Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 7, Issue 2 (December 2012), pp. 556 - 570

On The Numerical Solution of Linear Fredholm-Volterra İntegro Differential Difference Equations With Piecewise İntervals

Mustafa Gülsu and Yalçın Öztürk
Deparment of Mathematics
Mugla University
Mugla,Turkey
mgulsu@mu.edu.tr; yozturk@mu.edu.tr

Received:August 03, 2011; Accepted: October 6, 2012

Abstract

The numerical solution of a mixed linear integro delay differential-difference equation with piecewise interval is presented using the Chebyshev collocation method. The aim of this article is to present an efficient numerical procedure for solving a mixed linear integro delay differential difference equations. Our method depends mainly on a Chebyshev expansion approach. This method transforms a mixed linear integro delay differential-difference equations and the given conditions into a matrix equation which corresponds to a system of linear algebraic equation. The reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments and performed on the computer algebraic system Maple 10.

Keywords: Mixed linear integro delay differential-difference equations; Chebyshev polynomials and series; Approximation methods; Collocation points

MSC 2010 No.: 41A58; 39A10; 34K28; 41A10

1. Introduction

In recent years, the studies of mixed integro delay differential-difference equations have developed very rapidly. These equations may be classified into two types; the Fredholm integro-differential-difference equations and Volterra integro-differential-difference equations. The upper bound of the integral part of Volterra type is variable, while it is a fixed number for that of

Fredholm type. In this paper we focus on Fredholm Volterra integro differential difference equations with piecewise intervals. Integro-differential-difference equations are important, but are often harder to solve, even numerically, and progress on how to solve them has been slow. Problems involving these equations arise frequently in many applied areas including engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc. [Emler (2001,2002), Ren (1999), Rashed (2004), Kadalbajoo (2002,2004), Bainov (2000), Cao (2004),]: The study of integro differential difference equations has great interest in contemporary research work. Several numerical methods, such as the successive approximations, Adomian decomposition, Chebyshev and Taylor collocation, Haar Wavelet, Tau and Walsh series methods, etc. [Ortiz (1998), Hosseini (2003), Zhao (2006), Maleknejad (2006), Sezer (2005a, 2005b), Synder (1966), Gülsu (2010)] are used for their solution. Mainly we deal with the following integro delay differential-difference equation with piecewise intervals

$$\begin{aligned}&\sum_{k=0}^{m} P_k(x)y^{(k)}(x)+\sum_{s=0}^{n} H_s(x)y^{(s)}(x-\tau)=g(x)\\&\sum_{i=0}^{u}\lambda_i\int_{a_i}^{b_i}F_i(x,t)y(t)dt+\sum_{j=0}^{v}\mu_j\int_{a_j}^{x}K_j(x,t)y(t)dt\end{aligned} \tag{1}$$

$x\in[-\tau,0]$, $-1\le a_i,b_i,c_j\le -1$ under the mixed conditions

$$\sum_{k=0}^{m-1}\sum_{j=0}^{r} c^k{}_{ij}\, y^{(k)}(c_{ij})=\lambda_i, -\tau\le c_{ij}\le 0,\ i=0,1,...m-1, \tag{2}$$

where $y(x)$ is an unknown function, the known $P_k(x)$, $H_s(x)$, $F_i(x,t)$, $K_j(x,t)$ and $g(x)$ are defined on an interval and also c_{ij}^k, c_{ij}, λ_i and μ_s are appropriate constant. Our aim is to find an approximate solution expressed in the form

$$y(x)=\sum_{r=0}^{N} a_r T_r(x),\ 0\le i\le N, \tag{3}$$

where a_r, $r=0,1,2,...,N$, are unknown coefficients and N is any chosen positive integer such that $N\le m$. To obtained a solution in the form(3) of the problem (1) and (2), we may use the collocation points defined by

$$x_i=\frac{-\tau}{2}\left(1+\cos(\frac{i\pi}{N})\right), i=0,1,2,\dots,N. \tag{4}$$

The remainder of the paper is organized as follows: Higher-order linear mixed integro-delay-differential-difference equation with variable coefficients with piecewise intervals and fundamental relations are presented in Section 2. The method of finding approximate solution is described in Section 3. To support our findings, we present numerical results of some experiments using Maple10 in Section 4. Section 5 concludes this article with a brief summary.

2. Fundamental Matrix Relations

Let us write Eq.(1) in the form

$$D(x)+H(x)=g(x)+\sum_{i=0}^{u}\lambda_i I_i(x)+\sum_{j=0}^{v}\mu_j J_j(x),$$

where the differential part

$$D(x)=\sum_{k=0}^{m}P_k(x)y^{(k)}(x)$$

and the difference part

$$H(x)=\sum_{s=0}^{n}H_s(x)y^{(s)}(x-\tau)$$

the Fredholm integral part

$$I_i(x)=\int_{a_i}^{b_i}F_i(x,t)y(t)dt$$

and Volterra integral part

$$J_j(x)=\int_{c_j}^{x}K_j(x,t)y(t)dt\,.$$

We convert these equations and the mixed conditions in to the matrix form. Let us consider the Eq. (1) and find the matrix forms of each term of the equation. We first consider the solution $y(x)$ and its derivative $y^{(k)}(x)$ defined by a truncated Chebyshev series. Then we can put series in the matrix form

$$y(x)=\mathbf{T}(x)\mathbf{A}\,,\quad y^{(k)}(x)=\mathbf{T}^{(k)}(x)\mathbf{A}\,, \tag{5}$$

where

$$\mathbf{T}(x)=[T_0(x)\;T_1(x)\;...\;T_2(x)],\;\;\mathbf{T}^{(k)}(x)=[T_0^{(k)}(x)\;T_1^{(k)}(x)\;...\;T_N^{(k)}(x)],\;\mathbf{A}=[a_0\;a_1\;...\;a_N]^T\,.$$

On the other hand, it is well known that [Synder (1966)] the relation between the powers x^n and the Chebyshev polynomials $T_n(x)$ is

$$x^{2n} = 2^{-2n+1} \sum_{j=0}^{n} \binom{2n}{n-j} T_{2j}(x), \; -1 \le x \le 1, \tag{6}$$

and

$$x^{2n+1} = 2^{-2n} \sum_{j=0}^{n} \binom{2n+1}{n-j} T_{2j+1}(x), \; -1 \le x \le 1. \tag{7}$$

Using the expression (6) and (7) and taking $n = 0,1,...,N$, we obtain the corresponding matrix relation as follows:

$$\mathbf{X}^T(x) = \mathbf{D}\mathbf{T}^T(x) \;\text{ and }\; \mathbf{X}(x) = \mathbf{T}(x)(\mathbf{D}^T)^{-1}, \tag{8}$$

where

$$\mathbf{X} = [1 \; x^1 \; ... \; x^N]^T .$$

for odd N,

$$\mathbf{D} = \begin{bmatrix} \frac{1}{2}\binom{0}{0}2^1 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0}2^0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{1}2^{-1} & 0 & \binom{2}{0}2^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \binom{N}{(N-1)/2}2^{1-N} & 0 & \cdots & \binom{N}{0}2^{1-N} \end{bmatrix}$$

and for even N,

$$\mathbf{D} = \begin{bmatrix} \frac{1}{2}\binom{0}{0}2^1 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0}2^0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{1}2^{-1} & 0 & \binom{2}{0}2^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\binom{N}{N/2}2^{1-N} & 0 & \binom{N}{(N-2)/2}2^{1-N} & \cdots & \binom{N}{0}2^{1-N} \end{bmatrix}.$$

Then, by (8), we obtain

$$\mathbf{T}(x) = \mathbf{X}(x)(\mathbf{D}^T)^{-1} \tag{9}$$

and

$$\mathbf{T}^{(k)}(x) = \mathbf{X}^{(k)}(x)(\mathbf{D}^T)^{-1}, k = 0,1,2,\dots. \tag{10}$$

Moreover it is clearly seen that the relation between the matrix $\mathbf{X}(x)$ and its derivative $\mathbf{X}^{(k)}(x)$ is

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{B}^k, \tag{11}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

2.1. Matrix Representation for Differential and Difference Parts

Let us assume that the function $y(x)$ and its derivatives have truncated the Chebyshev expansion of the form

$$y^{(k)}(x) = \sum_{r=0}^{N} a_r T_r^{(k)}(x), \quad k = 0,1,2,\dots,m. \tag{12}$$

The derivative of the matrix $\mathbf{T}(x)$ defined in (10), and the relations (11), give

$$\mathbf{T}^{(k)}(x) = \mathbf{X}(x)\mathbf{B}^k(\mathbf{D}^T)^{-1}. \tag{13}$$

Substituting (13) into (5) we obtain

$$y^{(k)}(x) = \mathbf{T}^{(k)}(x)\mathbf{A} = \mathbf{X}(x)\mathbf{B}^k(\mathbf{D}^T)^{-1}\mathbf{A}, \tag{14}$$

where $y^{(0)}(x) \equiv y(x), T^{(0)}(x) \equiv T(x), T_0(x), T_1(x),\dots,T_N(x)$ are first-kind Chebyshev polynomial, $a_0, a_1,\dots, a_N$ are coefficients to be determined in (3). Now, the matrix representation of the differential part is given by

$$D(x) = \sum_{k=0}^{m} \mathbf{P}_k(x)\mathbf{X}(x)\mathbf{B}^k(\mathbf{M}^T)^{-1}\mathbf{A} . \tag{15}$$

To obtined the matrix form of the difference part

$$H(x) = \sum_{s=0}^{n} H_s(x) y^{(s)}(x-\tau) . \tag{16}$$

We know that;

$$\mathbf{X}(x-\tau) = \mathbf{X}(x)\mathbf{B}_{-\tau} , \tag{17}$$

where

$$\mathbf{B}_{-\tau} = \begin{bmatrix} \binom{0}{0}(-\tau)^0 & \binom{1}{0}(-\tau)^1 & \binom{2}{0}(-\tau)^2 & \cdots & \binom{N}{0}(-\tau)^N \\ 0 & \binom{1}{1}(-\tau)^0 & \binom{2}{1}(-\tau)^1 & \cdots & \binom{N-1}{0}(-\tau)^{N-1} \\ 0 & 0 & \binom{2}{2}(-\tau)^0 & \cdots & \binom{N-2}{0}(-\tau)^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}(-\tau)^0 \end{bmatrix} .$$

Using relation (11), we can write

$$\mathbf{X}^{(k)}(x-\tau) = \mathbf{X}(x)\mathbf{B}^k\mathbf{B}_{-\tau} . \tag{18}$$

In a similarly way as (14), we obtain

$$y^{(s)}(x-\tau) = \mathbf{T}^{(k)}(x-\tau)\mathbf{A} = \mathbf{X}(x)\mathbf{B}^s\mathbf{B}_{-\tau}(\mathbf{D}^T)^{-1}\mathbf{A} . \tag{19}$$

So that, the matrix representation of the difference part become

$$H(x) = \sum_{k=0}^{m} \mathbf{H}_s(x)\mathbf{X}(x)\mathbf{B}^s\mathbf{B}_{-\tau}(\mathbf{D}^T)^{-1}\mathbf{A} . \tag{20}$$

2.2. Matrix Representation for Fredholm Integral Part

Let assume that $F_i(x,t)$ can be expanded to univariate Chebyshev series with respect to t as follows:

$$F_i(x,t)=\sum_{r=0}^{N} f_{ir}(x)T_r(t)\,. \tag{21}$$

Then the matrix representations of the kernel function $F_i(x,t)$ is given by

$$F_i(x,t)=\mathbf{F}_i(x)\mathbf{T}^T(t)\,, \tag{22}$$

where

$$\mathbf{F}_i(x)=[f_{i0}(x) \quad f_{i1}(x) \quad f_{i2}(x) \quad \cdots \quad f_{iN}(x)]\,.$$

Substituting the relations (14) and (22) in the Fredholm part, we obtained

$$I_i(x)=\int_{a_i}^{b_i}\mathbf{F}_i(x,t)y(t)dt=\int_{a_i}^{b_i}\mathbf{F}_i(x)\mathbf{T}(t)\mathbf{T}(t)\mathbf{A}dt=\int_{a_i}^{b_i}\mathbf{F}_i(x)\mathbf{D}^{-1}\mathbf{X}^T(t)\mathbf{X}(t)(\mathbf{D}^T)^{-1}\mathbf{A}dt$$

$$=\mathbf{F}_i(x)\mathbf{D}^{-1}\left(\int_{a_i}^{b_i}\mathbf{X}^T(t)\mathbf{X}(t)dt\right)(\mathbf{D}^T)^{-1}\mathbf{A}=\mathbf{F}_i(x)\mathbf{D}^{-1}\mathbf{M}_i(\mathbf{D}^T)^{-1}\mathbf{A}$$

We say

$$\mathbf{M}_i=\int_{a_i}^{b_i}\mathbf{X}^T(t)\mathbf{X}(t)dt\,,$$

and

$$\mathbf{M}_i=[m_{pq}]=\frac{b_i^{\,p+q+1}-a_i^{\,p+q+1}}{b+q+1}, \quad p,q=0,1,\ldots,N.$$

Hence, the matrix representation of the Fredholm integral part is given by

$$I_i(x)=\sum_{i=0}^{u}\mathbf{F}_i(x)\mathbf{D}^{-1}\mathbf{M}_i(\mathbf{D}^T)^{-1}\mathbf{A}\,. \tag{23}$$

2.3. Matrix Representation for Volterra Integral Part

Similar to the previous section, suppose that the kernel functions $K_j(x,t)$ can be expanded to the univariate Chebyshev series with respect to t as follows:

$$K_j(x,t)=\sum_{r=0}^{N} k_{jr}(x)T_r(t)\,. \tag{24}$$

Then the matrix representations of the kernel function $K_j(x,t)$ become

$$K_j(x,t) = \mathbf{K}_j(x)\mathbf{T}^T(t)\,, \tag{25}$$

where

$$\mathbf{K}_j(x) = [k_{j0}(x) \quad k_{j1}(x) \quad k_{j2}(x) \quad \cdots \quad k_{jN}(x)].$$

Using (14) and (25), we obtain

$$J_j(x) = \int_{c_j}^{x} \mathbf{K}_j(x)\mathbf{T}^T(t)\mathbf{T}(t)\mathbf{A}dt = \int_{a_s}^{x} \mathbf{K}_j(x)\mathbf{D}^{-1}\mathbf{X}^T(t)\mathbf{X}(t)(\mathbf{D}^T)^{-1}\mathbf{A}dt$$

$$= \mathbf{K}_j(x)\mathbf{D}^{-1}\left(\int_{c_j}^{x} \mathbf{X}^T(t)\mathbf{X}(t)dt\right)(\mathbf{D}^T)^{-1}\mathbf{A} = \mathbf{K}_j(x)\mathbf{D}^{-1}\mathbf{L}_j(x)(\mathbf{D}^T)^{-1}\mathbf{A}\,,$$

where

$$\mathbf{L}_j(x) = \int_{c}^{x} \mathbf{X}^T(t)\mathbf{X}(t)dt\,,$$

and

$$\mathbf{L}_j = [l_{pq}] = \frac{x^{p+q+1} - c_j^{\,p+q+1}}{p+q+1}, \quad p,q = 0,1,\ldots,N.$$

So that,

$$J_j(x) = \sum_{i=0}^{v} \mathbf{K}_j(x)\mathbf{D}^{-1}\mathbf{L}_j(x)(\mathbf{D}^T)^{-1}\mathbf{A}\,. \tag{26}$$

2.4. Matrix Representation of the Conditions

Using the relation (14), the matrix form of the conditions defined by (2) can be written as

$$\sum_{k=0}^{m-1}\sum_{j=0}^{r} c^k{}_{ij}\mathbf{X}(c_{ij})\mathbf{B}^k(\mathbf{D}^T)^{-1}\mathbf{A} = [\lambda_i], -\tau \le c_{ij} \le 0\,, \tag{27}$$

where

$$\mathbf{X}(c_{ij}) = [c_{ij}^0 \quad c_{ij}^1 \quad c_{ij}^2 \quad \cdots \quad c_{ij}^N].$$

3. Method of Solution

We are now ready to construct the fundamental matrix equation corresponding to equation (1). For this purpose, substituting the matrix relations (15), (20), (23) and (26) into equation (1) we obtain

$$\left(\sum_{k=0}^{m}\mathbf{P}_k(x)\mathbf{X}(x)\mathbf{B}^k(\mathbf{D}^T)^{-1}+\sum_{s=0}^{n}\mathbf{H}_s(x)\mathbf{X}(x)\mathbf{B}^s\mathbf{B}_{-\tau}(\mathbf{D}^T)^{-1}\right.$$
$$\left.-\sum_{i=0}^{u}\lambda_i\mathbf{F}_i(x)\mathbf{D}^{-1}\mathbf{M}_i(x)(\mathbf{D}^T)^{-1}-\sum_{j=0}^{v}\mu_j\mathbf{K}_j(x)\mathbf{D}^{-1}\mathbf{L}_j(x)(\mathbf{D}^T)^{-1}\right)\mathbf{A}=g(x) \tag{28}$$

For computing the Chebyshev coefficient matrix $\mathbf{A}$ numerically, Chebyshev collocation points defined by

$$x_i=\frac{-\tau}{2}(1+\cos(\frac{r\pi}{N}),\quad r=0,1,2,...,N$$

are put in the above relation (28). We obtained

$$\left(\sum_{k=0}^{m}\mathbf{P}_k(x_r)\mathbf{X}(x_r)\mathbf{B}^k(\mathbf{D}^T)^{-1}+\sum_{s=0}^{n}\mathbf{H}_s(x_r)\mathbf{X}(x_r)\mathbf{B}^s\mathbf{B}_{-\tau}(\mathbf{D}^T)^{-1}\right.$$
$$\left.-\sum_{i=0}^{u}\lambda_i\mathbf{F}_i(x_r)\mathbf{D}^{-1}\mathbf{M}_i(x_r)(\mathbf{D}^T)^{-1}-\sum_{j=0}^{v}\mu_j\mathbf{K}_j(x_r)\mathbf{D}^{-1}\mathbf{L}_j(x_r)(\mathbf{D}^T)^{-1}\right)\mathbf{A}=g(x_r)\;. \tag{29}$$

so, the fundamental matrix equation is obtained

$$\left(\sum_{k=0}^{m}\mathbf{P}_k\mathbf{X}\mathbf{B}^k(\mathbf{D}^T)^{-1}+\sum_{s=0}^{n}\mathbf{H}_s\mathbf{X}\mathbf{B}^s\mathbf{B}_{-\tau}(\mathbf{D}^T)^{-1}\right.$$
$$\left.-\sum_{i=0}^{u}\lambda_i\mathbf{F}_i\mathbf{D}^{-1}\mathbf{M}_i(\mathbf{D}^T)^{-1}-\sum_{j=0}^{v}\mu_j\mathbf{K}_j\,\overline{\mathbf{D}^{-1}}\,\mathbf{L}_j\,\overline{(\mathbf{D}^T)^{-1}}\right)\mathbf{A}=\mathbf{G}\;, \tag{30}$$

where

$$\mathbf{P}_k=\begin{bmatrix}\mathbf{P}_k(x_0)&0&0&\cdots&0\\0&\mathbf{P}_k(x_1)&0&\cdots&0\\0&0&\mathbf{P}_k(x_2)&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\mathbf{P}_k(x_N)\end{bmatrix},\mathbf{H}_s=\begin{bmatrix}\mathbf{H}_s(x_0)&0&0&\cdots&0\\0&\mathbf{H}_s(x_1)&0&\cdots&0\\0&0&\mathbf{H}_s(x_2)&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\mathbf{H}_s(x_N)\end{bmatrix}$$

$$\mathbf{K}_j = \begin{bmatrix} \mathbf{K}_j(x_0) & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{K}_j(x_1) & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{K}_j(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{K}_j(x_N) \end{bmatrix}, \overline{\mathbf{D}^{-1}} = \begin{bmatrix} \mathbf{D}^{-1} & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{D}^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{D}^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{D}^{-1} \end{bmatrix}$$

$$\mathbf{L}_j = \begin{bmatrix} \mathbf{L}_j(x_0) & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{L}_j(x_1) & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{L}_j(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{L}_j(x_N) \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} 1 & x_0 & x_0^{\ 2} & \cdots & x_0^{\ N} \\ 1 & x_1 & x_1^{\ 2} & \cdots & x_1^{\ N} \\ 1 & x_2 & x_2^{\ 2} & \cdots & x_2^{\ N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^{\ 2} & \cdots & x_N^{\ N} \end{bmatrix}, \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{bmatrix}$$

$$\mathbf{F}_i = \begin{bmatrix} \mathbf{F}_i(x_0) \\ \mathbf{F}_i(x_1) \\ \mathbf{F}_i(x_2) \\ \vdots \\ \mathbf{F}_i(x_N) \end{bmatrix} \quad \overline{(\mathbf{D}^T)^{-1}} = \begin{bmatrix} (\mathbf{D}^T)^{-1} \\ (\mathbf{D}^T)^{-1} \\ (\mathbf{D}^T)^{-1} \\ \vdots \\ (\mathbf{D}^T)^{-1} \end{bmatrix}.$$

The fundamental matrix equation (30) for equation (1) corresponds to a system for the $(N+1)$ unknown coefficients a_0, $a_1, \ldots, a_N$. Briefly we can write equation (30) as

$$\mathbf{WA}=\mathbf{G} \text{ or } [\,\mathbf{W};\mathbf{G}\,], \tag{31}$$

so that

$$\begin{aligned} \mathbf{W} = [w_{pq}] &= \sum_{k=0}^{m} \mathbf{P}_k \mathbf{X} \mathbf{B}^k (\mathbf{D}^T)^{-1} + \sum_{s=0}^{n} \mathbf{H}_s \mathbf{X} \mathbf{B}^s \mathbf{B}_{-\tau} (\mathbf{D}^T)^{-1} \\ &- \sum_{i=0}^{u} \lambda_i \mathbf{F}_i \mathbf{D}^{-1} \mathbf{M}_i (\mathbf{D}^T)^{-1} - \sum_{j=0}^{v} \mu_j \mathbf{K}_j \overline{\mathbf{D}^{-1}} \mathbf{L}_j \overline{(\mathbf{D}^T)^{-1}} \end{aligned} \qquad p,q = 0,1,\ldots,N. \tag{32}$$

The matrix form for conditions (2) are then

$$\mathbf{C}_{\mathrm{i}}\mathbf{A} = [\ \lambda_i\ ] \text{ or } [\mathbf{C}_{\mathrm{i}}; \lambda_i] \quad \mathrm{i}=0,1,\ldots,\mathrm{m}\text{-}1, \tag{33}$$

where

$$\mathbf{C}_i = \sum_{k=0}^{m-1} c^k{}_{ij} \mathbf{X}(c_{ij}) \mathbf{B}^k (\mathbf{D}^T)^{-1} \equiv [u_{i0} u_{i1} \ldots u_{iN}].$$

To obtain the solution of equation (1) under the conditions (2), we replace the row matrices (33) by the last m rows of the matrix (31) to get the required augmented matrix

$$[\mathbf{W}^*;\mathbf{G}^*]=\begin{bmatrix} w_{00} & w_{01} & . & . & . & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & . & . & . & w_{1N} & ; & g(x_1) \\ \cdots & \cdots & & & & \cdots & ; & \cdots \\ w_{N-m,0} & w_{N-m,1} & . & . & . & w_{N-m,N} & ; & g(x_{N-m}) \\ u_{00} & u_{01} & . & . & . & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & . & . & . & u_{1N} & ; & \lambda_1 \\ \cdots & \cdots & & & & \cdots & ; & \cdots \\ u_{m-1,0} & u_{m-1,1} & . & . & . & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}$$

or the corresponding matrix equation

$$\mathbf{W}^*\mathbf{A}=\mathbf{G}^*. \tag{34}$$

If rank ($\mathbf{W}^*$) = rank [$\mathbf{W}^*;\mathbf{G}^*$]= $N+1$, then we can write

$$\mathbf{A}=(\mathbf{W}^*)^{-1}\mathbf{G}^*.$$

Thus, the coefficients $a_n, n=0,1,...,N$, are uniquely determined by equation (34). Also we can easily check the accuracy of the obtained solutions as follows:

Since the obtained first-kind Chebyshev polynomial expansion is an approximate solution of equation (1), when the function $y(x)$ and its derivatives are substituted in equation (1), the resulting equation must be satisfied approximately; that is, for $x=x_i \in [-1,1]$, i=0,1,2,…,

$$E(x_i)=\left|D(x_i)+H(x_i)-\sum_{i=0}^{u}\lambda_i I_i(x_i)-\sum_{j=0}^{v}\mu_j J_j(x_i)-g(x_i)\right|\cong 0\,.$$

4. Illustrative Examples

In this section, several numerical examples are given to illustrate the accuracy and effectiveness properties of the method and all of them were performed on the computer using a program written in Maple 9. The absolute errors in Tables are the values of $|y(x)-y_N(x)|$ at selected points.

Example4.1.

Let us first consider the second order linear Fredholm-Volterra integro-delay-differential-difference equation with piecewise interval,

$$y''(x) - xy'(x) + y(x) - xy'(x-1) + y(x-1) = (x+1)\sin(x) - 2(x+1)\sin(1) + x\sin(x-1)$$
$$+\int_{-1}^{0} y(t)dt + \int_{-1}^{1} xy(t))dt + \int_{-1}^{x} y(t))dt - 2\int_{0}^{x} y(t))dt$$

with mixed conditions $y(0)=1, y'(0)=0$ and seek the solution $y(x)$ as a truncated first-kind Chebyshev series

$$y(x) = \sum_{r=0}^{N} a_r T_r(x), \quad -1 \le x \le 0,$$

so that

$$P_0(x)=1,\ P_1(x)=-x,\ P_2(x)=1,\ H_0(x)=1, H_1(x)=-x\ , F_0(x,t)=1, F_1(x,t)=x\ ,$$

$$K_0(x,t)=1\ K_1(x,t)=1,\ g(x)=(x+1)\sin(x)-2(x+1)\sin(1)+x\sin(x-1)\,.$$

Then, for $N=5$, the collocation points are

$$x_0=-1,\ x_1=-\frac{1}{2}-\frac{1}{2}\cos\left(\frac{\pi}{5}\right),\ x_2=--\frac{1}{2}-\frac{1}{2}\cos\left(\frac{2\pi}{5}\right)$$
$$x_3=-\frac{1}{2}+\frac{1}{2}\cos\left(\frac{3\pi}{5}\right),\ x_4=-\frac{1}{2}+\frac{1}{2}\cos 4\left(\frac{\pi}{5}\right),\ x_5=0$$

and the fundamental matrix equation of the problem is

$$\begin{aligned}\mathbf{W} &= \mathbf{P}_0\mathbf{X}(\mathbf{D}^T)^{-1} + \mathbf{P}_1\mathbf{XB}(\mathbf{D}^T)^{-1} + \mathbf{P}_2\mathbf{XB}^2(\mathbf{D}^T)^{-1} + \mathbf{H}_0\mathbf{XB}_{\text{-}1}(\mathbf{D}^T)^{-1} + \mathbf{H}_1\mathbf{XBB}_{\text{-}1}(\mathbf{D}^T)^{-1} \\ &- \lambda_0\mathbf{F}_0\mathbf{D}^{-1}\mathbf{M}_0(\mathbf{D}^T)^{-1} - \lambda_1\mathbf{F}_1\mathbf{D}^{-1}\mathbf{M}_1(\mathbf{D}^T)^{-1} - \mu_0\mathbf{K}_0\,\overline{\mathbf{D}^{-1}}\,\mathbf{L}_0\,\overline{(\mathbf{D}^T)^{-1}} - \mu_1\mathbf{K}_1\,\overline{\mathbf{D}^{-1}}\,\mathbf{L}_1\,\overline{(\mathbf{D}^T)^{-1}}\end{aligned} \quad . \quad (35)$$

With the following matrices for conditions

$$\mathbf{X}(0)(\mathbf{D}^T)^{-1} = \begin{bmatrix}1 & 0 & -1 & 0 & 1 & 0\end{bmatrix}\mathbf{A} = [1],$$

$$\mathbf{X}(0)\mathbf{B}(\mathbf{D}^T)^{-1} = \begin{bmatrix}0 & 1 & 0 & -3 & 0 & 5\end{bmatrix}\mathbf{A} = [0],$$

where

$$\mathbf{P}_0=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\mathbf{P}_1=\begin{bmatrix}1&0&0&0&0&0\\0&0.9045&0&0&0&0\\0&0&0.6545&0&0&0\\0&0&0&0.3454&0&0\\0&0&0&0&0.0954&0\\0&0&0&0&0&0\end{bmatrix},\ \mathbf{P}_2=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$$

$$\mathbf{H}_0=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\mathbf{H}_1=\begin{bmatrix}1&0&0&0&0&0\\0&0.9045&0&0&0&0\\0&0&0.6545&0&0&0\\0&0&0&0.3454&0&0\\0&0&0&0&0.0954&0\\0&0&0&0&0&0\end{bmatrix},\mathbf{B}=\begin{bmatrix}0&1&0&0&0&0\\0&0&2&0&0&0\\0&0&0&3&0&0\\0&0&0&0&4&0\\0&0&0&0&0&5\\0&0&0&0&0&0\end{bmatrix}$$

$$\mathbf{B}_{-1}=\begin{bmatrix}1&-1&1&-1&1&-1\\0&1&-2&3&-4&05\\0&0&1&-3&6&-10\\0&0&0&1&-4&10\\0&0&0&0&1&-5\\0&0&0&0&0&1\end{bmatrix},\mathbf{F}_0=\begin{bmatrix}1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\end{bmatrix},\mathbf{F}_1=\begin{bmatrix}-1&0&0&0&0&0\\-0.9045&0&0&0&0&0\\-0.6545&0&0&0&0&0\\-0.3454&0&0&0&0&0\\-0.0954&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

$$\mathbf{K}_0=\begin{bmatrix}1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\end{bmatrix},\mathbf{K}_1=\begin{bmatrix}1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\end{bmatrix}$$

$$\mathbf{D}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\\frac{1}{2}&0&\frac{1}{2}&0&0&0\\0&\frac{3}{4}&0&\frac{1}{4}&0&0\\\frac{3}{8}&0&\frac{1}{2}&0&\frac{1}{8}&0\\0&\frac{5}{8}&0&\frac{5}{16}&0&\frac{1}{16}\end{bmatrix}.$$

If these matrices are substituted in (34), we obtain the linear algebraic system and the approximate solution of the problem for $N=5$ as

$$y(x)=1.000000{+}0.503090x^2{+}0.004434x^3{+}0.045501x^4{+}0.005079x^5.$$

The exact solution of this problem is $y(x) = \cos(x)$. Figure 1 shows the comparison between the exact solution and the approximate different for various N Chebshev collocation method solution of the system. In Table 1, we show that when N is increasing, N_e is decreasing.

Table 1: Numerical solution of Example 4.1 for different N.

		Present Method					
x	Exact Solution	$N=5$	$N_e=5$	N=7	$N_e=7$	N=9	$N_e=9$
0.0	1.000000	1.000000	0.000000	1.000000	0.00000	1.000000	0.00000
-0.1	0.995004	0.994977	0.262E-4	0.995007	0.339E-5	0.995003	0.376E-5
-0.2	0.980066	0.979982	0.838E-4	0.980054	0.116E-4	0.980065	0.133E-5
-0.3	0.955336	0.955197	0.139E-3	0.955314	0.216E-4	0.955333	0.260E-5
-0.4	0.921060	0.920900	0.160E-3	0.921030	0.302E-4	0.921057	0.387E-5
-0.5	0.877582	0.877464	0.117E-3	0.877548	0.342E-4	0.877577	0.484E-5
-0.6	0.825582	0.825344	0.875E-5	0.825304	0.311E-4	0.825330	0.524E-5
-0.7	0.764842	0.765073	0.231E-3	0.764822	0.192E-4	0.764837	0.487E-5
-0.8	0.696706	0.697260	0.553E-3	0.696709	0.235E-5	0.696703	0.359E-5
-0.9	0.621609	0.622576	0.966E-3	0.621643	0.336E-4	0.621608	0.137E-5
-1.0	0.540602	0.541757	0.142E-2	0.540375	0.734E-4	0.540304	0.172E-5

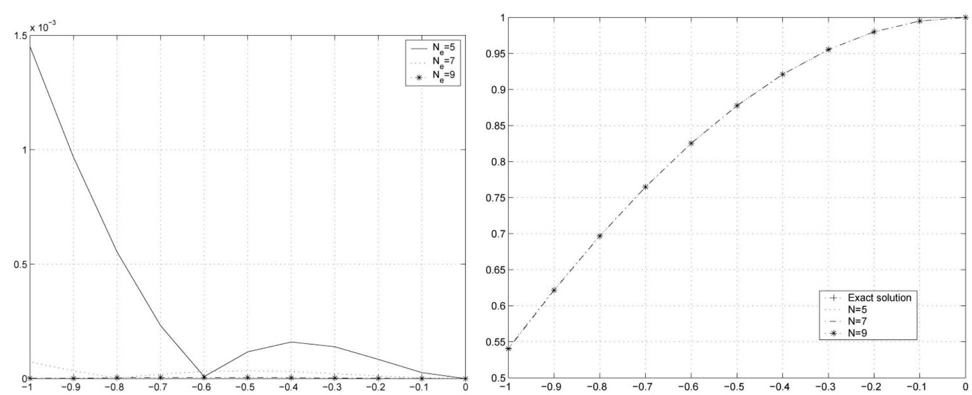

Figure 1. Error function of Example 4.1 for various N

Example 4.2.

Let us consider the second order linear Fredholm-Volterra integro delay differential-difference equation with piecewise intervals,

$$x^2 y''(x) - xy'(x) + (x-1)y(x) + y(x-0.5) - xy'(x-0.5) = -\frac{2}{3}x^4 + 6x^3 - 17x^2 + \frac{23}{2}x + \frac{13}{3}$$
$$+ \int_0^1 (x+t)y(t)dt + \int_{-1}^0 (x-t)y(t)dt + \int_0^x xy(t)dt$$

with conditions $y(0)=5$, $y'(0)=-4$ and its exact solution is $y(x)=2x^2-4x+5$. We obtained the approximate solution of the problem for $N=5$ which are the same with the exact solution.

Example 4.3.

Consider the second order linear Fredholm-Volterra integro delay differential-difference equation with piecewise intervals,

$$y'''-e^{-x}y''+(1+e^{-x})y'-y+y'(x-1)-(x-1)y(x-1)=(3-x)e^{x-1}+e^{x+1}+e^{-x}+e^{1-x}+(0.5-x)e^{0.5}$$
$$+(1.5-x)e^{-0.5}+\frac{1}{e}+\int_{-1}^{1}e^{x}y(t)dt+\int_{0}^{1}e^{x}y(t)dt+\int_{-1}^{x}(x-t)y(t)dt-\int_{0.5}^{x}(x-t)y(t)dt+\int_{0.5}^{x}(x+t)y(t)dt$$

with mixed conditions $y(0)=1$, $y'(0)=1$, $y''(0)=1$ and its exact solution is $y(x)=e^{x}$. We obtain the approximate solution of the problem for $N=4$, $N=5$, $N=6$ which are tabulated and graphed in Table 2 and Figure 2 respectively.

Table 2: Numerical solution of Example 4.3 for different N

		Present Method					
x	Exact Solution	$N=4$	$N_e=4$	N=5	$N_e=5$	N=6	$N_e=6$
0.0	1.000000	1.000000	0.000000	0.999999	0.400E-9	1.000000	0.000000
-0.1	0.904837	0.904836	0.106E-5	0.904837	0.806E-7	0.904837	0.557E-7
-0.2	0.818730	0.818719	0.112E-4	0.818731	0.847E-6	0.818731	0.453E-6
-0.3	0.740818	0.740775	0.430E-4	0.740822	0.400E-5	0.740819	0.153E-5
-0.4	0.670320	0.670215	0.104E-3	0.670332	0.128E-4	0.670323	0.354E-5
-0.5	0.606530	0.606336	0.193E-3	0.606562	0.321E-4	0.606537	0.643E-5
-0.6	0.548811	0.548524	0.287E-3	0.548878	0.671E-4	0.548821	0.971E-5
-0.7	0.496585	0.496247	0.337E-3	0.496707	0.121E-3	0.496597	0.123E-4
-0.8	0.449328	0.449061	0.357E-3	0.449525	0.196E-3	0.449341	0.128E-4
-0.9	0.406569	0.406609	0.450E-3	0.406856	0.287E-3	0.406579	0.937E-5
-1.0	0.367879	0.368617	0.638E-3	0.368256	0.377E-3	0.367880	0.565E-6

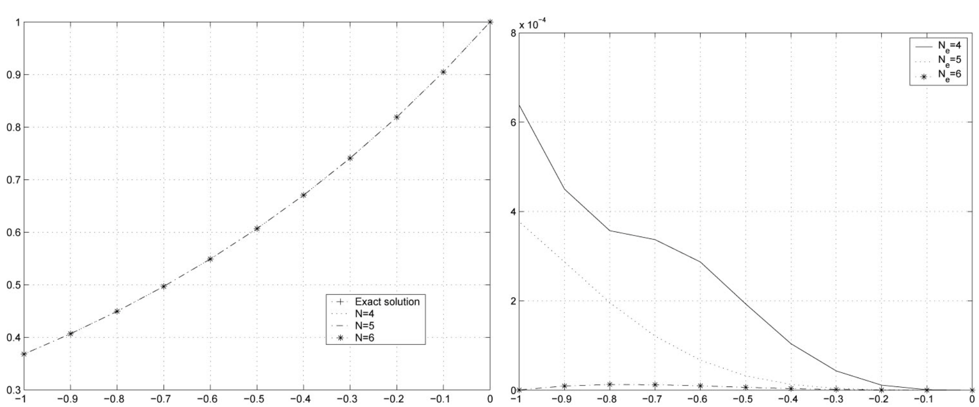

Figure 2. Error function of Example 4.3 for various N

Example 4.4.

Consider the linear third order Fredholm-Volterra integro delay differential-difference equation,

$$(x-1)y'''(x)+12y'(x)+(x-1)y(x)-y''(x-1)+y'(x-1)=\frac{293}{20}+\frac{206}{15}x-\frac{157}{3}x^3-\frac{1}{3}x^4+\frac{1}{5}x^5+\frac{1}{5}x^6$$
$$+\int_{-1}^{0}y(t)dt+\int_{0}^{1}(x-t)y(t)dt+\int_{0}^{x}y(t)dt+\int_{-1}^{x}xy(t)dt$$

with conditions $y(0)=0$, $y'(0)=0$, $y''(0)=2$ and its exact solution is $y(x)=x^2-x^4$. We obtained the approximate solution of the problem for N = 5 which are the same with the exact solution.

Example 4.5.

Consider the first order linear Fredholm-Volterra integro-differential equation,

$$y'-y=e^x-e+\int_{0}^{1}y(t)dt+\int_{0}^{x}y(t)dt$$

with nonlocal boundary condition

$$y(0)+\int_{0}^{1}y(t)dt=e$$

and its exact solution is $y(x)=e^x$. We obtain the approximate solution of the problem for $N=4$, $N=5$, $N=6$ which are tabulated and graphed in Table 3 and Figure 3 respectively.

Table 3: Numerical solution of Example 4.5 for different N

		Present Method					
x	Exact Solution	$N=4$	$N_e=4$	N=5	$N_e=5$	N=6	$N_e=6$
0.0	1.000000	1.001618	0.161E-2	1.000329	0.329E-3	1.000059	0.593E-4
-0.1	0.904837	0.904837	0.131E-2	0.905105	0.268E-3	0.904885	0.483E-4
-0.2	0.818730	0.818730	0.105E-2	0.818946	0.215E-3	0.818769	0.388E-4
-0.3	0.740818	0.740818	0.841E-3	0.740988	0.170E-3	0.740848	0.305E-4
-0.4	0.670320	0.670320	0.660E-3	0.670450	0.130E-3	0.670343	0.233E-4
-0.5	0.606530	0.606530	0.494E-3	0.606625	0.949E-4	0.606530	0.170E-4
-0.6	0.548811	0.548811	0.332E-3	0.548875	0.633E-4	0.548823	0.116E-4
-0.7	0.496585	0.496585	0.171E-3	0.496621	0.364E-4	0.496592	0.680E-5
-0.8	0.449328	0.449328	0.229E-4	0.449342	0.139E-4	0.449331	0.246E-5
-0.9	0.406569	0.406569	0.782E-4	0.406563	0.616E-5	0.406569	0.153E-5
-1.0	0.367879	0.367879	0.730E-4	0.367849	0.302E-4	0.367874	0.495E-5

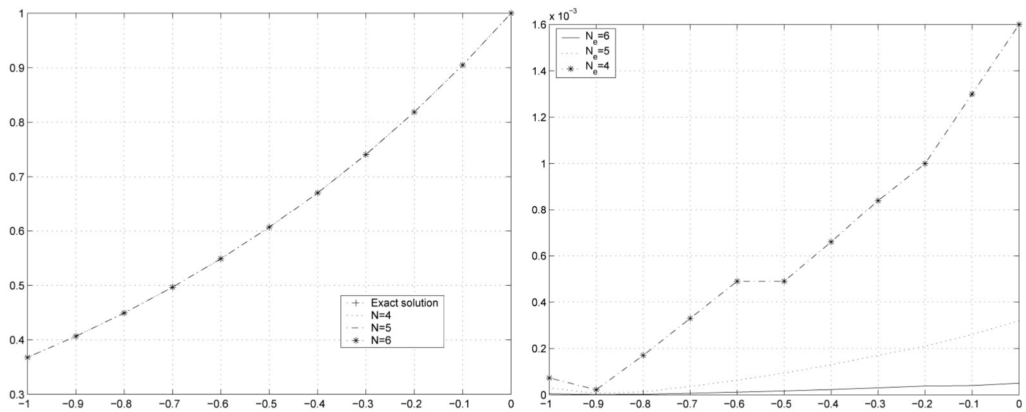

Figure 3. Error function of Example 4.5 for various N

5. Conclusion

The Chebyshev collocation methods are used to solve the linear integrodifferential- difference equation numerically. A considerable advantage of the method is that the Chebyshev polynomial coefficients of the solution are found very easily by using computer programs. Shorter computation time and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy. Illustrative examples are included to demonstrate the validity and applicability of the technique and performed on the computer using a program written in Maple 9. To get the best approximating solution of the equation, we take more forms from the Chebyshev expansion of functions, with, the truncation limit N chosen large enough. In addition, an interesting feature of this method is finding the analytical solutions if the equation has an exact solution that is a polynomial function. Illustrative examples with the satisfactory results are used to demonstrate the application of this method. Suggested approximations make this method very attractive and contribute to the good agreement between approximate and exact values in the numerical example.

REFERENCES

Bainov, D. D., Dimitrova, M. B. and Dishliev, A. D. (2000). Oscillation of the bounded solutions of impulsive differential-difference equations of second order, Appl. Math. Comput. 114, 61-68.

Cao, J. and Wang, J. (2004). Delay-dependent robust stability of uncertain nonlinear systems with time delay, Appl. Math. Comput. 154, 289-297.

Elmer, C. E., and van Vleck, E. S. (2001). Travelling wave solutions for bistable differential difference equations with periodic diffusion, SIAM J. Appl. Math. 61, 1648- 1679.

Elmer, C. E., and Van Vleck E. S. (2002). A variant of Newton's method for solution of traveling wave solutions of bistable differential-difference equation, J. Dyn. Differ. Equa. 14, 493-517.

Gülsu, M., Öztürk, Y. and Sezer M. (2010). A newcollocation method for solutionof the mixed linear integro-differential-difference equations, Appl. Math. Comput. 216, 2183-2198.

Hosseini, S. M. and Shahmorad, S. (2003). Numerical solution of a class of integrodifferential equations by the Tau method with an error estimation, Appl. Math. Comput. 136:559-570.

Kadalbajoo, M. K. and Sharma, K. K. (2002). Numerical analysis of boundary-value problems for singularly-perturbed differential-difference equations with small shifts of mixed type, J. Optim. Theory Appl. 115:145-163.

Kadalbajoo, M. K.and Sharma, K. K. (2004). Numerical analysis of singularly-perturbed delay differential equations with layer behavior, App. Math. Comput. 157:11-28.

Maleknejad, K. and Mirzaee, F. (2006). Numerical solution of integro-differential equations by using rationalized Haar functions method, Kybernetes Int. J. Syst. Math. 35:1735-1744.

Ortiz, E. L., and Aliabadi, M. H. (1998). Numerical treatment of moving and free boundaryvalue problems with the Tau method, Comp. Math. Appl. 35(8), 53- 61.

Rashed, M. T. (2004). Numerical solution of functional differential, integral and integro-differential equations, Appl. Numer. Math. 156, 485-492.

Ren, Y., Zhang, B. and Qiao, H. (1999). A simple Taylor-series expansion method for a class of second kind integral equations, J. Comp. Appl. Math. 110, 15-24.

Sezer, M. and Gulsu, M. (2005). Polynomial solution of the most general linear Fredhol integro-differential-difference equation by means of Taylor matrix method, Int. J. Complex Var. 50(5):367-382.

Sezer, M. and Gülsu, M. (2005). A new polynomial approach for solving difference and Fredholm integro-difference equations with mixed argument, Appl. Mat. Comp. 171, 332-344.

Synder, M. A. (1966). Chebyshev methods in Numerical Approximation, Prentice Hall, Inc., London.

Zhao, J. and Corless, R. M. (2006). Compact finite difference method for integrodifferential equations, Appl. Math. Comput. 177:271-288.