V+\mathbf{E}\Psi,\ \mathbf{\Omega}=\nabla\times\nabla\times\mathbf{z},\ \mathbf{E}=\nabla\times\mathbf{z}, \tag{9}$$

for the divergence-free vector field $\mathbf{u}$ (Chandrasekhar 1961). Here V and ψ are the poloidal and toroidal functions for $\mathbf{u}$, respectively. Taking the vertical components of the curl and the double –curl of the Darcy's momentum equation (3a) and using (5)-(9) in (3)-(4), we find the system for the dependent variables of the disturbances superimposed on the basic state for the flow in the dendrite layer. It is

$$\nabla^2[K(\phi_B+\varepsilon\phi)\Delta_2 V]+(\partial/\partial z)[\mathbf{\Omega}V.\nabla K(\phi_B+\varepsilon\phi)]+(\partial/\partial z)\{[\nabla K(\phi_B+\varepsilon\phi)\times\nabla\psi].\mathbf{z}\}-R\Delta_2\theta+$$

$$T(\partial/\partial z)\{[1/(1-\phi_B-\varepsilon\phi)]\Delta_2\psi+[\nabla(\partial/\partial z)V\times\nabla(1/(1-\phi_B-\varepsilon\phi))].\mathbf{z}+\nabla_2\psi.\nabla_2(1/(1-\phi_B-\varepsilon\phi))\}=0, \tag{10a}$$

$$K(\phi_B+\varepsilon\phi)\Delta_2\psi+[\nabla(\partial/\partial z)V\times\nabla K(\phi_B+\varepsilon\phi)].\mathbf{z}+\nabla_2\psi.\nabla_2 K(\phi_B+\varepsilon\phi)-T\{[1/(1-\phi_B-\varepsilon\phi)]\Delta_2(\partial/\partial z)V-$$

$$[\nabla\psi\times\nabla(1/(1-\phi_B-\varepsilon\phi))].\mathbf{z}+\Delta_2(\partial/\partial z)V.\nabla_2[1/(1-\phi_B-\varepsilon\phi)]\}=0, \tag{10b}$$

$$(\partial/\partial t-\delta\partial/\partial z)[-\theta+(S/\delta)\phi]+R(d\theta_B/dz)\Delta_2 V+\nabla^2\theta=\varepsilon R(\mathbf{\Omega}V+\mathbf{E}\Psi).\nabla\theta, \tag{10c}$$

$$(\partial/\partial t-\delta\partial/\partial z)[(-1+\phi_B)\theta+\theta_B\phi+\varepsilon\theta\phi-C\phi/\delta]+R(d\theta_B/dz)\Delta_2 V=\varepsilon R(\mathbf{\Omega}V+\mathbf{E}\Psi).\nabla\theta, \tag{10d}$$

$$\theta=V=0 \quad \text{at} \quad z=0, \tag{10e}$$

$$\partial\theta/\partial z-(S/C)\eta=\theta=V=\phi=0 \quad \text{at} \quad z=1, \tag{10f}$$

where

$$\eta=h-\delta,\ \Delta_2=\partial^2/\partial x^2+\partial^2/\partial y^2,\ \nabla_2=(\partial/\partial x,\ \partial/\partial y,\ 0) \tag{10g}$$

and

$$\varepsilon(\theta,\ \phi)=(\tilde{\theta}-\theta_B,\ \tilde{\phi}-\phi_B) \tag{10h}$$

In summary, we developed a mathematical model for the investigation of flow instabilities within a horizontal dendrite layer with a deformed upper boundary during alloy solidification and in the presence or absence of rotation. This model is based on the system (10a)-(10f) of nonlinear partial differential equations and appropriate boundary condition for the dependent variables of the disturbances, which will be considered in the following sections.

3. Analysis and secondary solutions

In this section we briefly describe weakly nonlinear and stability analyses using the method of approach in Riahi (2002, 2003) to determine the finite-amplitude secondary solutions and their stability for the present problem. The reader is referred to Riahi (2002, 2003) for details of the procedure. The analysis to determine the secondary solutions is based on double series expansions in powers of the two small parameters δ and ε which satisfy the condition given in (5a). Since we consider general types of modes, which can be oscillatory or stationary, the appropriate expansions are for the dependent variables of the perturbation system, R, η and the frequency ω of the oscillatory modes. These expansions are

$$(V, \psi, \theta, \phi, \eta, R, \omega)=[(V_{00}+\delta V_{01}+\ldots), (\psi_{00}+\delta\psi_{01}+\ldots), (\theta_{00}+\delta\theta_{01}+\ldots), (\phi_{00}+\delta\phi_{01}+\ldots),$$

$$(\eta_{00}+\delta\eta_{01}+\ldots), (R_{00}+\delta R_{01}+\ldots), (\omega_{00}+\delta\omega_{01}+\ldots)]+\varepsilon[(V_{10}+\delta V_{11}+\ldots), (\psi_{10}+\delta\psi_{11}+\ldots),$$

$$(\theta_{10}+\delta\theta_{11}+\ldots), (\phi_{10}+\delta\phi_{11}+\ldots), (\eta_{10}+\delta\eta_{11}+\ldots), (R_{10}+\delta R_{11}+\ldots), (\omega_{10}+\delta\omega_{11}+\ldots)]+\varepsilon^2$$

$$[(V_{20}+\delta V_{21}+\ldots), (\psi_{20}+\delta\psi_{21}+\ldots), (\theta_{20}+\delta\theta_{21}+\ldots), (\phi_{2(-1)}/\delta+\phi_{20}+\delta\phi_{21}+\ldots), (\eta_{20}+\delta\eta_{21}+\ldots), (R_{20}+$$

$$\delta R_{21}+\ldots), (\omega_{20}+\delta\omega_{21}+\ldots)]+\ldots \qquad (11)$$

Here the expansion of ϕ is singular at order ε^2 as $\delta\to 0$, but it turns out that $O(1/\delta)$ is needed only in the stability analysis of the secondary solutions since the $O(\varepsilon^2)$ problem is found to be forced by a term of $O(1/\delta)$ in the solute equation for the disturbances superimposed on the secondary solutions.

Considering the system (10a)-(10f) to the lowest order in ε, we find the linear problem. At order ε^0/δ, the linear system yield $\omega_{00}=0$. At order $\varepsilon^0\delta^0$, the same system yields the following results:

$$V_{00}=[(\pi^2+a^2)/(R_{00}\, a^2\, G)]\sin(\pi z)\sum_{n=-N}^{N} U_n,\ U_n\equiv(A_n^+W_n^++A_n^-W_n^-), \qquad (12a)$$

$$\psi_{00}=T\pi[(\pi^2+a^2)/(R_{00}\, a^2 G)]\cos(\pi z)\sum_{n=-N}^{N} U_n, \qquad (12b)$$

$$\theta_{00}= -\sin(\pi z)\sum_{n=-N}^{N} U_n, \qquad (12c)$$

$$\phi_{00}=\{-\pi(\pi^2+a^2)/[CG(-\omega_{01}^2+\pi^2)]\}\sum_{n=-N}^{N} [f_n(z)U_n^++f_n^*(z)U_n^-], \qquad (12d)$$

$$\eta_{00}=[\pi/(G-1)]\sum_{n=-N}^{N} U_n, \qquad (12e)$$

where

$$f_n(z)=\{(i\omega_{01}S_n)/\pi]\sin(\pi z)+\cos(\pi z)+\exp[i\omega_{01}S_n(z-1)]\}, \qquad (12f)$$

$$U_n^{\pm} \equiv A_n^{\pm} W_n^{\pm},\ W_n^{\pm} \equiv \exp[i(\mathbf{a}_n.\mathbf{r} \pm S_n\omega t)], \qquad (12g)$$

$$R_{00}^2=(\pi^2+a^2)[(\pi^2+a^2)+(\pi T)^2]/(a^2G), \qquad (12h)$$

$$S_n=1 \text{ for } n>0 \text{ and } -1 \text{ for } n<0. \qquad (12i)$$

Here i is the pure imaginary number, subscript 'n' takes only non-zero integers from –N to N, N is a positive integer representing the number of distinct modes, $\mathbf{r}$ is the position vector. The horizontal wave number vectors $\mathbf{a}_n$ satisfy the properties

$$\mathbf{a}_n.\mathbf{z}=0,\ |\mathbf{a}_n|=a,\ \mathbf{a}_{-n}= -\mathbf{a}_n. \qquad (13)$$

The coefficients A_n^+ and A_n^- are constants and satisfy the conditions

$$\sum_{n=-N}^{N} (A_n^+ A_n^{+*}+A_n^- A_n^{-*}) =2,\ A_n^{\pm*} =A_{-n}^{\pm}, \qquad (14)$$

where the asterisk indicates the complex conjugate. Minimizing the expression for R_{00} given in (12h), with respect to the wave number a, we find

$$R_{00c}= \pi[1+(1+T^2)^{1/2}]/ \sqrt{G}, \qquad (15a)$$

$$a_c= \pi(1+T^2)^{1/4}, \qquad (15b)$$

where R_{00c} is the minimum value of R_{00} achieved at $a=a_c$.

Considering the governing system (10a)-(10f) in the order $\varepsilon^0\delta^1$, eliminating ψ_{01} and ϕ_{01} and applying the existence condition of the type carried out in Riahi (2002), we find that the real and imaginary parts of this condition yield

$$(R_{01c}/R_{00c})=[K_1/(4C)]\{[\pi^2+a^2-(\pi T)^2]/[\pi^2+a^2+(\pi T)^2]+(\pi T)^2/[2C(\pi^2+a^2)+2C(\pi T)^2]+ GG_t\{1/4+$$

$$\pi^2[1+\cos(\omega_{01})]/(\pi^2-\omega_{01}^2)^2\}, \qquad (16a)$$

$$\omega_{01}\{1+G_t[(\pi^2+a^2)/(\pi^2 -\omega_{01}^2)][1-2\pi^2\sin\omega_{01}/(\omega_{01}\pi^2-\omega_{01}^3)]\}, \qquad (16b)$$

where $G_t=(G-1)/(CG^2)$.

Hence, the critical Rayleigh number R_c can be written as

$$R_c=R_{00c}+\delta R_{01c}+ O(\delta^2). \qquad (17)$$

Next, we analyze the nonlinear system (10a)-(10f) for the secondary solutions. At order ε/δ, we find $\omega_{10}=0$. We follow the procedure given in Riahi (2002, 2003) to determine the solvability conditions, which are the necessary conditions for the existence of the secondary solutions (Drazin and Reid 1981), for the systems in orders ε and $\varepsilon\delta$. We then find that $R_{10}=R_{11}=0$ for the oscillatory modes as well as for the steady non-hexagonal modes (Busse 1978). For the case of non-rotating steady hexagonal mode (Busse 1978), we find $R_{10}=0$, but the expression for R_{11}, in general, is non-zero. In both rotating and non-rotating cases, the expression for R_{11}, is found to be very lengthy and involves a number of finite integrals in the vertical variable and, thus, will not be given here. In the rotating steady hexagonal case, the expression for R_{10} is found to be

$$R_{10}=-\pi T^2(\pi^2 + a_c^2)^2/(R_{00c} a_c^2 C\sqrt{6}). \tag{18}$$

The leading order solutions to the ε-system, which is derived from (10a)-(10f), are then found. These solutions are found to be functions of independent variables and non-dimensional parameters as well as on $\Phi_{mn} \equiv \mathbf{a}_m.\mathbf{a}_n/a^2$ and $\Psi_{mn} \equiv \mathbf{a}_m \times \mathbf{a}_n/a^2$ (m, n = -N, …, -1, 1, …, N).

We now consider the system (10a)-(10f) in the order ε^2. The solvability conditions for this system yield the expression for R_{20}, which together with (14), were used to study the steady and oscillatory solutions. Just like R_{11}, the expression for R_{20} is found to be very lengthy and involves many finite integrals in the vertical variable and, thus, will not be given here. We restrict the analyses to regular or semi-regular solutions where (14) yield, for each n

$$|A_n^+|^2=|A_n^-|^2=1/(2N),\ |A_n^+|^2=1/N\ (|A_n^-|=0),\ |A_n^-|^2=1/N\ (|A_n^+|=0),\ (|A_n^+|^2, |A_n^-|^2)=[(0.5-b), (05+b)]/N, \tag{19}$$

for steady (or standing), left-traveling, right traveling and general traveling waves, respectively. Here a given value of the parameter b in its appropriate range b<0.5 (Riahi 2002) provides particular general traveling waves. Here by a right traveling wave, it is meant that the phase velocity of the wave is in the direction of the component of the position vector along the wave number vector. By a left traveling wave it is meant that the phase velocity of the wave is in the direction opposite to that of the component of the position vector along the wave number vector. In later references for a semi-regular secondary solution in the form of rectangles, an angle γ, which is less than 90°, is defined to be the angle between two adjacent wave number vectors of any rectangular cell.

To distinguish the physically realizable secondary solution among all the possible steady and oscillatory secondary solutions, the stability of the secondary solution is investigated by using the equations (10a)-(10f). We replace each of the dependent variables in these equations by the corresponding dependent variable for the sum of the secondary solutions and the perturbations of infinitesimal amplitude and with addition of a time dependence of the form exp(σt) for the perturbations. Here σ is the growth rate of the disturbances. Following the procedure carried out in Riahi (2002, 2003), we then obtain from the above resulting equations the stability system for the disturbances, which was solved by an expansion similar to (11) for the secondary solutions.

Using the stability system described in the previous paragraph, it is found that the growth rate to the lowest order in ε of the most critical disturbances is zero. Solvability conditions to the orders ε, εδ and ε^2 of the stability system are then combined to determine the expression for σ^*, where

$$\sigma^*=\varepsilon\sigma_{10} + \varepsilon\delta\sigma_{11} + \varepsilon^2\sigma_{20} \tag{20}$$

as the leading-order growth rate. The coefficient σ_{10} is found to be non-zero only for the steady secondary solutions in the form of hexagons in the rotating system. The coefficient σ_{11} is found to be non-zero only for the steady secondary solutions in the form of hexagons, and the coefficient σ_{20} is found to be generally non-zero for either steady or oscillatory secondary solutions.

4. Results and discussion

The linear system derived from the linear version of (10a)-(10f) and its associated eigenvalue problem, which led to the results (12)-(17), are, in general, functions of the parameters G, G_t (or equivalently C and S) and T. Here and thereafter value of δ=0.1 is chosen to evaluate R_c and other quantities whose values may depend on δ. The well known stabilizing effect of the Coriolis force on convective flow (Chandrasekhar 1961) can be seen from the expressions (15), (16a) and (17) for R_c and a_c. However, it can be seen from (15b) that a_c is independent from G and G_t, while R_c given by (17) in combination with (15a) and (16a) depends strongly on G and only weakly on G_t. To calculate the values of the frequency ω_{01}, we used the equation (16b) for the frequency, which involves the parameters T and G_t. It should be noted that for any given values of T and G_t, this equation can admits solution for $|\omega_{01}|$ but with arbitrary sign, provided this equation has non-zero solution for the frequency. We first assigned various values for T and the frequency and then used (16b) to find the corresponding values for G_t. Next, for given values of T and G_t, we carried out a Newton type numerical method of iteration (Anderson et al. 1984) to search for all possible values of the frequency. We found that for given values of T and G_t, there is a unique non-zero value for the magnitude $|\omega_{01}|$ if G_t is not less than some critical value G_c. This critical value was found to be 0.5 in the absence of rotation and decreases slowly with increasing T. Thus, no oscillatory secondary solution exists if $G_t<G_c$. For $G_t \geq G_c$, both oscillatory and steady secondary solutions can exist. However, we found that the critical value R_c of the Rayleigh number, which is the stability parameter in the convective flow problems of the type we studied in this paper, is smaller for the oscillatory solutions in this regime. Thus, oscillatory solutions are preferred over the steady secondary solutions if $G_t \geq G_c$. Here the preferred secondary solutions for the linear problem are those which correspond to the lowest value of the critical Rayleigh number R_c for a given set of values of the parameters. For the linear version of the system (10a)-(10f) infinite number of steady secondary solutions for $G_t \leq 0.5$ and infinite number of oscillatory secondary solutions for $G_t>0.5$ are possible. However, such linear degeneracy is reduced significantly by the consideration of the nonlinear system (10a)-(10f) of the problem. In both non-rotating and rotating cases, ω_{01} was found to be unaffected with respect to the variation of G if G_t is implicitly kept fixed. For $G_t \geq G_c$, the magnitude of the frequency was found to increase with T or with G_t. Thus in this regime the period of the oscillatory mode decreases with increasing both of these parameters. The results (15a), (16a) and (17) for R_c as functions of G, G_t and T are found to indicate that R_c increases with both T and G_t but decreases with increasing G. The result (15b) for the critical wave number indicates that the wavelength of the secondary flow decreases with increasing the rate of rotation and is independent with respect to either G or G_t.

The results given in (10f)-(10g) indicate that the deformation of the upper boundary of the dendrite layer is caused by the temperature or equivalently compositional variations of the secondary flow. Due to the degeneracy of the linear version of the system (10a)-(10f), the linear results for the flow within the dendrite layer and for the shape of the deformed upper boundary of the layer are applicable to both two- and three-dimensional cases. The result (12e) indicates that the shape of the deformed upper boundary of the dendrite layer due to the linear system is proportional to $[1/(G-1)]\sum_{n=-N}^{N} U_n$. Since $G>1$, the deformation of the upper boundary is amplified significantly as G approaches the value 1, which is the case when the scaled Stephen number S is very small or when the scaled concentration ratio C is relatively large. The result for the shape of the upper boundary also indicates that there is not any preference on the shape of this boundary due to the linear degeneracy of the problem. Thus any shape with the participation of any arbitrary N number of modes is possible.

Important quantities due to the nonlinear effects are the coefficients $R_1 \equiv R_{10}+\delta R_{11}$ and R_{20}, which are computed in the present study from the corresponding solvability conditions described in the previous section. These coefficients represent leading contributions to the change in R required to obtain finite amplitude ε for a nonlinear solution. In terms of these coefficients, the amplitude $|\varepsilon|$ is found to be of order

$$|\varepsilon|=\{\pm |R_1| \pm [R_1^2+4R_{20}(R-R_c)]^{1/2} \}/(2R_{20}). \qquad (21)$$

As can be seen from (21), there are 4 expressions for $|\varepsilon|$ if $R_1 \neq 0$, which turned out to correspond to steady hexagonal mode cases only. For the case $\varepsilon R_1<0$, the two roots with plus sign in front of $|R_1|$ provide the expressions for $|\varepsilon|$, while the two roots with a negative sign preceding $|R_1|$ provide the expressions for $|\varepsilon|$ if $\varepsilon R_1>0$. For either case $\varepsilon R_1<0$ or case $\varepsilon R_1>0$, the expression with a plus sign in front of the square-root term corresponds to the case where R_{20} is positive, while the expression with negative sign in front of the square-root term corresponds to the case where R_{20} is negative.

The expression (21) provides some qualitative nonlinear results about the variation of the $|\varepsilon|$ with respect to R. We assume that initially $|\varepsilon|$ is sufficiently small. For $\varepsilon R_1<0$ and $R_{20}>0$, $|\varepsilon|$ increases first with decreasing R and then increases with R beyond some value. For $\varepsilon R_1<0$ and $R_{20}<0$, $|\varepsilon|$ increases indefinitely with decreasing R, while for $\varepsilon R_1>0$ and $R_{20}>0$, $|\varepsilon|$ increases indefinitely with increasing R. For $\varepsilon R_1>0$ and $R_{20}<0$, $|\varepsilon|$ increases first with R and then decreases with increasing R beyond some value.

Using the equation for η given in (10f) in the orders ε and ε^2, the preferred flow structure and the shape of the deformed upper boundary of the layer are found to be caused and determined by the secondary flow solutions that correspond to the lowest value of R. Further more, the amplitude $|\varepsilon|$ for the preferred secondary solution is found to increase with R, a result that corresponds to the observation and agrees with the experimental expectation that the heat or solute transported by the convective flow (proportional to ε^2) should increase with R (Joseph 1976). Our stability analysis verified that those bifurcation branches for which the amplitude decreases with increasing R are unstable and, thus, not physically realizable. So the information about R_1 and R_{20} can be useful in the sense that they could indicate preference and stability of particular secondary solutions and vice versa.

For the case of R_1=0, which was found to correspond to either secondary oscillatory solutions or secondary steady non-hexagonal type solutions, the sign of R_{20} determines whether any such solution exists for values of R above or below R_c. For supercritical flow, where $R>R_c$, then the result (21) indicates that the amplitude of the convective flow is largest, provided the value of R_{20} is smallest among all the secondary solutions to the nonlinear problem.

In the present problem, R_1 and R_{20} are found to be due to the nonlinear convective terms in the temperature equation, the interaction between the flow velocity and the permeability and in the rotating case also due to the interactions between the solid fraction and the Coriolis term in the equation (3a). It should also be noted that the variations of R_1 with respect to different parameters provide, in particular, information about various destabilizing and stabilizing features for the steady hexagonal convection, while the information for R_{20} can be useful, in particular, in calculating the heat or solute flux.

The shape of the deformed upper boundary of the layer to the order ε^2 for the nonlinear system (10a)-(10f) was found to become a linear combination of $\sum_{n=-N}^{N} U_n$ and $\varepsilon\sum_{m,\ n=-N}^{N} U_m\ U_n$. Making use of the solvability conditions in the orders ε, $\varepsilon\delta$ and ε^2, we computed the coefficients R_{10}, R_{11} and R_{20} for different plan forms, such as rolls, rectangles, squares and hexagons, for either steady or oscillatory secondary flow, and for different values of the parameters T, G and G_t. Since the expressions for both R_{11} and R_{20} involve many finite integrals in z, the computations of these two coefficients for different types of solutions required evaluation of those integrals numerically. In the steady case, the preferred and stable form of either the flow structure or the shape of the deformed upper boundary was found to be that of the steady hexagons for sufficiently small $|\varepsilon|$ and that of the steady non-hexagons for $|\varepsilon|$ beyond some value. Depending on the parameter values, the preferred steady non-hexagons can be in the form of two-dimensional rolls, squares or rectangular cells, and the preferred steady hexagons can be in the form of either up-hexagons or down-hexagons. In the case of up-hexagons, the flow is upward at the cells' centers and downward at the cells' boundaries, while in the case of down-hexagons, the flow is downward at the cells' centers and upward at the cells' boundaries.

In the oscillatory case and depending on the parameter values, the preferred and stable form of either the flow structure or the shape of the deformed upper boundary was found to be one of those which are given as follows. They are oscillatory hexagons in the form of traveling up-hexagons, standing up-hexagons, traveling down-hexagons or standing down-hexagons, and traveling or standing non-hexagons in the form of rolls, squares or rectangles.

The results presented in this section provide the details of the types of secondary flows and various stabilizing and destabilizing features that often exist in various applications of the convective flow during alloy solidification. In particular, these results can be used in the design of the manufacturing of the solidified alloy to improve the quality and strength of the produced crystal. For example, the results for particular parameter regime that were provided in this section can be used to aid in elimination or at least reduction of the original instabilities that can occur during alloy solidification. Such instabilities tend to enhance the convective flow which can reduce the quality of the produced alloy crystal.

5. Some remarks

The present results, which are based on a more realistic mathematical model for the flow during alloy solidification, indicate that the effects of the Coriolis force, due to a moderate rotation rate, can generally be stabilizing and, hence, beneficial for the associated convective flow. Such convective flow can occur in a number of industrial and engineering applications. Presence of rotation is found to reduce the strength of such convective flow, which can reduce the tendency for the chimney formation in the dendrite layer. It is known that formation of chimneys in the dendrite layer and the convective flow within such chimneys can lead to freckle defects in the final produced alloy crystal in the solidification system (Copley et al.1970; Fowler1985; Tait and Jaupart1992). The results of the present study indicate that imposing an external constraint of rotation on the solidification system, which can weaken the convective flow and, thus, reduce the tendency for the chimney formation within the dendrite layer. Hence, imposing an external constraint of rotation upon such flow system appears to be a useful controlling approach that needs to be implemented to improve the quality of the produced solidified alloy.

In regard to the relevance of the present results to the geophysical applications, the outstanding unsolved problem of the observed anisotropy lines on the surface of the earth's inner core (Bergman 1997), can be an example. Bergman (1997) proposed that the inner core's anisotropy and the depth dependence of the anisotropy may be due to the solidification texturing that result from the growth of crystals within a dendrite layer. Such dendrite layer is known be over at least some surface area of the inner core and below the liquid zone of the outer core. Our present results indicate that the deformation of the surface on the top of the dendrite layer is a direct result of the realized flow within the dendrite layer, and the shape of the deformed surface is a copy of that due to the stable flow structure within the underneath layer. However, since this kind of geophysical application of the present problem requires significantly higher values of the Rayleigh number in the dendrite layer (Bergman 1997), computational studies will be needed in future to determine the extend about which the present results are valid.

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