Available at
http://pvamu.edu/pages/398/asp

Vol. 1, No. 1 (2006) pp. 36 – 60

Applications and Applied Mathematics (AAM):
An International Journal

ANALYSIS OF AN SIRS AGE-STRUCTURED EPIDEMIC MODEL WITH VACCINATION AND VERTICAL TRANSMISSION OF DISEASE

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Received July 22, 2005; revised Febuary 8, 2006; accepted April 18, 2006

Abstract

An SIRS age-structured epidemic model for a vertically as well as horizontally transmitted disease under vaccination is investigated when the fertility, mortality and removal rates depend on age and the force of infection of proportionate mixing assumption type, and vaccination wanes over time. We prove the existence and uniqueness of solution to the model equations, and show that solutions of the model equations depend continuously on the initial age-distributions. Furthermore, we determine the steady states and obtain an explicitly computable threshold condition, in terms of the demographic and epidemiological parameters of the model; we then study the stability of the steady states. We also compare the behavior of the model with the one without vertical transmission.

Keywords: Vertical transmission,Horizontal transmission,Age-structure, Epidemic, Stability,Proportionate mixing

MSC 2000: 45K05; 45M10; 35A05; 35B30; 35B35; 35B45; 35L40; 92D30; 92D25.

Introduction

In this paper, we study an SIRS age-structured epidemic model, where age is assumed to be the chronological age i.e. the time since birth. The disease causes so few fatalities that they can be neglected, and is horizontally as well as vertically transmitted. Horizontal transmission is the passing of infection through some direct or indirect contact with infected individuals, for example, malaria and tuberculosis are horizontally transmitted diseases. Vertical transmission is the passing of infection from parents to newborn or unborn offspring, for example, AIDS, Chagas and Hapatitis B are vertically (as well as horizontally) transmitted diseases. Vertical transmission plays an important role in maintaining some diseases, for example, see Busenberg, et al. (1993), (1988, p. 1379), (1982), (1988, p. 181), (1990) and Busenberg (1986). In Fine (1975), several examples of vertically transmitted diseases are given, and in Busenberg, et al. (1993), a book is devoted for the study of the models and dynamics of vertically transmitted diseases.

Several recent papers have dealt with age-dependent vaccination models, for example, Hethcote (1983), (1989), (1997), (2000), Dietz, et al. (1985), Dietz (1981), Katzmann, et al. (1984), Schenzle (1984), Anderson, et al. (1999), El-Doma (2000), (2001), (2005), (2006), Müller (1994), (1998), Hadeler, et al. (1996), Knox (1980), McLean (1986), Coutinho, et al. (1993), Lopez, et al. (2000), Greenhalgh (1988),(1990), Castillo-Chavez, et al. (1998), Li, et al. (2004), and Thieme (2001). Much of this previous work assumed that the total population has a fixed size. As one might expect

there are situations, for example, in developing countries, where it is necessary to consider total population sizes that vary with time, mortality due to disease and reduction of birth rate due to infection. Some of the effects of such assumptions are considered by McLean (1986), El-Doma (2001), (2004) and Hadeler, et al. (1996).

We note that several papers have dealt with SIR age-structured epidemic models, but without vaccination, for example, Thieme (1990), Andreasen (1995), Chipot, et al. (1995), Inaba (1990), Cha, et al. (1998), Greenhalgh (1987) and Castillo-Chavez, et al. (1989). Also, we note that some vaccines wane over time giving rise to SIRS type models, for example, see, Li, et al. (2004).

In this paper, we study an SIRS age-structured epidemic model when the disease is vertically as well as horizontally transmitted, and therefore our results generalize those in Li, et al. (2004), were vertical transmission is not considered. We prove the existence and uniqueness of solution to the model equations and show that solutions of the model equations depend continuously on the initial age-distributions. We determine the steady states of the model by proving a threshold theorem and obtaining an explicitly computable threshold R_ν, in terms of the demographic and epidemiological parameters of the model, known as the reproduction number in the presence of vaccination strategy $\nu(a)$, as in Hadeler, et al. (1996) or the net replacement ratio, as in Thieme (2001). And hence, we show that R_ν increases with q, which is the probability of vertically transmitting the disease (see section 2 for definitions) and therefore, increases the likelihood that an endemic will occur; also R_ν is used to determine a critical vaccination coverage which will eradicate the disease with minimum vaccination coverage. In addition, we show that the model gives rise to a continuum of positive endemic equilibriums in the case of non-fertile infectiables. This situation does not occur if there is no vertical transmission, for example, see, Li, et al. (2004). We also study the stability of the steady states.

We study the stability of the disease-free equilibrium and show that the disease-free equilibrium is locally asymptotically stable if $R_\nu < 1$, and unstable if $R_\nu > 1$. Also, we show that the disease-free equilibrium is globally stable if $R_0 < 1$, where R_0 is the basic reproduction number, and is interpreted as the average number of secondary infections that occur when an infective is introduced into a totally susceptible population. For endemic equilibriums, we obtain a complicated characteristic equation, that allows us to prove local asymptotic stability, in some special cases. In addition, we derive general formulas for the characteristic equations, in terms of integral equations. Although we do not obtain explicit formulas for the solutions of these integral equations, we will use these integral equations to deduce that if the force of infection is sufficiently small, then an endemic equilibrium is always locally asymptotically stable.

The organization of this paper is as follows: in section 2 we describe the model and obtain the model equations; in section 3 we reduce the model equations to several subsystems and prove the existence and uniqueness of solution as well as the continuous dependence on initial age-distributions; in section 4 we determine the steady states; in section 5 we study the stability of the steady states; in section 6 we conclude our results.

The Model

We consider an age-structured population of variable size exposed to a communicable disease. The disease is vertically as well as horizontally transmitted and causes so few fatalities that they can be neglected. We assume the following.

• s(a,t), i(a,t) and r(a,t), respectively, denote the age-density for susceptible, infective and immune individuals of age a at time t. Then

$\int_{a_1}^{a_2} s(a,t)da =$ total number of susceptible individuals at time t of ages between a_1 and a_2,

$\int_{a_1}^{a_2} i(a,t)da =$ total number of infective individuals at time t of ages between a_1 and a_2. And similarly for r(a,t). We assume that the total population consists entirely of susceptible, infective and immune individuals.

• Let $k(a,a')$ denote the probability that a susceptible individual of age a is infected by an infective of age a'. We further assume that, $k(a,a') = k_1(a).k_2(a')$, which is know as the ``proportionate mixing assumption'', see Dietz, et al. (1985). Therefore, the horizontal transmission of the disease for the susceptible individuals occurs at the following rate:

$$k_1(a)s(a,t)\int_0^\infty k_2(a')i(a',t)da',$$

where $k_1(a)$ and $k_2(a)$ are bounded, nonnegative, continuous functions of a. The term $k_1(a)\int_0^\infty k_2(a)i(a,t)da,$ is called ``force of infection'' and we let

$$\lambda(t) = \int_0^\infty k_2(a)i(a,t)da.$$

And immune individuals are infected due to waning of vaccine over time at the following rate: $\varepsilon k_1(a)r(a,t)\lambda(t),$ where ε is a positive real number $\in [0,1]$.

• The fertility rate $\beta(a)$ is a nonnegative, continuous function, with compact support $[0,A]$, $(A>0)$. The number of births of susceptible individuals per unit time is given by

$s(0,t) = \int_0^\infty \beta(a)[s(a,t)+(1-q)i(a,t)+r(a,t)]da, q \in [0,1]$, where q is the probability of vertically transmitting the disease. Accordingly all newborns from susceptible and immune individuals are susceptible, but a portion q of newborns from infected parents are infective, i.e., they acquire the disease via birth (vertical transmission) and therefore, $i(0,t) = q\int_0^\infty \beta(a)i(a,t)da,$ and $r(0,t) = 0$.

• The death rate $\mu(a)$ is the same for susceptible, infective and immune individuals, and $\mu(a)$ is a nonnegative, continuous function and $\exists a_0 \in [0,\infty)$ such that $\mu(a) > \bar{\mu} > 0 \,\forall\, a > a_0$ and $\mu(a_2) > \mu(a_1) \,\forall\, a_2 > a_1 > a_0$.

• The cure rate $\gamma(a)$ is a bounded, nonnegative, continuous function of a.

• The vaccination rate $\nu(a)$ is a bounded, nonnegative, continuous function of a.

• The vaccination wanes in immune individuals and they become susceptible at a rate $\delta(a)$ which is a bounded, nonnegative, continuous function of a.

• The initial age distributions $s(a,0) = s_0(a),$ $i(a,0) = i_0(a),$ and $r(a,0) = r_0(a)$ are continuous, nonnegative and integrable functions of a $\in [0,\infty)$.

These assumptions lead to the following system of nonlinear integro-partial differential equations with non-local boundary conditions, which describes the dynamics of the transmission of the disease.

$$\begin{cases} \dfrac{\partial s(a,t)}{\partial a}+\dfrac{\partial s(a,t)}{\partial t}+[\mu(a)+\nu(a)]s(a,t)=-k_1(a)s(a,t)\lambda(t)+\delta(a)r(a,t), a>0, t>0, \\ \dfrac{\partial i(a,t)}{\partial a}+\dfrac{\partial i(a,t)}{\partial t}+[\mu(a)+\gamma(a)]i(a,t)=k_1(a)s(a,t)\lambda(t)+\varepsilon k_1(a)r(a,t)\lambda(t), a>0, t>0, \\ \dfrac{\partial r(a,t)}{\partial a}+\dfrac{\partial r(a,t)}{\partial t}+[\mu(a)+\delta(a)]r(a,t) \\ \qquad =\nu(a)s(a,t)+\gamma(a)i(a,t)-\varepsilon k_1(a)r(a,t)\lambda(t), a>0, t>0, \\ s(0,t)=\int_0^\infty \beta(a)[s(a,t)+(1-q)i(a,t)+r(a,t)]da, t\geq 0, \\ i(0,t)=q\int_0^\infty \beta(a)i(a,t)da, t\geq 0, \\ r(0,t)=0, t\geq 0, \\ \lambda(t)=\int_0^\infty k_2(a)i(a,t)da, t\geq 0, \\ s(a,0)=s_0(a), i(a,0)=i_0(a), r(a,0)=r_0(a), a\geq 0. \end{cases} \tag{1}$$

We note that problem (2.1) is an SIRS epidemic model, the same model but with $q=0$, i.e., the case of no vertical transmission, is dealt with in Li, et al. (2004), and the steady states are determined and the stability of the disease-free equilibrium is studied.

In what follows, we show that problem (2.1) has a unique solution that exists for all time. Furthermore, we show that solutions of problem (2.1) depend continuously on the initial age-distributions. Also, we determine the steady states and study their stability.

Reduction of the Model, the Existence and Uniqueness of Solution and Continuous Dependence on Initial Age-Distributions

In this section, we develop some preliminary formal analysis of problem (2.1) and show that problem (2.1) has a unique solution that exists for all time. Furthermore, we show that solutions of problem (2.1) depend contiuously on the initial age-distributions.

We define $p(a,t)$ by $p(a,t)=s(a,t)+i(a,t)+r(a,t)$. Then from (2.1), by adding the equations, we find that $p(a,t)$ satisfies the following McKendrick-Von Forester equation:

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial a}+\dfrac{\partial p(a,t)}{\partial t}+\mu(a)p(a,t)=0, a>0, t>0, \\ p(0,t)=B(t)=\int_0^\infty \beta(a)p(a,t)da, t\geq 0, \\ p(a,0)=p_0(a)=s_0(a)+i_0(a)+r_0(a), a\geq 0. \end{cases} \tag{2}$$

Note that problem (3.1) has a unique solution that exists for all time, see Bellman, et al. (1963), Feller (1941) and Hoppensteadt (1975). The unique solution is given by

$$p(a,t)=\begin{cases} p_0(a-t)\pi(a)/\pi(a-t), & a>t, \\ B(t-a)\pi(a), & a<t, \end{cases} \tag{3}$$

where $\pi(a)$ is given by

$$\pi(a)=e^{-\int_0^a \mu(\tau)d\tau},$$

and $B(t)$ has the following asymptotic behavior as $t \to \infty$:

$$B(t) = [c + \theta(t)]e^{p^* t}, \tag{4}$$

where p^* is the unique real number which satisfies the following characteristic equation:

$$\int_0^\infty \beta(a)\pi(a)e^{-p^* a}da = 1, \tag{5}$$

$\theta(t)$ is a function such that $\theta(t) \to 0$ as $t \to \infty$ and c is a constant.
Using (3.1)-(3.2), we obtain that $B(t)$ satisfies

$$B(t) = \int_0^\infty \beta(a)\pi(a)B(t-a)da + \int_t^\infty \beta(a)\, p_0(a-t)\frac{\pi(a)}{\pi(a-t)}da. \tag{6}$$

Using (3.5) and *Gronwall's* inequality, we obtain

$$B(t) \le \|\beta(a)\|_\infty \|p_0(a)\|_{L^1[0,\infty)} e^{(\|\beta(a)\|_\infty - \mu_*)t}, \tag{7}$$

where μ_* is given by

$$\mu_* = \inf_{a\in[0,\infty)} \mu(a). \tag{8}$$

From (3.2) and (3.6), we obtain the following a priori estimate:

$$\int_0^\infty p(a,t)da \le \|p_0(a)\|_{L^1[0,\infty)} e^{(\|\beta(a)\|_\infty - \mu_*)t}. \tag{9}$$

Also, from (2.1), s(a,t), i(a,t) and r(a,t) satisfy the following systems of equations:

$$\begin{cases} \dfrac{\partial s(a,t)}{\partial a} + \dfrac{\partial s(a,t)}{\partial t} + [\mu(a) + \nu(a)]s(a,t) = -k_1(a)s(a,t)\lambda(t) + \delta(a)r(a,t), a > 0, t > 0, \\ s(0,t) = \int_0^\infty \beta(a)[s(a,t) + (1-q)i(a,t) + r(a,t)]da, t \ge 0, \\ s(a,0) = s_0(a), a \ge 0, \end{cases} \tag{10}$$

$$\begin{cases} \dfrac{\partial i(a,t)}{\partial a} + \dfrac{\partial i(a,t)}{\partial t} + [\mu(a) + \gamma(a)]i(a,t) = k_1(a)s(a,t)\lambda(t) + \varepsilon k_1(a)r(a,t)\lambda(t), a > 0, t > 0, \\ i(0,t) = q\int_0^\infty \beta(a)i(a,t)da, t \ge 0, \\ i(a,0) = i_0(a), a \ge 0, \end{cases} \tag{11}$$

$$r(a,t) = p(a,t) - s(a,t) - i(a,t). \tag{12}$$

By integrating problem (3.9) along characteristic lines $t - a = const.$, we find that $s(a,t)$ satisfies

$$s(a,t)=\begin{cases} s_0(a-t)e^{-\int_0^t\left[\mu(a-t+\tau)+\nu(a-t+\tau)+\delta(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)\right]d\tau} \\ +\int_0^t e^{-\int_\sigma^t\left[\mu(a-t+\tau)+\nu(a-t+\tau)+\delta(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)\right]d\tau}\times \\ \delta(a-t+\sigma)\left[p(a-t+\sigma,\sigma)-i(a-t+\sigma,\sigma)\right]d\sigma, & a>t, \\ \left[B(t-a)-i(0,t-a)\right]\pi(a)e^{-\int_0^a\left[\nu(\tau)+\delta(\tau)+k_1(\tau)\lambda(t-a+\tau)\right]d\tau} \\ +\int_0^a e^{-\int_\sigma^a\left[\mu(\tau)+\nu(\tau)+\delta(\tau)+k_1(\tau)\lambda(t-a\tau)\right]d\tau}\times \\ \delta(\sigma)\left[p(\sigma,t-a+\sigma)-i(\sigma,t-a+\sigma)\right]d\sigma, & a<t, \end{cases} \tag{13}$$

By integrating problem (3.10) along characteristic lines $t-a=const.$, and using (3.12), we find that $i(a,t)$ satisfies

$$i(a,t)=\begin{cases} i_0(a-t)e^{-\int_0^t\left[\mu(a-t+\tau)+\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau} \\ +\int_0^t e^{-\int_\sigma^t\left[\mu(a-t+\tau)+\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau}k_1(a-t+\sigma)\lambda(\sigma)\times \\ [(1-\varepsilon)\{s_0(a-t)e^{-\int_0^\sigma\left[\mu(a-t+\tau)+\nu(a-t+\tau)+\delta(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)\right]d\tau} \\ +\int_0^\sigma e^{-\int_b^\sigma\left[\mu(a-t+\tau)+\nu(a-t+\tau)+\delta(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)\right]d\tau}\delta(a-t+b)\times \\ \left[p(a-t+b,b)-i(a-t+b,b)\right]db\}+\varepsilon p(a-t+\sigma,\sigma)]d\sigma, & a>t, \\ i(0,t-a)\pi_2(a)e^{-\varepsilon\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau} \\ +\int_0^a e^{-\int_\sigma^a\left[\mu(\tau)+\gamma(\tau)+\varepsilon k_1(\tau)\lambda(t-a+\tau)\right]d\tau}k_1(\sigma)\lambda(t-a+\sigma)\times \\ [(1-\varepsilon)\{\left[B(t-a)-i(0,t-a)\right]e^{-\int_0^\sigma\left[\mu(\tau)+\nu(\tau)+\delta(\tau)+k_1(\tau)\lambda(t-a+\tau)\right]d\tau} \\ +\int_0^\sigma e^{-\int_b^\sigma\left[\mu(\tau)+\nu(\tau)+\delta(\tau)+k_1(\tau)\lambda(t-a\tau)\right]d\tau}\delta(b)\times \\ \left[p(b,t-a+b)-i(b,t-a+b)\right]db\}+\varepsilon p(\sigma,t-a+\sigma)]d\sigma, & a<t, \end{cases} \tag{14}$$

where $\pi_2(a)$ is given by

$$\pi_2(a)=\pi(a)e^{-\int_0^a\gamma(\tau)d\tau}. \tag{15}$$

It is worth noting that if we can establish a solution for problem (3.13), then a solution for problem (3.12) is determined, and consequently a solution for problem (2.1) is determined by using equation

(3.11). To establish the existence and uniqueness of solution to problem (2.1), we define the following set E to satisfy:

$$E=\{i(a,t):i(.,t)\in L^1([0,\infty));C[0,t_0]),a\in[0,\infty),t\in[0,t_0],|||\,i(a,t)\,|||=\sup_{t\in[0,t_0]}\|i(a,t)\|_{L^1}\},$$

where $C[0,t_0]$ denotes the Banach space of continuous functions in $[0,t_0]$ and $L^1([0,\infty))$ denotes the space of equivalent classes of Lebesgue integrable functions. We note that E is a Banach space.

In order to facilitate our future calculations, we need the following lemma:

Lemma 1 *Suppose that* $x, y \geq 0,$ *then* $\left|e^{-y} - e^{-x}\right| \leq |y - x|.$

Proof: Let $f(x) = e^{-x},$ then use the mean value theorem to establish the required result.
Also, for the same purpose we note that by suitable changes of variables and reversing of the order of integration, we obtain that

$$\begin{aligned}
&\int_t^\infty \int_0^t \int_0^\sigma e^{-\int_\sigma^t \left[\mu(a-t+\tau)+\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau} k_1(a-t+\sigma)\lambda(\sigma)\delta(a-t+b)\times \\
&e^{-\int_b^\sigma \left[\mu(a-t+\tau)+\nu(a-t+\tau)+\delta(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)\right]d\tau} p(a-t+b,b)dbd\sigma da \\
&+\int_0^t \int_0^a \int_0^\sigma e^{-\int_\sigma^a \left[\mu(\tau)+\gamma(\tau)+\varepsilon k_1(\tau)\lambda(t-a+\tau)\right]d\tau} k_1(\sigma)\lambda(t-a+\sigma)\delta(b)\times \\
&e^{-\int_b^\sigma \left[\mu(\tau)+\nu(\tau)+\delta(\tau)+k_1(\tau)\lambda(t-a+\tau)\right]d\tau} p(b,t-a+b)dbd\sigma da \\
&= \int_0^t \int_0^\infty \int_{t-b}^t e^{-\int_{a+\sigma-t}^a \left[\mu(\tau)+\gamma(\tau)+\varepsilon k_1(\tau)\lambda(\tau+t-a)\right]d\tau} k_1(a+b+\sigma-t)\lambda(\sigma)\delta(a)\times \\
&p(a,t-b)e^{-\int_a^{a+b+\sigma-t} \left[\mu(\tau)+\nu(\tau)+\delta(\tau)+k_1(\tau)\lambda(\tau+t-a-b)\right]d\tau} d\sigma dadb.
\end{aligned} \tag{16}$$

In the next theorem, we prove the existence and uniqueness of solution to problem (2.1) via a fixed-point theorem.

Theorem 2 *Problem (2.1) has a unique solution that exists for all time.*

Proof: Define the set Q by $Q = \{i(a,t) \in E, i(a,t) \geq 0, |||\, i(a,t)\, ||| \leq M\},$ where M is a constant which satisfies the following:

$$M > \|p_0(a)\|_{L^1} e^{(\|\beta(a)\|_\infty - \mu_*)t_0}. \tag{17}$$

We note that Q is a closed set in E. Now, for fixed initial age-distributions $s_0(a), i_0(a), r_0(a)$ and $p_0(a),$ define the mapping $T: Q \subset E \to E$ by

$$
Ti(a,t)=\begin{cases}
i_0(a-t)e^{-\int_0^t\left[\mu(a-t+\tau)+\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau} \\
+\int_0^t e^{-\int_\sigma^t\left[\mu(a-t+\tau)+\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau} k_1(a-t+\sigma)\lambda(\sigma)\times \\
[(1-\varepsilon)\{s_0(a-t)e^{-\int_0^\sigma\left[\mu(a-t+\tau)+\nu(a-t+\tau)+\delta(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)\right]d\tau} \\
+\int_0^\sigma e^{-\int_b^\sigma\left[\mu(a-t+\tau)+\nu(a-t+\tau)+\delta(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)\right]d\tau} \delta(a-t+b)\times \\
\left[p(a-t+b,b)-i(a-t+b,b)\right]db\}+\varepsilon p(a-t+\sigma,\sigma)]d\sigma, & a>t, \\
i(0,t-a)\pi_2(a)e^{-\varepsilon\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau} \\
+\int_0^a e^{-\int_\sigma^a\left[\mu(\tau)+\gamma(\tau)+\varepsilon k_1(\tau)\lambda(t-a+\tau)\right]d\tau} k_1(\sigma)\lambda(t-a+\sigma)\times \\
[(1-\varepsilon)\{\left[B(t-a)-i(0,t-a)\right]e^{-\int_0^\sigma\left[\mu(\tau)+\nu(\tau)+\delta(\tau)+k_1(\tau)\lambda(t-a+\tau)\right]d\tau} \\
+\int_0^\sigma e^{-\int_b^\sigma\left[\mu(\tau)+\nu(\tau)+\delta(\tau)+k_1(\tau)\lambda(t-a\tau)\right]d\tau} \delta(b)\times \\
\left[p(b,t-a+b)-i(b,t-a+b)\right]db\}+\varepsilon p(\sigma,t-a+\sigma)]d\sigma. & a<t,
\end{cases} \tag{18}
$$

We notice that, by the a priori estimate (3.8), we obtain the following estimate for $i(a,t)$:

$$\int_0^\infty i(a,t)da \le \|p_0(a)\|_{L^1} e^{(\|\beta(a)\|_\infty-\mu_*)t_0}. \tag{19}$$

Accordingly, we see that, T maps Q into Q. Now, we look for a fixed point of this mapping to provide existence and uniqueness of solution for problem (2.1). To this end, we let $i(a,t)$ and $i_1(a,t)$ be elements of Q, then using (3.6)-(3.8), (3.15), and Lemma (3.1), we obtain the following:

$$\|Ti(.,t)-Ti_1(.,)\|_{L^1} \le K(M,t_0)\int_0^t \|i(.,\sigma)-i_1(.,\sigma)\|_{L^1} d\sigma, \tag{20}$$

where $K(M,t_0)$ is a constant which depends on M and t_0. Therefore,

$$|||\, Ti(.,t)-Ti_1(.,t)\,|||\le t_0K(M,t_0)\,|||\, i(.,\sigma)-i_1(.,\sigma)\,|||. \tag{21}$$

And thus, by induction, for each positive integer n, we obtain

$$|||\, T^n i(.,t)-T^n i_1(.,t)\,|||\le \frac{\left[t_0K(M,t_0)\right]^n}{n!}\,|||\, i(.,t)-i_1(.,t)\,|||. \tag{22}$$

Inequality (3.21) implies that there exists a positive integer N such T^N is a strict contraction on Q. Thus T has a unique fixed point in Q. Since t_0 is arbitrary, it follows that problem (2.1) has a unique solution that exists for all time. This completes the proof of the theorem.

In the next theorem, we show that solutions of problem (2.1) depend continuously on the initial age-distributions, therefore, problem (2.1) is well posed.

Theorem 3: *Let* $p(a,t)$ *and* $p_1(a,t)$ *be two solutions of problem (2.1) corresponding to initial age-distributions* $p_0(a), s_0(a), i_0(a), r_0(a)$ *and* $p_{01}(a), s_{01}(a), i_{01}(a), r_{01}(a)$, *respectively. Also, suppose that* $p(0,t) = B(t)$ *and* $p_1(0,t) = B_1(t)$, *and let* $i(a,t)$ *and* $i_1(a,t)$ *be the corresponding solutions of problem (3.10). Then the following properties hold:*

$$\begin{aligned} |B(t) - B_1(t)| &\leq \|\beta(a)\|_\infty \|p_0(a) - p_{01}(a)\|_{L^1} e^{(\|\beta(a)\|_\infty - \mu_*)t}, \\ \|p(.,t) - p_1(.,t)\|_{L^1} &\leq \|p_0(a) - p_{01}(a)\|_{L^1} e^{(\|\beta(a)\|_\infty - \mu_*)t}, \\ \|i(.,t) - i_1(.,t)\|_{L^1} &\leq \end{aligned} \tag{23}$$

$$[\|s_0(a) - s_{01}(a)\|_{L^1} + \|i_0(a) - i_{01}(a)\|_{L^1} + C\|p_0(a) - p_{01}(a)\|_{L^1}]e^{tK(M,t_0)}, \tag{24}$$

where C is a constant that depends on the parameters of the model and t_0.

Proof. Note that (3.22) and (3.23) follow directly from (3.6) and (3.8), respectively, by linearity. To obtain (3.24), first we use (3.13) and (3.15), and then (3.19) to obtain the following:

$$\begin{aligned} \|i(.,t) - i_1(.,t)\|_{L^1} &\leq [\|s_0(a) - s_{01}(a)\|_{L^1} + \|i_0(a) - i_{01}(a)\|_{L^1} + C\|p_0(a) - p_{01}(a)\|_{L^1}] \\ &+ K(M,t_0)\int_0^t \|i(.,\sigma) - i_1(.,\sigma)\|_{L^1} d\sigma. \end{aligned}$$

Now, the foregoing inequality yields (3.24) by the aid of *Gronwall's* inequality. This completes the proof of the theorem.

We note that (3.22)-(3.24), show that solutions of problem (2.1) depend continuously on the initial age-distributions, and therefore, problem (2.1) is well posed.

The Steady States

In this section, we look at the steady state solution of problem (2.1), under the assumption that the total population has already reached its steady state distribution $p_\infty(a) = c\pi(a)$, i.e., we assume that (3.4) is satisfied with $p^* = 0$, see, for example, Busenberg, et al. (1988, p. 1379). We consider the following transformations, called the age-profiles of susceptible and infective, respectively:

$$u(a,t) = \frac{s(a,t)}{p_\infty(a)}, v(a,t) = \frac{i(a,t)}{p_\infty(a)}.$$

Then with these transformations, (3.9)-(3.10) satisfy the following systems of integro-partial differential equations:

$$\begin{cases} \dfrac{\partial u(a,t)}{\partial a} + \dfrac{\partial u(a,t)}{\partial t} + \nu(a)u(a,t) \\ \qquad = -k_1(a)u(a,t)\lambda(t) + \delta(a)[1 - u(a,t) - v(a,t)], a > 0, t > 0, \\ u(0,t) = 1 - v(0,t), t \geq 0, \\ u(a,0) = u_0(a), a \geq 0, \end{cases} \tag{25}$$

$$\begin{cases} \dfrac{\partial v(a,t)}{\partial a} + \dfrac{\partial v(a,t)}{\partial t} + \gamma(a)v(a,t) \\ \qquad = k_1(a)u(a,t)\lambda(t) + \varepsilon k_1(a)\lambda(t)[1-u(a,t)-v(a,t)], a > 0, t > 0, \\ v(0,t) = q\int_0^\infty \beta(a)\pi(a)v(a,t)da, t \geq 0, \\ v(a,0) = v_0(a), a \geq 0. \end{cases} \tag{26}$$

A steady state $u^*(a), v^*(a),$ and λ^* must satisfy the following equations:

$$\begin{cases} \dfrac{du^*(a)}{da} + [\nu(a) + k_1(a)\lambda^* + \delta(a)]u^*(a) = \delta(a)[1-v^*(a)], a > 0, \\ u^*(0) = 1 - v^*(0), \end{cases} \tag{27}$$

$$\begin{cases} \dfrac{dv^*(a)}{da} + [\gamma(a) + \varepsilon k_1(a)\lambda^*]v^*(a) = \lambda^* k_1(a)[\varepsilon + (1-\varepsilon)u^*(a)], a > 0, \\ v^*(0) = q\int_0^\infty \beta(a)\pi(a)v^*(a)da, \end{cases} \tag{28}$$

$$\lambda^* = c\int_0^\infty k_2(a)\pi(a)v^*(a)da. \tag{29}$$

Anticipating our future needs, we define a threshold parameter R_ν, and is given by

$$\begin{aligned} R_\nu &= c\int_0^\infty \int_0^a k_2(a)\pi(a)e^{-\int_\sigma^a \gamma(\tau)d\tau} k_1(\sigma)D_f(\sigma)d\sigma da \\ &+ \frac{cq\int_0^\infty \int_0^a \beta(a)\pi(a)e^{-\int_\sigma^a \gamma(\tau)d\tau} k_1(\sigma)D_f(\sigma)d\sigma da\left[\int_0^\infty k_2(a)\pi_2(a)da\right]}{\left[1 - q\int_0^\infty \beta(a)\pi_2(a)da\right]}, \end{aligned} \tag{30}$$

where $D_f(a)$ and $\pi_2(a)$ are defined as

$$D_f(a) = 1 - (1-\varepsilon)\int_0^a \nu(\sigma)e^{-\int_\sigma^a [\nu(\tau)+\delta(\tau)]d\tau} d\sigma, \tag{31}$$

$$\pi_2(a) = \pi(a)e^{-\int_0^a \gamma(\tau)d\tau}. \tag{32)(33}$$

Here, we note that the threshold parameter R_ν, known as the reproduction number in the presence of the vaccination strategy $\nu(a)$, as in Hadeler, et al. (1996) or the net replacement ratio, as in Thieme (2001). And if we set $\nu(a) = 0,$ in the formula for R_ν, then we obtain R_0, usually called the basic reproduction number, and is interpreted as the expected number of secondary cases produced, in a lifetime, by an infective, in a totally susceptible population. Also, from equations (4.6)-(4.7), we can see that R_ν decreases with $\nu(a),$ and hence $R_\nu < R_0$.

In the following result, we determine special case of the steady state solution of problem (2.1), and, in particular, we look at what is called disease-free equilibrium, and we show that it is possible that $\lambda^* = 0$ and a positive endemic equilibrium exists, in fact, a continuum of positive endemic

equilibriums. This behavior is solely created by vertical transmission and is not present when $q = 0$.

Theorem 4

• Suppose that the following conditions hold: (i) $k_2(a)$ is identically zero, (ii) $q = 1$, and (iii) the support of $\gamma(a)$ lies to the right of the support of $\beta(a)$. Then, problem (2.1) gives rise to a continuum of endemic equilibriums of the form:

$$u^*(a) = (1 - v^*(0))e^{-\int_0^a [\nu(\tau)+\delta(\tau)]d\tau} + \int_0^a \delta(\sigma)e^{-\int_\sigma^a [\nu(\tau)+\delta(\tau)]d\tau}\left[1 - v^*(\sigma)\right]d\sigma, \tag{34}$$

$$v^*(a) = v^*(0)e^{-\int_0^a \gamma(\tau)d\tau}, \tag{35}$$

where the real number $v^*(0) \in (0,1]$, is arbitrary.

• If any one of the conditions in (1) does not hold then, if $\lambda^* = 0$, then the steady state of problem (2.1) is the disease-free equilibrium:

$$v^*(a) = 0, u^*(a) = e^{-\int_0^a [\nu(\tau)+\delta(\tau)]d\tau} + \int_0^a \delta(\sigma)e^{-\int_\sigma^a [\nu(\tau)+\delta(\tau)]d\tau} d\sigma. \tag{36}$$

• Suppose that the following conditions hold: (i) $k_2(a)$ is not identically zero,(ii) $q = 1$, and (iii) the supports of $\gamma(a)$ and $k_1(a)$ lie to the right of the support of $\beta(a)$. Then problem (2.1) gives rise to a continuum of endemic equilibriums of the form:

$$u^*(a) = (1 - v^*(0))e^{-\int_0^a [\nu(\tau)+\delta(\tau)+\lambda^* k_1(\tau)]d\tau} + \int_0^a \delta(\sigma)e^{-\int_\sigma^a [\nu(\tau)+\delta(\tau)+\lambda^* k_1(\tau]d\tau}\left[1 - v^*(\sigma)\right]d\sigma, \tag{37}$$

$$v^*(a) = v^*(0)e^{-\int_0^a [\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} + \lambda^* \int_0^a k_1(\sigma)e^{-\int_\sigma^a [\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} F(\sigma)d\sigma, \tag{38}$$

where the real number $v^*(0) \in (0,1]$, is arbitrary, and $F(\sigma)$ is defined as

$$F(\sigma) = \left[\varepsilon + (1-\varepsilon)u^*(\sigma)\right]. \tag{39}$$

Proof: To prove (1), we note that if we solve the system of the ordinary differential equations (4.3)-(4.4), then we obtain (4.12)-(4.13). Then, if we set $\lambda^* = 0$ in (4.13) and used the resulting equation and equation (4.4) to find $v^*(0)$, we find that $v^*(0)$ is undetermined by (ii) and (iii).

Now, using equation (4.5), we obtain the following equation for λ^* and $v^*(0)$:

$$\begin{aligned}\lambda^* = cv^*(0)\int_0^\infty k_2(a)\pi(a)e^{-\int_0^a [\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} da \\ + c\lambda^* \int_0^\infty \int_0^a k_2(a)\pi(a)k_1(\sigma)e^{-\int_\sigma^a [\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} F(\sigma)d\sigma da.\end{aligned} \tag{40}$$

Therefore, if we set $\lambda^* = 0$, then we see that $v^*(0)$ is undetermined by (i). And accordingly, for arbitrary fixed $v^*(0) \in (0,1]$, we obtain an endemic equilibrium given by (4.9)-(4.10). This completes the proof of (1).

To prove (2), we note that it is easy to see that $v^*(0) = 0$, either from (4.15) and $\lambda^* = 0$, if $k_2(a)$ is

not identically zero or if any of the other conditions in (1) is not satisfied, then $q\int_0^\infty \beta(a)\pi_2(a)da < 1$ and therefore, using (4.4), we see that $v^*(0) = 0.$ Now, if we use equations (4.12)-(4.13), we obtain (4.11). This completes the proof of (2).

To prove (3), we note that using (4.4) and (4.13), it is easy to see that $v^*(0)$ is undetermined, by assumptions (i)-(iii). Therefore, for arbitrary fixed $v^*(0) \in (0,1],$ we use equation (4.15) to determine at least one $\lambda^* > 0.$ To this end, we rewrite equation (4.15) in the following form:

$$\lambda^*\left[1 - c\int_0^\infty \int_0^a k_2(a)\pi(a)k_1(\sigma)e^{-\int_\sigma^a [\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} F(\sigma)d\sigma da\right] \tag{41}$$

$$= \quad cv^*(0)\int_0^\infty k_2(a)\pi(a)e^{-\int_0^a [\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} da.$$

We can easily see that the right-hand side of (4.16) is a decreasing function of $\lambda^*,$ with a value greater than zero when $\lambda^* = 0,$ since $k_2(a)$ is not identically zero, and tends to zero if $\lambda^* \to +\infty.$ On the other hand, the left-hand side of (4.16) has a value equal to zero when $\lambda^* = 0$ and approaches $+\infty,$ when $\lambda^* \to +\infty.$ Therefore, equation (4.16) has a solution $\lambda^* > 0,$ and this value of λ^* gives rise to an endemic equilibrium via equations (4.12)-(413). Here, we note that the system of ODEs (4.3)-(4.4) has a unique solution for a fixed λ^* and a known $v^*(0).$ Also, we can easily see that $u^*(a)$ satisfies the following integral equation, which has a unique solution for a fixed λ^* and a known $v^*(0)$:

$$u^*(a) = f(a;\lambda^*;v^*(0)) + \int_0^a K(s,a;\lambda^*)u^*(s)ds, \tag{42}$$

where $f(a;\lambda^*;v^*(0))$ and $K(s,a;\lambda^*)$ are functions that depend on the parameters of the model only, for a fixed λ^* and a known $v^*(0),$ and they are defined as follows:

$$f(a;\lambda^*;v^*(0)) = (1-v^*(0))e^{-\int_0^a \left[\nu(a)+\delta(a)+\lambda^* k_1(\tau)\right]d\tau} + \int_0^a \delta(\sigma)e^{-\int_\sigma^a [\nu(\tau)+\delta(\tau)+\lambda^* k_1(\tau)]d\tau} d\sigma$$

$$- \quad v^*(0)\int_0^a \delta(\sigma)e^{-\int_\sigma^a [\nu(\tau)+\delta(\tau)+\lambda^* k_1(\tau)]d\tau} e^{-\int_0^\sigma \left[\gamma(\tau)+\varepsilon k_1(\tau)\right]d\tau} d\sigma$$

$$- \quad \varepsilon\lambda^*\int_0^a \delta(\sigma)e^{-\int_\sigma^a [\nu(\tau)+\delta(\tau)+\lambda^* k_1(\tau)]d\tau}\left(\int_0^\sigma k_1(s)e^{-\int_s^\sigma \left[\gamma(\tau)+\varepsilon k_1(\tau)\right]d\tau} ds\right)d\sigma, \tag{43}$$

$$K(s,a;\lambda^*) = -(1-\varepsilon)\lambda^* k_1(s)\int_s^a \delta(\sigma)e^{-\int_\sigma^a \left[\nu(\tau)+\delta(\tau)+\lambda^* k_1(\tau)\right]d\tau} e^{-\int_s^\sigma \left[\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)\right]d\tau} d\sigma. \tag{44}$$

This completes the proof of (3) and therefore, the proof of the theorem is complete.
In the next result, we prove the existence of an endemic equilibrium when $R_\nu > 1,$ however, this

endemic equilibrium may not be unique due to possible lack of monotonicity, we note that this is also the case for several age-structured epidemic models, for example, see, Castillo-Chavez, et al. (1998), Cha, et al. (1998) and El-Doma (2006).

Theorem 5 *Suppose that* $q \neq 1$, *and* $R_\nu > 1$, *then* $\lambda^* = 0$ *and* $\lambda^* > 0$ *are possible steady states for problem (2.1).*

Proof: Note that if $q \neq 1$, then we can use (4.4) and (4.13) to obtain the following:

$$v^*(0) = \frac{q\lambda^* \int_0^\infty \int_0^a \beta(a)\pi(a)k_1(\sigma)e^{-\int_\sigma^a [\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} F(\sigma)d\sigma da}{\left[1 - q\int_0^\infty \beta(a)\pi_2(a)e^{-\varepsilon\lambda^* \int_0^a k_1(\tau)d\tau} da\right]}. \tag{45}$$

Now, we can use (4.15) and (4.20) to obtain that either $\lambda^* = 0$ or λ^* satisfies the following equation:

$$1 = c\int_0^\infty \int_0^a k_2(a)\pi(a)e^{-\int_\sigma^a [\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} k_1(\sigma)F(\sigma)d\sigma da + \frac{cq\int_0^\infty \int_0^a \beta(a)\pi(a)e^{-\int_\sigma^a [\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} k_1(\sigma)F(\sigma)d\sigma da \left[\int_0^\infty k_2(a)\pi_2(a)e^{-\varepsilon\lambda^* \int_0^a k_1(\tau)d\tau} da\right]}{\left[1 - q\int_0^\infty \beta(a)\pi_2(a)e^{-\varepsilon\lambda^* \int_\sigma^a k_1(\tau)d\tau} da\right]}, \tag{46}$$

where $F(\sigma)$ is defined by equation (4.14).

Noticing that, if $q \neq 1$, i.e., $q < 1$ then from equation (4.4), we see that $u^*(0) > 0$ and therefore, $v^*(a) < 1 \,\forall\, a \in [0,\infty)$. Accordingly, from equation (4.5), we see that, $\lambda^* < c\int_0^\infty k_2(a)\pi(a)da$. Now, using this value for λ^* in equation (4.21) and the fact that $v^*(a) < 1 \,\forall\, a \in [0,\infty)$, and equation (4.13), we can deduce that the right-hand side of equation (4.21) is less than one at this value of λ^*. Also, it is easy to see that the right-hand side of (4.21) is equal to $R_\nu > 1$, when $\lambda^* = 0$. Therefore, equation (4.21) has at least one solution $\lambda^* > 0$. This completes the proof of the theorem.

Here, we note that from theorem (4.2), an endemic equilibrium would exists if $R_\nu > 1$, and from equation (4.6) the effect of vertical transmission via its parameter q which is the probability of vertically transmitting the disease, is seen R_ν increases with $q \in [0,1)$, and therefore increases the likelihood that an endemic will occur. So, in order to prevent an outbreak and control the spread of the disease, we need to reduce R_ν to a value less than one. If $\nu(a)$ is constant then there exists a unique value for $\nu(a)$ which reduces R_ν to one, but if $\nu(a)$ is age-dependent and not constant then $\nu(a)$ can be chosen according to some constraint that reduces the cost of vaccination or in general to obtain what is called an optimal vaccination strategy, for example, see Müller (1994), (1998), Hadeler, et al. (1996), Castillo-Chavez, et al. (1998) and Li, et al. (2004).

If $q=1$, and other conditions hold, for example, see Theorem (4.1), then problem (2.1) gives rise to a continuum of endemic steady states, and this situation does not occur when there is no vertical transmission. Also, if $q=1$, and $\gamma(a)$ is identically zero, then problem (2.1) has the steady state as the total population consisting of infective only. Also, it is easy to see that this steady state is actually, by uniqueness, the solution for problem (2.1), in this special case.

The effects of certain vaccination strategies for the eradication of important communicable diseases such as measles, rubella, pertussis and tuberculosis are dealt with in several papers, for example, see Hethcote (1983), (2000), (1997), (1989), Dietz (1981), Knox (1980), McLean (1986), Katzmann, et al. (1984), Schenzle (1984), Coutinho, et al. (1993), Greenhalgh (1990) and Anderson, et al (1999).

Stability of the Steady States

In this section, we study the stability of the steady states of problem (2.1), and, in particular, we study the stability of the disease-free equilibrium, and the endemic equilibriums.

Theorem 6 *The disease-free equilibrium, given by equation (4.11), is locally asymptotically stable if* $R_v<1$, *and unstable if* $R_v>1$.

Proof. Straightforward linearization of equations (4.1)-(4.2) around the disease-free equilibrium yields the following characteristic equation:

$$1=c\int_0^\infty\int_0^a k_2(a)\pi(a)e^{-\int_\sigma^a[\gamma(\tau)+\xi]d\tau}k_1(\sigma)F(\sigma)d\sigma da$$
$$+\frac{cq\int_0^\infty\int_0^a\beta(a)\pi(a)e^{-\int_\sigma^a[\gamma(\tau)+\xi]d\tau}k_1(\sigma)F(\sigma)d\sigma da\left[\int_0^\infty k_2(a)\pi_2(a)e^{-\xi a}da\right]}{\left[1-q\int_0^\infty\beta(a)\pi_2(a)e^{-\xi a}da\right]}, \tag{47}$$

where $F(\sigma)$ is given by equation (4.14) and $u^*(\sigma)$ in the definition of $F(\sigma)$ is defined as in equation (4.11), and ξ is a complex number.

We note that, when ξ is real, then the right-hand side of equation (5.1) is a decreasing function of ξ, and approaches zero as $\xi\to+\infty$, and equals R_v when $\xi=0$. Therefore, equation (5.1) has a solution $\xi>0$ if $R_v>1$. Accordingly, the trivial equilibrium is unstable if $R_v>1$. And if $R_v<1$, then it is clear from (4.6) that the only possible solutions of equation (5.1) must satisfy $\xi<0$. The local asymptotic stability of the disease-free equilibrium is completed by observing that the real root of equation (5.1) has the dominant real part, and this is obtained by considering absolute values. This completes the proof of the theorem.

In the next result, we show that the disease-free equilibrium is globally stable when $R_0<1$. We note that $R_v<R_0$.

Theorem 7 *The disease-free equilibrium is globally stable when* $R_0<1$.

Proof. By using equation (3.13), we find that $i(a,t)$ satisfies

$$i(a,t)=\begin{cases} i_0(a-t)e^{-\int_0^t\left[\mu(a-t+\tau)+\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau}+c\pi(a)\int_0^t k_1(a-t+\sigma)\times \\ e^{-\int_\sigma^t\left[\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau}\lambda(\sigma)F_1(a-t+\sigma,\sigma)d\sigma, & a>t, \\ i(0,t-a)\pi_2(a)e^{-\varepsilon\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}+c\pi(a)\int_0^a k_1(\sigma)\times \\ \lambda(t-a+\sigma)e^{-\int_\sigma^a\left[\gamma(\tau)+\varepsilon k_1(\tau)\lambda(t-a+\tau)\right]d\tau}F_1(\sigma,t-a+\sigma)d\sigma, & a<t, \end{cases} \tag{48}$$

where $F_1(a-t+\sigma,\sigma)$ is defined as

$$F_1(a-t+\sigma,\sigma)=\varepsilon+(1-\varepsilon)u(a-t+\sigma,\sigma). \tag{49}$$

From problem (2.1), $i(0,t)=q\int_0^\infty\beta(a)i(a,t)da$, then using (5.2), we obtain the following:

$$\begin{aligned} i(0,t)=q\{&\int_0^t\beta(a)\pi_2(a)i(0,t-a)e^{-\varepsilon\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}da. \\ &+c\int_0^t\int_0^a\beta(a)\pi(a)k_1(\sigma)\lambda(t-a+\sigma)e^{-\int_\sigma^a\left[\gamma(\tau)+\varepsilon k_1(\tau)\lambda(t-a+\tau)\right]d\tau}F_1(\sigma,t-a+\sigma)d\sigma da \\ &+c\int_t^\infty\int_0^t\beta(a)\pi(a)k_1(a-t+\sigma)\lambda(\sigma)e^{-\int_\sigma^t\left[\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau}F_1(a-t+\sigma,\sigma)d\sigma da \\ &\bmod 0.5cm.+\int_t^\infty\beta(a)\pi(a)i_0(a-t)e^{-\int_0^t\left[\mu(a-t+\tau)+\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau}da\}. \end{aligned} \tag{50}$$

Also, from problem (2.1), $\lambda(t)=\int_0^\infty k_2(a)i(a,t)da$, then using (5.2), and changing the order of integration several times and making appropriate changes of variables yields

$$\begin{aligned} \lambda(t) \quad = \quad &\int_0^t k_2(a)\pi_2(a)i(0,t-a)e^{-\varepsilon\int_0^a k_1(\tau)\lambda(t-a+\tau)d\tau}da. \\ &+c\int_0^t\int_\sigma^\infty k_2(a)k_1(a-\sigma)\pi(a)\lambda(t-\sigma)F_1(a-\sigma,t-\sigma)e^{-\int_{a-\sigma}^a\left[\gamma(\tau)+\varepsilon k_1(\tau)\lambda(t-a+\tau)\right]d\tau}dad\sigma \\ &\bmod 0.5cm.+\int_t^\infty k_2(a)(a)i_0(a-t)e^{-\int_0^t\left[\mu(a-t+\tau)+\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau}da. \end{aligned} \tag{51}$$

Note that by Assumptions 2-5 of section 2 and the dominated convergence theorem, we obtain

$$\int_t^\infty k_2(a)i_0(a-t)e^{-\int_0^t\left[\mu(a-t+\tau)+\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau}da\to 0,\ \text{as}\ t\to\infty.$$

Also, by similar reasoning as above, we obtain

$$\int_t^\infty\beta(a)i_0(a-t)e^{-\int_0^t\left[\mu(a-t+\tau)+\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau}da\to 0,\ \text{as}\ t\to\infty.$$

And

$$c\int_t^\infty\int_0^t \beta(a)\pi(a)k_1(a-t+\sigma)\lambda(\sigma)e^{-\int_\sigma^t\left[\gamma(a-t+\tau)+\varepsilon k_1(a-t+\tau)\lambda(\tau)\right]d\tau}F_1(a-t+\sigma,\sigma)d\sigma da \to 0,$$

as $t\to\infty$.

Now, we let $i^\infty = \limsup_{t\to\infty} i(0,t)$ and $\lambda^\infty = \limsup_{t\to\infty}\lambda(t)$, then from equations (5.4)-(5.5) and *Fatou's* Lemma, we obtain the following:

$$i^\infty \le qi^\infty\int_0^\infty \beta(a)\pi_2(a)da + cq\lambda^\infty\int_0^\infty\int_0^a \beta(a)\pi(a)k_1(a-\sigma)d\sigma da,$$

$$\lambda^\infty \le i^\infty\int_0^\infty k_2(a)\pi_2(a)da + c\lambda^\infty\int_0^\infty\int_0^a k_2(a)\pi(a)k_1(\sigma)e^{-\int_\sigma^a\gamma(\tau)d\tau}d\sigma da.$$

Therefore, $\lambda^\infty \le \lambda^\infty R_0 < \lambda^\infty$, since $R_0 < 1$, which gives $\lambda^\infty = 0$. Accordingly, the disease-free equilibrium is globally stable, if $R_0 < 1$. This completes the proof of the theorem.

In order to study the stability of an endemic equilibrium, we linearize the system of equations (4.1)-(4.2) by considering perturbations $w(a,t)$ and $\eta(a,t)$ defined by

$$\begin{aligned} w(a,t) &= u(a,t)-u^*(a), \\ \eta(a,t) &= v(a,t)-v^*(a). \end{aligned}$$

Accordingly, we obtain the following systems of integro-partial differential equations:

$$\begin{cases} \dfrac{\partial w(a,t)}{\partial a}+\dfrac{\partial w(a,t)}{\partial t}+\left[\nu(a)+\delta(a)+k_1(a)\lambda^*\right]w(a,t) \\ \qquad = -\delta(a)\eta(a,t)-k_1(a)u^*(a)\psi, a>0, t>0, \\ w(0,t) = -\eta(0,t), t\ge 0, \\ w(a,0) = w_0(a) = u_0(a)-u^*(a), a\ge 0, \end{cases} \tag{52}$$

$$\begin{cases} \dfrac{\partial \eta(a,t)}{\partial a}+\dfrac{\partial \eta(a,t)}{\partial t}+\left[\gamma(a)+\varepsilon\lambda^* k_1(a)\right]\eta(a,t) = (1-\varepsilon)\lambda^* k_1(a)w(a,t) \\ +k_1(a)\left[(1-\varepsilon)u^*(a)+\varepsilon(1-v^*)\right]\psi, a>0, t>0, \\ \eta(0,t) = q\int_0^\infty \beta(a)\pi(a)\eta(a,t)da, t\ge 0, \\ \eta(a,0) = \eta_0(a) = v_0(a)-v^*(a), a\ge 0, \end{cases} \tag{53}$$

where $\psi(t)$ is given by

$$\psi(t) = c\int_0^\infty k_2(a)\pi(a)\eta(a,t)da. \tag{54}$$

Now, we assume that

$$\begin{aligned} w(a,t) &= f(a)e^{\xi t}, \\ \eta(a,t) &= g(a)e^{\xi t}, \end{aligned}$$

where ξ is a complex number. Accordingly, we obtain the following systems of ODEs:

$$\begin{cases} f'(a)+\left[\xi+\nu(a)+\lambda^* k_1(a)+\delta(a)\right] f(a) = -\delta(a)g(a)-k_1(a)u^*(a)\psi^*, \\ f(0) = -g(0), \end{cases} \tag{55}$$

$$\begin{cases} g'(a)+\left[\xi+\gamma(a)+\varepsilon\lambda^* k_1(a)\right] g(a) = (1-\varepsilon)\lambda^* k_1(a) f(a)+ \\ k_1(a)\left[(1-\varepsilon)u^*(a)+\varepsilon(1-v^*)\right]\psi^*, \\ g(0) = q\int_0^\infty \beta(a)\pi(a)g(a)da. \end{cases} \tag{56}$$

where ψ^* is defined as

$$\psi^* = c\int_0^\infty k_2(a)\pi(a)g(a)da.$$

Using (5.9)-(5.10), we obtain the following characteristic equation, in the case $\psi^* \neq 0$:

$$1 = c\int_0^\infty\int_0^a k_2(a)\pi(a)e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} k_1(\sigma)F_2(\sigma)d\sigma da + \frac{cq\int_0^\infty\int_0^a \beta(a)\pi(a)e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} k_1(\sigma)F_2(\sigma)d\sigma da\left[\int_0^\infty k_2(a)\pi_2(a)e^{-\int_0^a\left[\xi+\varepsilon\lambda^* k_1(\tau)\right]d\tau}da\right]}{\left[1-q\int_0^\infty \beta(a)\pi_2(a)e^{-\int_0^a\left[\xi+\varepsilon\lambda^* k_1(\tau)\right]d\tau}da\right]}, \tag{57}$$

where $F_2(a)$ is defined as

$$F_2(a) = \frac{(1-\varepsilon)\lambda^* f(a)}{\psi^*}+\left[(1-\varepsilon)u^*(a)+\varepsilon(1-v^*(a)\right]. \tag{58}$$

We note that if we set $\lambda^* = 0$ in (5.11), we obtain (5.1).
In the following result, we establish the local asymptotic stability of an endemic equilibrium, in the special case $\varepsilon = 1$.

Theorem 8: *Suppose that* $R_\nu > 1$, $q \neq 1$, *and* $\varepsilon = 1$, *then an endemic equilibrium is locally asymptotically stable.*

Proof: We note that if $\psi^* = 0$, then $g(a) = 0$ from equation (5.10), since $g(0) = 0$ because $\varepsilon = 1$. Therefore, it follows from equation (5.9) that $f(a) = 0$, and hence, stability follows in this case. If $\psi^* \neq 0$, we take ξ in equation (5.11) to be real, and then it is clear from the characteristic equation (4.21) that $\xi < 0$, since $v^*(a)$ satisfies equation (4.13), and $R_\nu > 1$, therefore the local asymptotic stability of an endemic equilibrium follows from the fact that the real root of the characteristic equation has the dominant real part. This completes the proof of the theorem.

In the next result, we will prove the local asymptotic stability of an endemic equilibrium, when $q = 0$ and $\delta(a)$ is identically zero. This result will allow us to deduce the stability of the endemic

equilibrium of the SIR age-structured epidemic model, note that the endemic equilibrium is unique in this special case because of the monotonicity of the right-hand side of equation (4.21).

Theorem 9: *Suppose that* $R_v > 1$, $q = 0$ *and* $\delta(a)$ *is identically zero. Then the endemic equilibrium is locally asymptotically stable.*

Proof: Note that if $\psi^* = q = 0$, and $\delta(a) \equiv 0$, then it is easy to see that $f(a) = 0$ from (5.9), since $f(0) = -g(0) = 0$. Accordingly, it follows that $g(a) = 0$. And hence, if $\psi^* \neq 0$, then the characteristic equation (5.11) takes the following form:

$$1 = c\int_0^\infty \int_0^a k_2(a)\pi(a)e^{-\int_\sigma^a [\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} k_1(\sigma)F_2(\sigma)d\sigma da, \tag{59}$$

where $F_2(a)$ is given by equation (5.12).

Now, (5.13) can be rewritten in the following form:

$$1 = c\int_0^\infty \int_0^a k_2(a)\pi(a)k_1(\sigma)e^{-\int_\sigma^a [\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} \times$$
$$\left\{F(\sigma) - \left[\lambda^*(1-\varepsilon)\int_0^\sigma e^{-\int_s^\sigma [\xi+\nu(\tau)+\lambda^* k_1(\tau)]d\tau} k_1(s)u^*(s)ds + \varepsilon v^*(\sigma)\right]\right\} d\sigma da, \tag{60}$$

where $F(\sigma)$ is given by (4.14).

Therefore, if we assume that ξ is real in (5.14), then from equation (4.21), we conclude that $\xi < 0$, and accordingly, the local asymptotic stability follows from the fact that the real root of the characteristic equation has the dominant real part. This completes the proof of the theorem.

Note that if $\delta(a) = q = \varepsilon = 0$, then we obtain the SIR age-structured epidemic model, and theorem (5.4) shows that the endemic equilibrium of the SIR age-structured epidemic model is locally asymptotically stable.

In the next result, we prove the local asymptotic stability of an endemic equilibrium in the case $q \neq 0$, and $\delta(a) \equiv 0$.

Theorem 10: *Suppose that the following hold: (i)* $R_v > 1$, *(ii)* $\delta(a) \equiv 0$, *and (iii)* $(1-\varepsilon)\lambda^* \int_0^\infty k_1(\sigma)e^{-\int_0^\sigma \left[\nu(\tau)+(1-\varepsilon)\lambda^* k_1(\tau)-\gamma(\tau)\right]d\tau} d\sigma < 1$. *Then an endemic equilibrium is locally asymptotically stable.*

Proof: If $\psi^* = 0$, then from equations (5.9)-(5.10), we obtain the following:

$$cg(0)\int_0^\infty k_2(a)\pi(a)e^{-\int_0^a \left[\xi+\varepsilon\lambda^* k_1(\tau)+\gamma(\tau)\right]d\tau} \times$$
$$[1-(1-\varepsilon)\lambda^* \int_0^a k_1(\sigma)e^{-\int_0^\sigma \left[\nu(\tau)+(1-\varepsilon)\lambda^* k_1(\tau)-\gamma(\tau)\right]d\tau} d\sigma]da = 0. \tag{61}$$

From equation (5.15) and assumption (iii), we deduce that $g(0) = 0$, and then it follows that $f(0) = 0$, and from assumption (ii) and equation (5.9), we obtain that $f(a) = 0$. From $f(a) = 0$ and equation (5.10), we obtain that $g(a) = 0$.

Now, if $\psi^* \neq 0$, then using equations (5.9)-(5.10) and solving for $\frac{g(0)}{\psi^*}$, we obtain the following characteristic equation:

$$\left[1-q\int_0^\infty \beta(a)\pi_2(a)e^{-\int_0^a\left[\xi+\varepsilon\lambda^* k_1(\tau)\right]d\tau}da\right]\times$$

$$\{1-c\int_0^\infty\int_0^a k_2(a)\pi(a)e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}k_1(\sigma)F(\sigma)d\sigma da-$$

$$\frac{cq\int_0^\infty\int_0^a \beta(a)\pi(a)e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}k_1(\sigma)F(\sigma)d\sigma da\left[\int_0^\infty k_2(a)\pi_2(a)e^{-\int_0^a\left[\xi+\varepsilon\lambda^* k_1(\tau)\right]d\tau}da\right]}{\left[1-q\int_0^\infty \beta(a)\pi_2(a)e^{-\int_0^a\left[\xi+\varepsilon\lambda^* k_1(\tau)\right]d\tau}da\right]}\}$$

$$+c\left[1-q\int_0^\infty \beta(a)\pi_2(a)e^{-\int_0^a\left[\xi+\varepsilon\lambda^* k_1(\tau)\right]d\tau}da\right]\times$$

$$\{\varepsilon\int_0^\infty\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}k_2(a)\pi(a)k_1(\sigma)v^*(\sigma)d\sigma da+\lambda^*(1-\varepsilon)\int_0^\infty\int_0^a\int_0^\sigma\times$$

$$e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}k_2(a)\pi(a)k_1(\sigma)k_1(s)u^*(s)e^{-\int_s^\sigma[\xi+\nu(\tau)+\lambda^* k_1(\tau)]d\tau}dsd\sigma da\}+$$

$$q\lambda^*(1-\varepsilon)\int_0^\infty\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}\beta(a)\pi(a)k_1(\sigma)e^{-\int_0^\sigma[\xi+\nu(\tau)+\lambda^* k_1(\tau)]d\tau}d\sigma da\{1-c\int_0^\infty\times$$

$$\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}k_2(a)\pi(a)k_1(\sigma)\left[F(\sigma)-\varepsilon v^*(\sigma)\right]d\sigma da+c\lambda^*(1-\varepsilon)\int_0^\infty\int_0^a\int_0^\sigma\times$$

$$e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}k_2(a)\pi(a)k_1(\sigma)k_1(s)u^*(s)e^{-\int_s^\sigma[\xi+\nu(\tau)+\lambda^* k_1(\tau)]d\tau}dsd\sigma da\}=-\varepsilon qc\int_0^\infty\times$$

$$\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}\beta(a)\pi(a)k_1(\sigma)v^*(\sigma)d\sigma da(\int_0^\infty k_2(a)\pi_2(a)e^{-\int_0^a\left[\xi+\varepsilon\lambda^* k_1(\tau)\right]d\tau}da)$$

$$-qc\lambda^*(1-\varepsilon)\int_0^\infty\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}k_2(a)\pi(a)k_1(\sigma)e^{-\int_0^\sigma[\xi+\nu(\tau)+\lambda^* k_1(\tau)]d\tau}d\sigma da\times$$

$$(\int_0^\infty\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}\beta(a)\pi(a)k_1(\sigma)[F(\sigma)-\varepsilon v^*(\sigma)]d\sigma da)-qc\lambda^*(1-\varepsilon)\times$$

$$\int_0^\infty\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}\beta(a)\pi(a)k_1(\sigma)(\int_0^\sigma e^{-\int_s^\sigma[\xi+\nu(\tau)+\lambda^* k_1(\tau)]d\tau}k_1(s)u^*(s)ds)d\sigma da(\int_0^\infty\times$$

$$e^{-\int_0^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} k_2(a)\pi(a)[1-\lambda^*(1-\varepsilon)\int_0^a k_1(\sigma)e^{-\int_0^\sigma[\nu(\tau)+(1-\varepsilon)\lambda^* k_1(\tau)-\gamma(\tau)]d\tau} d\sigma]da). \quad (62)$$

Now, suppose that ξ is real, then from the characteristic equation (5.16) and equation (4.21), and the fact that $R_\nu > 1$, we obtain that $\xi < 0$, since for $\xi \geq 0$, the right-hand side of (5.16) is non-positive while the left-hand side is positive. Therefore, an endemic equilibrium is locally asymptotically stable, since the real root has the dominant real part. This completes the proof of the theorem.

We note that if, in addition to the assumptions of Theorem (5.5), we assume that $\varepsilon = 0$ in the model, then we obtain the SIR age-structured epidemic model studied in Cha, et al. (1998) and therefore, Theorem (5.5) establishes the local asymptotic stability for the endemic equilibrium of that model. Also, we note that if $\delta(a) \equiv 0$, and $\varepsilon = 0$, then condition (iii) of Theorem (5.5) guarantees the uniqueness of the endemic equilibrium, and this follows from the monotonicity of the right-hand side of equation (4.21).

Now, we look at the case in which $\delta(a)$ is not identically zero but $q = 0$, for example, see Li, et al. (2004). In this case, we first use the system of ODEs (5.9)-(5.10) to obtain that if $\psi^* = 0$, then $f(a) = g(a) = 0$, and that is because $f(0) = g(0) = 0$ and the uniqueness of solution of the system of ODEs. Also, if we assume that $\psi^* \neq 0$, then using the same system of equations, we can deduce that $G(a) = \dfrac{g(a)}{\psi^*}$ satisfies the following Volterra integral equation:

$$G(a) = \int_0^a K(a,s)G(s)ds + d(a), \quad (63)$$

where $K(a,s)$ and $d(a)$ are given by

$$K(a,s) = -\lambda^*(1-\varepsilon)\delta(s)\int_s^a k_1(\sigma)e^{-\int_s^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} e^{-\int_s^\sigma[\xi+\nu(\tau)+\delta(\tau)+(1-\varepsilon)\lambda^* k_1(\tau)-\gamma(\tau)]d\tau} d\sigma, \quad (64)$$

$$d(a) = -\lambda^*(1-\varepsilon)\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} k_1(\sigma)(\int_0^\sigma (e^{-\int_s^\sigma[\xi+\nu(\tau)+\delta(\tau)+\lambda^* k_1(\tau)]d\tau} k_1(s)u^*(s)ds)d\sigma \quad (65)$$
$$+\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} k_1(\sigma)[F(\sigma)-\varepsilon v^*(\sigma)]d\sigma.$$

Also, in this case the characteristic equation is given by the following equation:

$$1 = c\int_0^\infty G(a)k_2(a)\pi(a)da. \quad (66)$$

We note that the Volterra integral equation (5.17) has a unique solution, and this fact can also be seen from the system of ODEs (5.9)-(5.10).

In the next result, we show that if the force of infection is sufficiently small, then an endemic equilibrium is always locally asymptotically stable.

Theorem 11: *Suppose that* $R_\nu > 1$, *and* $q = 0$. *If* λ^* *is sufficiently small, then an endemic equilibrium is locally asymptotically stable.*

Proof: We note that $G(a) = \dfrac{g(a)}{\psi^*} = e^{-\xi t}\dfrac{[v(a,t)-v^*(a)]}{\psi^*}$, and so for Re $\xi \geq 0$ and $\psi^* \neq 0$, $|G(a)|$ is bounded and its integral from 0 to ∞ is also bounded, because we are assuming that the total population has already reached its steady state. Now, since $\varepsilon > 0$, we can use equation (4.21) to show that the following integral is bounded:

$$c\int_0^\infty\int_0^a k_2(a)\pi(a)e^{-\int_\sigma^a[\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}k_1(\sigma)d\sigma da. \tag{67}$$

Using equations (5.17) and (5.20), we obtain that

$$1 = c\int_0^\infty\int_0^a K(a,s)k_2(a)\pi(a)G(s)dsda + c\int_0^\infty k_2(a)\pi(a)d(a)da. \tag{68}$$

Therefore, from equations (5.18)-(5.19), we obtain that the right-hand side of equation (5.22) approaches

$$c\int_0^\infty\int_0^a k_2(a)\pi(a)e^{-\int_\sigma^a[\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}k_1(\sigma)[F(\sigma)-\varepsilon v^*(\sigma)]d\sigma da \text{ as } \lambda^* \to 0. \tag{69}$$

And therefore, in view of equations (4.21) and (4.13), the right-hand side of equation (5.22) is strictly less than one for any ξ with Re $\xi \geq 0$, and λ^* sufficiently small. Accordingly, an endemic equilibrium is locally asymptotically stable if λ^* is sufficiently small. This completes the proof of the theorem.

Similarly, we obtain the following integral equation in the case $\delta(a)$ is not identically zero, $q \neq 0$, and Re $\xi \geq 0$:

$$\begin{aligned} G_q(a) &= \int_0^a K(a,s)G_q(s)ds + d(a) \\ &+ q\frac{H(a)}{\Delta}\{\int_0^\infty\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}\beta(a)\pi(a)k_1(\sigma)\left[F(\sigma)-\varepsilon v^*(\sigma)\right]d\sigma da \\ &- \lambda^*(1-\varepsilon)\int_0^\infty\int_0^a\int_0^\sigma e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}\beta(a)\pi(a)k_1(\sigma)e^{-\int_s^\sigma[\xi+\nu(\tau)+\delta(\tau)+\lambda^* k_1(\tau)]d\tau} \times \\ &\left[\delta(s)G_q(s)+k_1(s)u^*(s)\right]dsd\sigma da\}, \end{aligned} \tag{70}$$

where $K(a,s)$ and $d(a)$ are as before, are given by equations (5.18)-(5.19), and $H(a)$ and Δ are defined as follows

$$\begin{aligned} H(a) &= e^{-\int_0^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} \\ &- \lambda^*(1-\varepsilon)\int_0^a e^{-\int_\sigma^a[\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau}k_1(\sigma)e^{-\int_0^\sigma[\xi+\nu(\tau)+\delta(\tau)+\lambda^* k_1(\tau)]d\tau}d\sigma, \end{aligned} \tag{71}$$

$$\Delta = 1 - q\int_0^\infty e^{-\int_0^a [\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} \beta(a)\pi(a)da$$
$$+ q\lambda^*(1-\varepsilon)\int_0^\infty\int_0^a e^{-\int_\sigma^a [\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} \beta(a)\pi(a)k_1(\sigma)e^{-\int_0^\sigma [\xi+\nu(\tau)+\delta(\tau)+\lambda^* k_1(\tau)]d\tau} d\sigma da. \tag{72}$$

Also, in this case, the characteristic equation is given by

$$1 = c\int_0^\infty G_q(a)k_2(a)\pi(a)da. \tag{73}$$

We note that if we put $q = 0$ in equation (5.24), then we obtain the Volterra integral equation (5.17). Also, we note that in the case $q \neq 0$, $\delta(a)$ not identically zero, and Re $\xi \geq 0$, then if we assume that $\psi^* = 0$, then from equations (5.9)-(5.10), we obtain the following two formulas for g(0):

$$g(0) = -\frac{\lambda^*(1-\varepsilon)\int_0^\infty\int_0^a k_2(a)\pi(a)k_1(\sigma)f(\sigma)e^{-\int_\sigma^a [\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} d\sigma da}{\int_0^\infty k_2(a)\pi_2(a)e^{-\int_0^a [\xi+\varepsilon\lambda^* k_1(\tau)]d\tau} da}, \tag{74}$$

$$g(0) = \frac{q\lambda^*(1-\varepsilon)\int_0^\infty\int_0^a \beta(a)\pi(a)k_1(\sigma)f(\sigma)e^{-\int_\sigma^a [\xi+\gamma(\tau)+\varepsilon\lambda^* k_1(\tau)]d\tau} d\sigma da}{\left[1 - q\int_0^\infty \beta(a)\pi_2(a)e^{-\int_0^a [\xi+\varepsilon\lambda^* k_1(\tau)]d\tau} da\right]}. \tag{75}$$

And therefore, if we assume that $\beta(a)$ and $k_2(a)$ are constants in equations (5.28)-(5.29), we find that g(0) = 0, and accordingly, f(0) = 0, and hence, f(a) = g(a) = 0, by the uniqueness of solution for the system of ODEs (5.9)-(5.10).

In the next result, we show that an endemic equilibrium of problem (2.1), is locally asymptotically stable, if the force of infection is sufficiently small.

Theorem 12 *Suppose that* $R_\nu > 1$, *and* $k_2(a)$ *and* $\beta(a)$ *are constants independent of age. If* λ^* *is sufficiently small, then an endemic equilibrium is locally asymptotically stable.*

Proof: The proof of this theorem is similar to the proof of Theorem (5.6), and therefore, we omit the proof.

Conclusion

We studied an age-structured SIRS epidemic model, when the disease is vertically as well as horizontally transmitted and the force of infection of proportionate mixing assumption type, susceptible individuals are vaccinated with vaccine that wanes over time, and therefore immune individuals are susceptible with some resistance to the disease. The mortality and fertility rates are age-dependent.

We proved the existence and uniqueness of solution to the model equations and showed that solutions of the model equations depend continuously on the initial age-distributions, and therefore,

the well posedness of the problem is proved.

Furthermore, we determined the steady states of the model and examined their stability, when $q \neq 1$, by determining an explicitly computable threshold parameter R_ν, in terms of the demographic and epidemiological parameters of the model, known as the reproduction number in the presence of vaccination strategy $\nu(a)$, as in Hadeler, et al. (1996) or the net replacement ratio, as in Thieme (2001). And hence, we showed that R_ν increases with q, which is the probability of vertically transmitting the disease, and therefore, increases the likelihood that an endemic will occur; also R_ν decreases with $\nu(a)$, and is used to determine a critical vaccination coverage which will eradicate the disease with minimum vaccination coverage.

If $R_\nu \leq 1$, then the only steady state of problem (2.1) is the disease-free equilibrium, and is locally asymptotically stable if $R_\nu < 1$, and globally stable if $R_0 < 1$. If $R_\nu > 1$, then a disease-free equilibrium and an endemic equilibrium are possible steady states, the disease-free equilibrium is unstable. The question of uniqueness of an endemic equilibrium is an open problem, and this is also the case for several age-structured epidemic models, for example, see Castillo-Chavez, et al. (1998), Cha, et al. (1998), and El-Doma (2006).

Concerning the stability of an endemic equilibrium, we obtained a complicated characteristic equation that allowed to prove local asymptotic stability, in some special cases, which are: (i) the case $\varepsilon = 1$, (ii) the case $q = 0$, and $\delta \equiv 0$, (iii) the case $\delta \equiv 0$, and $(1-\varepsilon)\lambda^* \int_0^\infty k_1(\sigma) e^{-\int_0^\sigma \left[\nu(\tau) + (1-\varepsilon)\lambda^* k_1(\tau) - \gamma(\tau)\right] d\tau} d\sigma < 1$. And we derived general formulas for the characteristic equations, in terms of integral equations. Although explicit formulas for the solutions of these integral equations are not obtained, we are able to use them to deduce that if the force of infection is sufficiently small, then an endemic equilibrium is always locally asymptotically stable.

If $q = 1$, and other conditions hold, for example, see Theorem (4.1), then problem (2.1) gives rise to a continuum of endemic steady states, and this situation does not occur when there is no vertical transmission. Also, if $q = 1$, and $\gamma(a)$ is identically zero, then problem (2.1) has the total population consisting of infective individuals only as the steady state.

Acknowledgments

The author wrote this paper while he was visiting the Institute of Mathematics of the University of Potsdam, Potsdam, Germany, and he would like to thank the German Academic Exchange Service, DAAD: Deutscher Akademischer Austausch Dienst e.V., for support and Prof. Dr. N. Tarkhanov and Prof. Dr. B.-W. Schulze for an invitation and hosbitality during his stay in the Institute of Mathematics.

This work is completed while the author is an Arab Regional Fellow at the Center for Advanced Mathematical Sciences (CAMS), American University of Beirut, Beirut, Lebanon, he is supported by a grant from the Arab Fund for Economic and Social Development, and he would like to thank the Director of CAMS, Prof. Dr. Wafic Sabra, for an invitation and hospitality during his stay in CAMS.

He would also like to thank Professor Mimmo Iannelli and Dr. Xue-Zhi Li for sending references, and two anonymous referees for helpful comments and valuable suggestions on the manuscript.

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