Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 4, Issue 1 (June 2009) pp. 52 – 61
(Previously, Vol. 4, No. 1)

Applications and Applied Mathematics:
An International Journal
(AAM)

Adomian Decomposition Method for Solving the Equation Governing the Unsteady Flow of a Polytropic Gas

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Received: May 22, 2008; Accepted: January 22, 2009

Abstract

In this article, we have discussed a new application of Adomian decomposition method on nonlinear physical equations. The models of interest in physics are considered and solved by means of Adomian decomposition method. The behavior of Adomian solutions and the effects of different values of time are investigated. Numerical illustrations that include nonlinear physical models are investigated to show the pertinent features of the technique.

Keywords: Partial Differential Equations; Differential Transform Method; Approximation Method

MSC (2000) No.: 34K28; 35G25; 34K17

1. Introduction

Nonlinear phenomena that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics can be modeled by partial differential equation. A broad class of analytical solutions methods and numerical solutions methods were used in handle these problems. The Adomian decomposition method has been proved to be effective and reliable for handling differential equations, linear or nonlinear.

Various methods for seeking explicit travelling solutions to nonlinear partial differential equations are proposed such as Wadati et al. (1992), Wadati et al. (1975), Wadati (2001), Wadati (1972), Drazin et al. (1997). In the beginning of the 1980, a so-called Adomian decomposition method (ADM), which appeared in Adomian (1994), Adomian and Serrano (1998), Adomian et al. (1995), Deeba and Khuri (1996), Oldham (1974), Podlubny (1999), Wazwaz (2002), Wazwaz (2000), ElWakil et al. (in press), Abdou (2005), Kaya and El-Sayed (2003), Seng and Abbaoui (1996), and Lesnic (2006) has been used to solve effectively, easily, and accurately a large class of linear and nonlinear equations, solutions partial, deterministic or stochastic differential equations with approximates which converge (see Figure 2).

Unlike classical techniques, the nonlinear equations are solved easily and elegantly without transforming the equation by using the ADM. The technique has many advantages over the classical techniques, mainly, it avoids linearization and perturbation in order to find explicit solutions of a given nonlinear equations. To give a clear view to our study, we have chosen the equation governing the unsteady flow of a polytropic gas to illustrate the analysis of the Adomian.

2. The Adomian decomposition

For the purpose of illustration of the methodology to the proposed method, using **ADM**, we begin by considering the differential equation

$$Lu + Ru + Nu = g, \tag{1}$$

with prescribed conditions, where u is the unknown function, L is the highest order derivative which is assumed to be easily invertible, *R* is a linear differential operator of less order than *L* (operator *L* is linear also), *Nu* represents the nonlinear term and g is the source term. Assuming the inverse operator *L* exists and it can be taken as the definite integral with respect to *t* from t_0 to *t*, i.e.

$$L^{-1} = \int_{t_0}^{t} (\square) dt. \tag{2}$$

Applying the inverse operator L^{-1} to both sides of equation (1) and using the initial conditions we find

$$u = f - L^{-1}[Ru + Nu], \tag{3}$$

where the function $f(x, y)$ represents the term arising from integrating the source term *g* and from using the given initial or boundary conditions, all are assumed to be prescribed.

The nonlinear operator $[Nu]$ can be decomposed by an infinite series of polynomials given by

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \cdots, u_n), \tag{4}$$

where $A_n(u_0, u_1, \cdots, u_n)$ the appropriate Adomian's polynomials are defined by Adomian G.(1994), Adomian G, Serrano SE.(1998).

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0}, \quad n > 0. \tag{5}$$

This formula is easy to compute by using Mathematica software or by setting a computer code to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions. The Adomian decomposition method assumes a series that the unknown function $u(x,y,t)$ can be expressed by an infinite series of the form

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t). \tag{6}$$

Identifying the zeros component $u(x,y,0)$ the remaining components where $n > 1$ can be determined by using the recurrence relation

$$u_0(x,y) = f(x,y), \tag{7}$$

$$u_{n+1}(x,y,t) = -L^{-1}[R(u_n) - A_n], \quad n \geq 0 \ . \tag{8}$$

Then other polynomials can be generated in a similar way. The scheme (8) can easily determine the components $u_n(x,y,t)$ It is in principle, possible to calculate more components in the decomposition series to enhance the approximation. One cannot compute an infinite number of terms; only a quite limited number of terms are determined of the series $\sum_{n=0}^{\infty} u_n(x,y,t)$ and hence the solution $u(x,y,t)$ is readily obtained. It is interesting to note that we obtained the solution by using the initial condition only.

3. Application

For simplicity, we are interested to deal with Adomian decomposition solution associated with the operator L^{-1} rather than the other operators in our example.

The equation governing the unsteady flow of a polytropic gas in two dimensions is given by Feng X (1996), Billingham (2004), Rogers and Ames (1988).

$$u_t + uu_x + vu_y + \frac{p_x}{\rho} = 0, \tag{9}$$

$$v_t + uv_x + vv_y + \frac{p_y}{\rho} = 0, \tag{10}$$

$$\rho_t + u\rho_x + v\rho_y + \rho(u_x + v_y) = 0, \tag{11}$$

$$p_t + up_x + vp_y + \gamma p(u_x + v_y) = 0, \tag{12}$$

where ρ is the density, p the pressure, u and v the velocity components in the x and y directions, respectively, and the adiabatic index γ is the ratio of the specific heats.
With the initial data:

$$u(x,y,0)=e^{x+y}, \tag{13}$$

$$v(x,y,0)=-1-e^{x+y}, \tag{14}$$

$$\rho(x,y,0)=e^{x+y}, \tag{15}$$

$$p(x,y,0)=c. \tag{16}$$

Note that the selection of equations (9-12) that are obtained from Billingham (2004) the fluid is incompressible and invisid (no viscose), because we assumed the sentences $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\neq 0,\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\neq 0$ that appear in equation (9), which is hard to solve.

Equations (9-12) can be written in an operator form as

$$Lu=-[N_1(u,u_x)+K_1(v,u_y)+H_1(\rho,p_x)], \tag{17}$$

$$Lv=-[N_2(u,v_x)+K_2(v,v_y)+H_2(\rho,p_y)], \tag{18}$$

$$L\rho=-[N_3(u,\rho_x)+K_3(v,\rho_y)+H_3(\rho,u_x)+G_3(\rho,v_y)], \tag{19}$$

$$Lu=-[N_4(u,p_x)+K_4(v,p_y)+\gamma H_4(p,u_x)+\gamma G_4(p,v_y)], \tag{20}$$

where $L=\frac{\partial}{\partial t}$.

The Adomian Decomposition Method **(ADM)** assumes a series solution of the unknown functions

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y,t), \tag{21}$$

$$v(x,y,t)=\sum_{n=0}^{\infty}v_n(x,y,t), \tag{22}$$

$$\rho(x,y,t)=\sum_{n=0}^{\infty}\rho_n(x,y,t), \tag{23}$$

$$p(x,y,t)=\sum_{n=0}^{\infty}p_n(x,y,t). \tag{24}$$

Substituting Equations (21-24) with initial conditions into Equations (17-20) yields

$$\sum_{n=0}^{\infty}u(x,y,t)=u(x,y,0)-L^{-1}\left[N_1(u,u_x)+K_1(v,u_y)+H_1(\rho,p_x)\right], \tag{25}$$

$$\sum_{n=0}^{\infty}v(x,y,t)=v(x,y,0)-L^{-1}\left[N_2(u,v_x)+K_2(v,v_y)+H_2(\rho,p_y)\right], \tag{26}$$

$$\sum_{n=0}^{\infty}\rho(x,y,t)=\rho(x,y,0)-L^{-1}\left[N_3(u,\rho_x)+K_3(v,\rho_y)+H_3(\rho,u_x)+G_3(\rho,v_y)\right], \tag{27}$$

$$\sum_{n=0}^{\infty}p(x,y,t)=p(x,y,0)-L^{-1}\left[N_4(u,p_x)+K_4(v,p_y)+\gamma H_4(p,u_x)+\gamma G_4(p,v_y)\right], \tag{28}$$

where the functions

$N_1(u,u_x), K_1(v,u_y), H_1(\rho,p_x), N_2(u,v_x), K_2(v,v_y), H_2(\rho,p_y), N_3(u,\rho_x), K_3(v,\rho_y), H_3(\rho,u_x),$

$G_3(\rho,v_y), N_4(u,p_x), K_4(v,p_y), H_4(p,u_x)$ and $G_4(p,v_y)$

are:

$$N_1(u,u_x)=uu_x=\sum_{n=0}^{\infty}A_{1n}(u,u_x)=u_0u_{0x}+\left(u_1u_{0x}+u_0u_{1x}\right)+\cdots=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_m u_{(n-m)x} \tag{29}$$

$$K_1(v,u_y)=vu_y=\sum_{n=0}^{\infty}B_{1n}(v,u_y)=v_0u_{0y}+\left(v_1u_{0y}+v_0u_{1y}\right)+\cdots=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}v_m u_{(n-m)y} \tag{30}$$

$$H_1(\rho,p_x)=\rho^{-1}p_x=\sum_{n=0}^{\infty}C_{1n}(\rho,p_x)=\frac{p_{0x}}{\rho_0}+\frac{\rho_0p_{1x}-\rho_1p_{0x}}{\rho_0}+\cdots \tag{31}$$

$$N_2(u,v_x)=uv_x=\sum_{n=0}^{\infty}A_{2n}(u,v_x)=u_0v_{0x}+\left(u_1v_{0x}+u_0v_{1x}\right)+\cdots=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_m v_{(n-m)x} \tag{32}$$

$$K_2(v,v_y)=vv_y=\sum_{n=0}^{\infty}B_{2n}(v,v_y)=v_0v_{0y}+\left(v_1v_{0y}+v_0v_{1y}\right)+\cdots=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}v_m v_{(n-m)y} \tag{33}$$

$$H_2(\rho,p_y)=\rho^{-1}p_y=\sum_{n=0}^{\infty}C_{2n}(\rho,p_y)=\frac{p_{0y}}{\rho_0}+\frac{\rho_0p_{1y}-\rho_1p_{0y}}{\rho_0}+\cdots \tag{34}$$

$$N_3(u,\rho_x)=u\rho_x=\sum_{n=0}^{\infty}A_{3n}(u,\rho_x)=u_0\rho_{0x}+\left(u_1\rho_{0x}+u_0\rho_{1x}\right)+\cdots=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_m\rho_{(n-m)x} \tag{35}$$

$$K_3(v,v_y)=v\rho_y=\sum_{n=0}^{\infty}B_{3n}(v,\rho_y)=v_0\rho_{0y}+\left(v_1\rho_{0y}+v_0\rho_{1y}\right)+\cdots=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}v_m\rho_{(n-m)y} \tag{36}$$

$$H_3(\rho,u_x)=\rho u_x=\sum_{n=0}^{\infty}C_{3n}(\rho,u_x)=\rho_0u_{0x}+\left(\rho_1u_{0x}+\rho_0u_{1x}\right)+\cdots=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho_m u_{(n-m)x} \tag{37}$$

$$G_3(\rho,v_y)=\rho v_y=\sum_{n=0}^{\infty}D_{3n}(\rho,v_y)=\rho_0v_{0y}+\left(\rho_1v_{0y}+\rho_0v_{1y}\right)+\cdots=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho_m v_{(n-m)y} \tag{38}$$

$$N_4(u,p_x)=up_x=\sum_{n=0}^{\infty}A_{4n}(u,p_x)=u_0p_{0x}+\left(u_1p_{0x}+u_0p_{1x}\right)+\cdots=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_m p_{(n-m)x} \tag{39}$$

$$K_4(v,p_y)=vp_y=\sum_{n=0}^{\infty}B_{4n}(v,p_y)=v_0p_{0y}+\left(v_1p_{0y}+v_0p_{1y}\right)+\cdots=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}v_m p_{(n-m)y} \tag{40}$$

$$H_4(p,u_x)=\gamma pu_x=\gamma\sum_{n=0}^{\infty}C_{4n}(p,u_x)=\gamma\left[p_0u_{0x}+\left(p_1u_{0x}+p_0u_{1x}\right)+\cdots\right]=\gamma\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}p_m u_{(n-m)x}\,. \tag{41}$$

Identifying the zeros components of u_0, v_0, ρ_0 and p_0 the remaining components $u_n(x,y,t), v_n(x,y,t), \rho_n(x,y,t)$ and $p_n(x,y,t),\ n>1$ can be determined by using recursive relations given by

$$u(x,y,0)=e^{x+y},$$

$$v(x,y,0)=-1-e^{x+y},$$

$$\rho(x,y,0)=e^{x+y},$$

$$p(x,y,0)=c,$$

$$u_{n+1}(x,y,t)=-L^{-1}\left[N_1(u_n,u_{nx})+K_1(v_n,u_{ny})+H_1(\rho_n,p_{nx})\right], \tag{43}$$

$$v(x,y,t)=-L^{-1}\left[N_2(u_n,v_{nx})+K_2(v_n,v_{ny})+H_2(\rho_n,p_{ny})\right], \tag{44}$$

$$\rho(x,y,t)=-L^{-1}\left[N_3(u_n,\rho_{nx})+K_3(v_n,\rho_{ny})+H_3(\rho_n,u_{nx})+G_3(\rho_n,v_{ny})\right], \tag{45}$$

$$p(x,y,t)=-L^{-1}\left[N_4(u_n,p_{nx})+K_4(v_n,p_{ny})+\gamma H_4(p_n,u_{nx})+\gamma G_4(p_n,v_{ny})\right]. \tag{56}$$

The remaining components u_n, v_n, ρ_n and p_n can be completely determined such that each term that determined by using the previous terms, and the series solutions thus entirely evaluated.

$$u_1(x,y,t)=\frac{t}{1!}e^{t+y},\quad v_1(x,y,t)=-\frac{t}{1!}e^{x+y},\quad \rho_1(x,y,t)=\frac{t}{1!}e^{x+y} \tag{47}$$

$$u_2(x,y,t)=\frac{t^2}{2!}e^{t+y},\quad v_2(x,y,t)=-\frac{t^2}{2!}e^{x+y},\quad \rho_2(x,y,t)=\frac{t^2}{2!}e^{x+y} \tag{48}$$

$$u_3(x,y,t)=\frac{t^3}{3!}e^{t+y},\quad v_3(x,y,t)=-\frac{t^3}{3!}e^{x+y},\quad \rho_3(x,y,t)=\frac{t^3}{3!}e^{x+y} \tag{49}$$

$$u_4(x,y,t)=\frac{t^4}{4!}e^{t+y},\quad v_4(x,y,t)=-\frac{t^4}{4!}e^{x+y},\quad \rho_4(x,y,t)=\frac{t^4}{4!}e^{x+y} \tag{50}$$

$$p_n(x,y,t)=0 \quad ,n=1,2,3,\cdots, \tag{51}$$

etc.

In general we have

$$u_n(x,y,t)=\frac{t^n}{n!}e^{x+y} \tag{52}$$

$$v_n(x,y,t)=-\frac{t^n}{n!}e^{x+y} \tag{53}$$

$$\rho_n(x,y,t)=\frac{t^n}{n!}e^{x+y} \tag{54}$$

$$p_n(x,y,t)=0 \quad ,n=1,2,3,\cdots \tag{55}$$

The solution of $u(x,y,t), v(x,y,t), \rho(x,y,t)$ and $p(x,y,t)$ are

$$u(x,y,0)=e^{x+y+t}, \tag{56}$$

$$v(x,y,0)=-1-e^{x+y+t}, \tag{57}$$

$$\rho(x,y,0)=e^{x+y+t}, \tag{58}$$

$$p(x,y,0)=c. \tag{59}$$

Adomian solutions coincides with the exact solution

$$(u, v, \rho, p) = (e^{x+y+t}, -1 - e^{x+y+t}, e^{x+y+t}, c) . \tag{60}$$

With different values of time *t*, it is shown from Figures 1a and 1b, the exact solution (60). Also the behavior of the solution is shown in Figure 2.

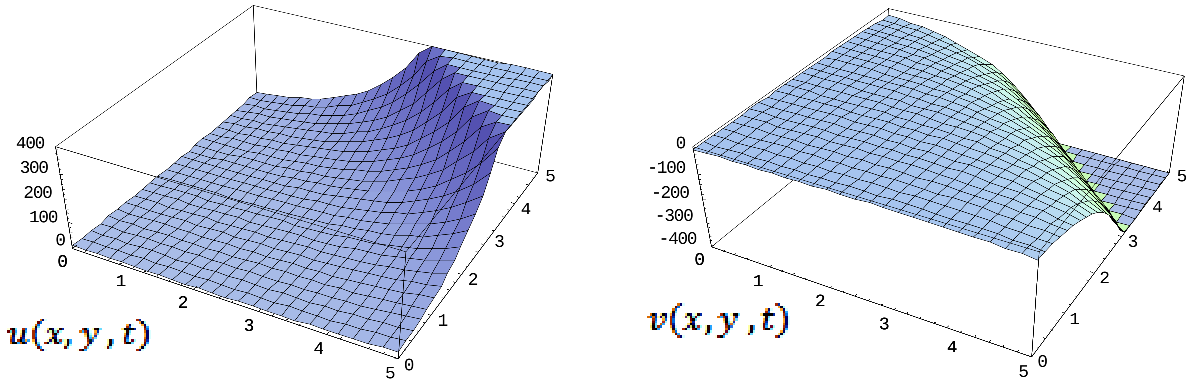

Figure 1a. Exact solution of $u(x, y, t)$ and $v(x, y, t)$ for $t : 0 \rightarrow 5$, $x : 0 \rightarrow 5$, $y = -2, c = 4$

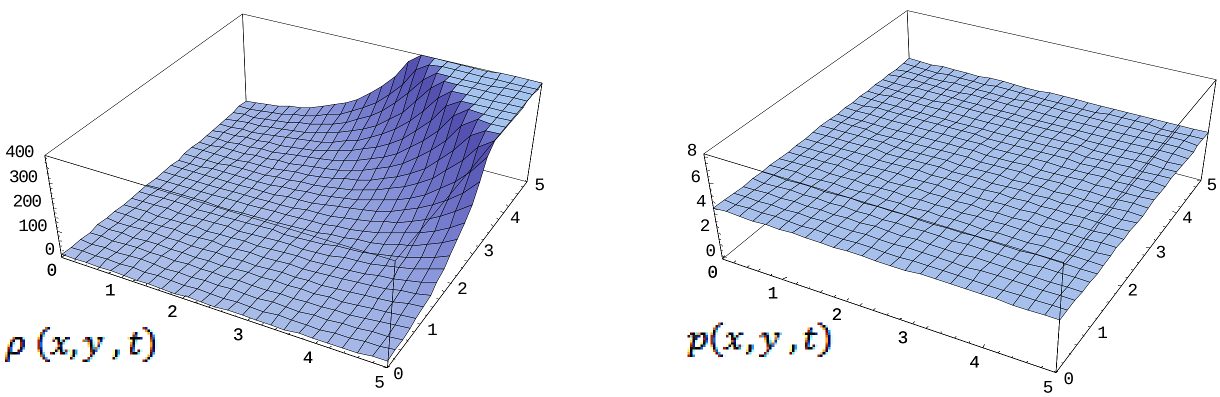

Figure 1b. Exact solution of $\rho(x, y, t)$ and $p(x, y, t)$ for $t : 0 \rightarrow 5$, $x : 0 \rightarrow 5$, $y = -2, c = 4$

The following figures show the difference between numerical and exact solution which show the dependency of the error to the number of terms *M* since from the following figures, when we increase the number of terms the solution converges to the exact solution

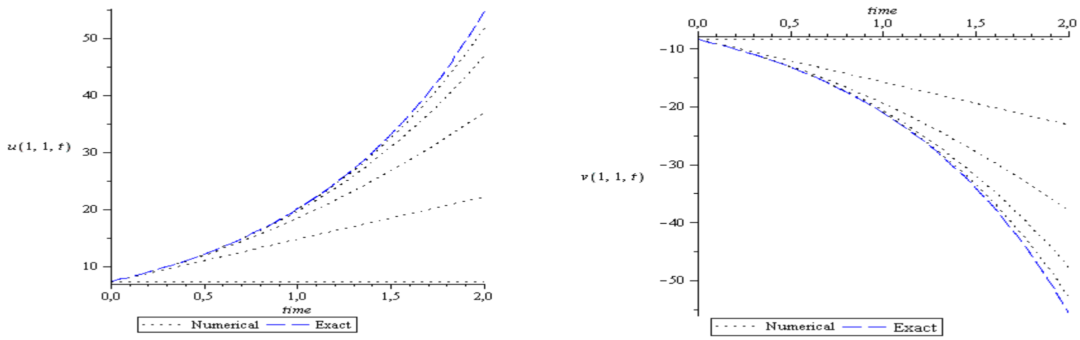

Figure 2. comparison between the exact solution and the behavior of the solution obtained by ADM method

By the same manner the functions $\rho(1,1,t)$ and $p(1,1,t)$ can be obtained to which gives us the same results that we got it from the last figures.

4. Conclusion

In this article, Adomian decomposition method for approximating the solutions of the equation governing the unsteady flow of a polytropic gas is implemented. By using this scheme, explicit exact solutions arising in nonlinear physics are calculated in form of a convergent power series with easily computable components. To illustrate the application of this method, numerical results were derived by using the calculated components of the decomposition series. Numerical illustrations are investigated to show the pertinent features of the technique. The results reported here provide further evidence of the usefulness of Adomian decomposition method **(ADM)**. The **ADM** was clearly very efficient and powerful technique in finding the solutions the equation governing the unsteady flow of a polytropic gas since we can retch to the exact solution after few iterations of using Adomian decomposition method **(ADM)**. It is clear that this method avoids linearization and biologically unrealistic assumptions, and provides an efficient numerical solution.

Acknowledgements

The author is thankful to anonymous referees for their useful suggestions, which led to the present form of the paper and I am highly grateful to Professor Dr. A. M. Haghighi for his constructive comments.

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