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Remarks on the Stability of some Size-Structured Population Models III: The Case of Constant Inflow of Newborns

Mohammed El-doma

Faculty of Mathematical Sciences
University of Khartoum
P. O. Box 321, Khartoum, Sudan
biomath2004@yahoo.com

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Abstract

The stability of some size-structured population dynamics models are investigated. We determine the steady states and study their stability. We also give examples that illustrate the stability results. The results in this paper generalize previous results, for example, see Calsina, et al. (2003), El-Doma (2006) and El-Doma (2008).

Keywords: Population; Stability; Steady state; Size-structure.

MSC 2000: 45K05; 45M10; 35B35; 35L60; 92D25.

1. Introduction

In this paper, we study the following size-structured population dynamics model:

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial t} + \dfrac{\partial}{\partial a}(V(a,P(t))p(a,t)) + \mu(a,P(t))p(a,t) = 0, \;\; a \in [0,l), l \leq +\infty, \;\; t > 0, \\ V(0,P(t))p(0,t) = C + \displaystyle\int_0^l \beta(a,P(t))p(a,t)da, \;\; t \geq 0, \\ p(a,0) = p_0(a), \;\; a \in [0,l), \\ P(t) = \displaystyle\int_0^l p(a,t)da, \;\; t \geq 0, \end{cases} \tag{1}$$

where $p(a,t)$ is the density of the population with respect to size $a \in [0,l)$ at time $t \geq 0$, where, $l \leq +\infty$, is the maximum size an individual in the population can attain; $P(t) = \int_0^l p(a,t)da$ is the total population size at time t; $\beta(a,P(t)), \mu(a,P(t))$ are, respectively, the birth rate i.e. the average number of offspring, per unit time, produced by an individual of size a when the population size is $P(t)$, and the mortality rate i.e. the death rate at size a, per unit population, when the population size is $P(t)$; $0 < V(a,P)$ is the individual growth rate at the population size P; $p(0,t) = \int_0^l \beta(a,P(t))p(a,t)da$ is the number of births, per unit time, when the population size is $P(t)$; and, $C > 0$, is a constant that represents the inflow of newborns from an external source, for example, seeds, when carried by winds in plants or, eggs of fish, when carried by water.

We study problem (1) under the following general assumptions:

$$0 \leq p_0(a) \in L^1([0,l)) \cap L_\infty[0,l), \mathbb{R}^+ = [0,\infty);$$

$$V(a,P(t)), \beta(a,P(t)), \mu(a,P(t)) \geq 0, \quad \& \in C([0,l) \times \mathbb{R}^+);$$

$$V_P(a,P), V_{Pa}(a,P), \beta_P(a,P), \mu_P(a,P) \text{ exist for } \forall a \geq 0, P \geq 0;$$

$$V_P(.,P), V_{Pa}(.,P), \beta(.,P), \beta_P(.,P), \mu(.,P), \mu_P(.,P) \in C([0,l) : L_\infty(\mathbb{R}^+)).$$

Models of size-structured populations were first derived in Sinko, et al. (1967) where the population density and the vital rates depend on age, size and time. Due to its complication, this type of model has been ignored by mathematicians, for example, see Metz, et al. (1986). Problem (1) is a special case of the classical model given in Sinko, et al. (1967). However problem (1) generalizes those given in Calsina, et al. (2003) and El-Doma (2006) where the vital rates are taken to depend on the population size only; and also El-Doma (2008), and Hagen, et al (2007) where, C, is assumed to be zero; and Hagen, et al. (2008) where the population is subdivided into adults and juveniles but only steady states are determined.

Mimura, et al. (1988) studied a model that is similar to problem (1) with, $C = 0$, and the dependence on the population size $P(t)$ is changed to a dependence on a weighted population size $r(t)$ i.e., $r(t) = \int_0^l \omega(a)p(a,t)da, \omega \geq 0$, and the growth rate V is of separable form that is a special case of problem (1); and they proved the global existence and uniqueness of non-negative solutions, and obtained some stability results when the death rate μ depends on the weighted population size $r(t)$ only. Calsina, et al. (1995) studied problem (1) with, C, depends on time t, and proved the existence and uniqueness of solution; and the existence of a global attractor when the inflow, C, is a constant.

Further generalization of size-structured population dynamics models involved the additional assumption of subdividing the population into subgroups based on growth rates, these growth rates can be finite in number leading to a finite number of subgroups, for example, see Ackleh, et al (2005) or infinitely many different growth rates, for example, see Huyer (1994). These studies proved existence and uniqueness results; and provided numerical results as in Huyer (1994), and numerical and statistical results as in Ackleh, et al. (2005).

In this paper, we study problem (1), which is a generalization of our previous study in El-Doma (2008) for the case, $C = 0$, and determine its steady states and examine their stability. We prove that the trivial steady state is not a steady state and that there are as many nontrivial steady states, P_∞, as the positive solutions of the equation, $R(P_\infty) = 1$, see Section 2 for the definition of $R(P)$. Then we determine sufficient conditions for the local asymptotic stability of a nontrivial steady state, P_∞, and show that if, $R'(P_\infty) > 0$, then a nontrivial steady state is unstable.

We note that we can retain all the stability results that we have proved in El-Doma (2008), for the special case, $C = 0$, and show that if a nontrivial steady state of problem (1) with, $C = 0$, is locally asymptotically stable, then it is locally asymptotically stable with, $C > 0$.

The organization of this paper as follows: in Section 2 we determine the steady states; in Section 3 we study the stability of the steady states and give several examples that illustrate our theorems; in Section 4 we conclude our results.

2. The Steady States

In this section, we determine the steady states of problem (1). A steady state of problem (1) satisfies the following:

$$\begin{cases} \dfrac{d}{da}[V(a,P_\infty)p_\infty(a)] + \mu(a,P_\infty)p_\infty(a) = 0, \quad a \in [0,l), \\ V(0,P_\infty)p_\infty(0) = C + \displaystyle\int_0^l \beta(a,P_\infty)p_\infty(a)da, \\ P_\infty = \displaystyle\int_0^l p_\infty(a)da. \end{cases} \tag{2}$$

From (2), by solving the differential equation, we obtain that

$$p_\infty(a) = p_\infty(0)V(0,P_\infty)\frac{\pi(a,P_\infty)}{V(a,P_\infty)}, \tag{3}$$

where $\pi(a,P_\infty)$ is defined as

$$\pi(a,P) = e^{-\int_0^a \frac{\mu(\tau,P)}{V(\tau,P_\infty)}d\tau}.$$

Also, from (2) and equation (3), we obtain the following:

$$p_\infty(0)V(0,P_\infty) = C + p_\infty(0)V(0,P_\infty)\int_0^l \frac{\beta(a,P_\infty)}{V(a,P_\infty)}\pi(a,P_\infty)da. \tag{4}$$

It is easy to see that, $p_\infty(0) = 0$, does not satisfy equation (4) since, $C > 0$, and accordingly, by equation (3), $p_\infty(a) = 0$, is not a steady state.

From equation (3), we obtain that $p_\infty(0)$ satisfies $p_\infty(0) = \dfrac{P_\infty}{V(0,P_\infty)\displaystyle\int_0^l \frac{\pi(a,P_\infty)}{V(a,P_\infty)}da}$, accordingly, from equation (4), we obtain that, P_∞, satisfies

$$1 = \int_0^l \frac{\beta(a,P_\infty)}{V(a,P_\infty)}\pi(a,P_\infty)da + \frac{C}{P_\infty}\int_0^l \frac{\pi(a,P_\infty)}{V(a,P_\infty)}da. \tag{5}$$

In order to facilitate our writing, we define a threshold parameter $R(P)$ by

$$R(P) = \int_0^l \frac{\beta(a,P)}{V(a,P)}\pi(a,P)da + \frac{C}{P}\int_0^l \frac{\pi(a,P)}{V(a,P)}da, \tag{6}$$

which when $V \equiv 1, C \equiv 0$, and a is age (the age-structured case) is interpreted as the number of children expected to be born to an individual, in a life time, when the population size is P.

We note that a steady state for problem (1) is completely determined by a solution $P_\infty > 0$ of equation (5).

In the following theorem, we describe the steady states of problem (1), the proof of the theorem is straightforward and therefore, is omitted.

Theorem 1.

(1) *Problem (1) has no trivial steady state,* $P_\infty = 0$, *as a steady state.*
(2) *All positive solutions of,* $R(P_\infty) = 1$, *are nontrivial steady states of problem (1).*

3. Stability of the Steady States

In this section, we study the stability of the steady states for problem (1) as given by Theorem 1.

To study the stability of a steady state $p_\infty(a)$, which is a solution of (2) and is given by equation (3), we linearize problem (1) at $p_\infty(a)$ in order to obtain a characteristic equation, which in turn will determine conditions for the stability. To that end, we consider a perturbation $\omega(a,t)$ defined by $\omega(a,t) = p(a,t) - p_\infty(a)$, where $p(a,t)$ is a solution of problem (1). Accordingly, we obtain that $\omega(a,t)$ satisfies the following:

$$\begin{cases} \dfrac{\partial \omega(a,t)}{\partial t} + \dfrac{\partial}{\partial a}\Big(V(a,P_\infty)\omega(a,t)\Big) + \Big[\dfrac{\partial}{\partial a}\Big(V_P(a,P_\infty)p_\infty(a)\Big) + p_\infty(a)\mu_P(a,P_\infty)\Big]W(t) \\ +\mu(a,P_\infty)\omega(a,t) = 0,\ a \in [0,l),\ t > 0, \\ \omega(0,t)V(0,P_\infty) = \displaystyle\int_0^l \beta(a,P_\infty)\omega(a,t)da + W(t)\int_0^l \beta_P(a,P_\infty)p_\infty(a)da \\ -p_\infty(0)V_P(0,P_\infty)W(t),\ t \geq 0, \\ \omega(a,0) = p_0(a) - p_\infty(a), \quad a \in [0,l), \\ W(t) = \displaystyle\int_0^l \omega(a,t)da, \quad t \geq 0. \end{cases} \tag{7}$$

By substituting $\omega(a,t) = f(a)e^{\xi t}$ in (7), where ξ is a complex number, and straightforward calculations, we obtain the following characteristic equation:

$$1 = \frac{1}{V(0,P_\infty)} \int_0^l e^{-\int_0^a E(\tau)d\tau} \beta(a,P_\infty)da$$

$$+\frac{\displaystyle\int_0^l e^{-\int_0^a E(\tau)d\tau}da}{V(0,P_\infty)\Big[1+\displaystyle\int_0^l\int_0^a e^{-\int_\sigma^a E(\tau)d\tau}g(\sigma)d\sigma da\Big]} \tag{8}$$

$$\times\Big[\int_0^l \beta_P(a,P_\infty)p_\infty(a)da - p_\infty(0)V_P(0,P_\infty) - \int_0^l\int_0^a e^{-\int_\sigma^a E(\tau)d\tau}\beta(a,P_\infty)g(\sigma)d\sigma da\Big],$$

where $g(\sigma)$ and $E(\sigma)$ are, respectively, given by

$$g(\sigma) = \frac{\frac{\partial}{\partial\sigma}\Big(V_P(\sigma,P_\infty)p_\infty(\sigma)\Big)+p_\infty(\sigma)\mu_P(\sigma,P_\infty)}{V(\sigma,P_\infty)},$$

$$E(\sigma) = \frac{\xi+V_\sigma(\sigma,P_\infty)+\mu(\sigma,P_\infty)}{V(\sigma,P_\infty)}.$$

In the next theorem, we give a condition for the instability of a nontrivial steady state.

Theorem 2. *A nontrivial steady state is unstable if* $R'(P_\infty) > 0$.

Proof: We note that the characteristic equation (8) can be rewritten as

$$\begin{aligned}
1 &= \frac{1}{V(0,P_\infty)}\int_0^l e^{-\int_0^a E(\tau)d\tau}\beta(a,P_\infty)da\Big(1+\int_0^l\int_0^a e^{-\int_\sigma^a E(\tau)d\tau}g(\sigma)d\sigma da\Big) \\
&- \int_0^l\int_0^a e^{-\int_\sigma^a E(\tau)d\tau}g(\sigma)d\sigma da + \frac{1}{V(0,P_\infty)}\int_0^l e^{-\int_0^a E(\tau)d\tau}da\ \times \qquad (9) \\
&\Big[\int_0^l \beta_P(a,P_\infty)p_\infty(a)da - p_\infty(0)V_P(0,P_\infty) - \int_0^l\int_0^a e^{-\int_\sigma^a E(\tau)d\tau}\beta(a,P_\infty)g(\sigma)d\sigma da\Big] \\
&:= G(\xi).
\end{aligned}$$

Now, suppose that $R'(P_\infty) > 0$, then from the characteristic equation (9), we obtain that $G(0) = 1 + R'(P_\infty)P_\infty > 1$, and $G(\xi) \longrightarrow 0$ as $\xi \longrightarrow +\infty$. Accordingly, $\exists\xi^* > 0$ such that $G(\xi^*) = 1$, and hence a nontrivial steady state is unstable. This completes the proof of the theorem. ■

In the next theorem, we prove that, $\xi = 0$, is a root of the characteristic equation (9) iff, $R'(P_\infty) = 0$.

Theorem 3. $\xi = 0$, *is a root of the characteristic equation (9) iff,* $R'(P_\infty) = 0$.

Proof: We note that if, $\xi = 0$, then using equation (5), the characteristic equation (9) becomes, $R'(P_\infty) = 0$. This completes the proof of the theorem. ■

To obtain further stability results, we note that by suitable changes of the variables of the integrations, we can rewrite the characteristic equation (9) in the following form:

$$\begin{aligned} 1 &= \frac{1}{V(0,P_\infty)}\int_0^l e^{-\int_0^a E(\tau)d\tau}\Big[\beta(a,P_\infty)+\int_0^l \beta_P(b,P_\infty)p_\infty(b)db - p_\infty(0)V_P(0,P_\infty)\Big]da \\ &+ \frac{1}{V(0,P_\infty)}\int_0^l\int_0^l\int_0^a e^{-\int_0^b E(\tau)d\tau}e^{-\int_\sigma^a E(\tau)d\tau}g(\sigma)\Big[\beta(b,P_\infty)-\beta(a,P_\infty)\Big]d\sigma da db \\ &- \int_0^l\int_0^a e^{-\int_\sigma^a E(\tau)d\tau}g(\sigma)d\sigma da. \end{aligned} \tag{10}$$

In the next theorem, we give a sufficient condition for the local asymptotic stability of a nontrivial steady state. We note that this result is for the general problem(1), and in the sequel we give other conditions which are for special cases of problem (1). We also note that the proof of the theorem is exactly as in the special case when, $C = 0$, which is given in El-Doma (2008), and therefore, is omitted.

Theorem 4. *Suppose that the following holds:*

$$\begin{aligned} &\int_0^l \frac{\pi(a,P_\infty)}{V(a,P_\infty)}\Big|\Big[\beta(a,P_\infty)+\int_0^l \beta_P(b,P_\infty)p_\infty(b)db - p_\infty(0)V_P(0,P_\infty)\Big]\Big|da \\ &+ \int_0^l\int_0^l\int_0^a e^{-\int_\sigma^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)}d\tau}\frac{V(\sigma,P_\infty)\pi(b,P_\infty)}{V(b,P_\infty)V(a,P_\infty)}\Big|g(\sigma)\Big[\beta(b,P_\infty)-\beta(a,P_\infty)\Big]\Big|d\sigma da db \\ &+ \int_0^l\int_0^a e^{-\int_\sigma^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)}}\frac{V(\sigma,P_\infty)}{V(a,P_\infty)}\Big|g(\sigma)\Big|d\sigma da < 1. \end{aligned} \tag{11}$$

Then a nontrivial steady state is locally asymptotically stable.

In the next result, we give a corollary to Theorem 4, and the proof of this corollary is straightforward, and therefore, is omitted.

Corollary 5. *Suppose that the following hold:*
1. $\beta(a,P) = \beta(P), \forall a \geq 0,$

2. $$\int_0^l \frac{\pi(a,P_\infty)}{V(a,P_\infty)}\Big|\Big[\beta(P_\infty)+\beta'(P_\infty)P_\infty - p_\infty(0)V_P(0,P_\infty)\Big]\Big|da$$

$$+\int_0^l\int_0^a e^{-\int_\sigma^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)}}\frac{V(\sigma,P_\infty)}{V(a,P_\infty)}\Big|g(\sigma)\Big|d\sigma da < 1.$$

Then a nontrivial steady state is locally asymptotically stable.

We note that we can produce further stability results for two special cases, namely, the case when

$$l = +\infty, V(a,P) = V(a), \mu(a,P) = \mu(P), \int_0^\infty \frac{d\tau}{V(\tau)} = +\infty,$$

and the case when

$$l = +\infty, V(a,P) = V(a), \mu(a,P) = \mu(a), \int_0^\infty \frac{\mu(\tau)}{V(\tau)} d\tau = +\infty.$$

The case: $\underline{l = +\infty, V(a,P) = V(a), \mu(a,P) = \mu(P), \int_0^\infty \frac{d\tau}{V(\tau)} = +\infty.}$

We note that the result for this case is a generalization of that given in Gurney, et al. (1980) and Weinstock, et al. (1987) for the classical age-structured population dynamics model of Gurtin, et al. (1974), which corresponds to problem (1) when, $V \equiv 1$, and, $C = 0$. In El-Doma (2008), we obtained results for the size-structured case, when there is no inflow of newborns from an external source i.e., when, $C = 0$.

We also note that if $\mu(P_\infty) = 0$, then from equation (3), we obtain $P_\infty = +\infty$. Therefore, we assume that $\mu(P_\infty) > 0$.

By straightforward calculations in the characteristic equation (9), we obtain

$$\begin{aligned} 0 &= \frac{1}{\xi}\Big[\xi + \frac{p_\infty(0)\mu'(P_\infty)V(0)}{\mu(P_\infty)}\Big]\Big[\frac{1}{V(0)}\int_0^\infty \beta(a,P_\infty)e^{-\int_0^a E(\tau)d\tau}da - 1\Big] \\ &\quad + \frac{1}{[\xi + \mu(P_\infty)]}\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da + \frac{C\mu'(P_\infty)}{\xi[\xi + \mu(P_\infty)]}, \xi \neq 0. \end{aligned} \tag{12}$$

Now, let $\xi = x + iy$, then the real part of equation (12) gives

$$\begin{aligned} 1 &= \frac{A}{D}\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da + C\mu'(P_\infty)\frac{B}{D} - \frac{C}{p_\infty(0)V(0)} \\ &\quad + \int_0^\infty \frac{\beta(a,P_\infty)}{V(a)} e^{-\mu(P_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}\Big) e^{-x\int_0^a \frac{d\tau}{V(\tau)}} da + \frac{C}{p_\infty(0)V(0)}, \end{aligned} \tag{13}$$

where A, B, D, satisfy the following:

$$A = x[x + \mu(P_\infty)]\Big[x + \frac{p_\infty(0)V(0)\mu'(P_\infty)}{\mu(P_\infty)}\Big] + y^2\Big[x + \mu(P_\infty) + \frac{p_\infty(0)V(0)\mu'(P_\infty)}{\mu(P_\infty)}\Big], \tag{14}$$

$$B = (x + \mu(P_\infty))\Big(x + \frac{p_\infty(0)V(0)\mu'(P_\infty)}{\mu(P_\infty)}\Big) - y^2, \tag{15}$$

$$D = \Big[(x + \mu(P_\infty))^2 + y^2\Big]\Big[\Big(x + \frac{p_\infty(0)V(0)\mu'(P_\infty)}{\mu(P_\infty)}\Big)^2 + y^2\Big]. \tag{16}$$

In the next lemma, we give stability result for the special case

$$\mu'(P_\infty) = \int_0^\infty \beta_P(a, P_\infty) p_\infty(a) da = 0.$$

Lemma 6 *Suppose that* $\mu'(P_\infty) = \int_0^\infty \beta_P(a, P_\infty) p_\infty(a) da = 0$. *Then a nontrivial steady state is locally asymptotically stable.*

Proof: From equations (13) and (5), and assuming that $x \geq 0$, we obtain

$$\begin{aligned} 1 &= -\frac{C}{p_\infty(0)V(0)} + \int_0^\infty \frac{\beta(a, P_\infty)}{V(a)} e^{-\mu(P_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos\Big(y \int_0^a \frac{d\tau}{V(\tau)}\Big) e^{-x\int_0^a \frac{d\tau}{V(\tau)}} da \\ &\quad + \frac{C}{p_\infty(0)V(0)} \\ &\leq 1 - \frac{C}{p_\infty(0)V(0)} \\ &< 1. \end{aligned}$$

Accordingly, the characteristic equation (12) can not be satisfied for any ξ with, $Re\xi \geq 0$, and hence a nontrivial steady state is locally asymptotically stable. This completes the proof of the lemma. ■

In the next lemma, we give stability result for the special case

$$\mu'(P_\infty) > 0 = \int_0^\infty \beta_P(a, P_\infty) p_\infty(a) da.$$

Lemma 7. *Suppose that* $\mu'(P_\infty) > 0 = \int_0^\infty \beta_P(a, P_\infty) p_\infty(a) da$. *Then a nontrivial steady state is locally asymptotically stable.*

Proof: We assume that $x \geq 0$. From equations (13), we obtain

$$\begin{aligned} 1 &= C\mu'(P_\infty)\frac{B}{D} - \frac{C}{p_\infty(0)V(0)} \qquad (17) \\ &+ \int_0^\infty \frac{\beta(a, P_\infty)}{V(a)} e^{-\mu(P_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos\Big(y \int_0^a \frac{d\tau}{V(\tau)}\Big) e^{-x\int_0^a \frac{d\tau}{V(\tau)}} da + \frac{C}{p_\infty(0)V(0)}. \end{aligned}$$

Now, we note that by Theorem 3, $\xi = 0$, is a root of the characteristic equation (9) iff $R'(P_\infty) = 0$. And since, in this case, $R'(P_\infty) < 0$, we conclude that $(x, y) = (0, 0)$ is not a root of the characteristic equation. And also, in this case, if $y^2 \geq (x+\mu(P_\infty))\Big(x+\dfrac{p_\infty(0)V(0)\mu'(P_\infty)}{\mu(P_\infty)}\Big)$, then by similar arguments as in Lemma 6, we obtain the result since $\dfrac{B}{D} \leq 0$ and $\mu'(P_\infty) > 0$. Accordingly, we assume that $y^2 < (x+\mu(P_\infty))\Big(x+\dfrac{p_\infty(0)V(0)\mu'(P_\infty)}{\mu(P_\infty)}\Big)$, and, in this case, we can estimate $\dfrac{B}{D}$ as follows

$$\frac{B}{D} \leq \frac{1}{(x+\mu(P_\infty))\Big(x+\frac{p_\infty(0)V(0)\mu'(P_\infty)}{\mu(P_\infty)}\Big)}. \tag{18}$$

From (18), we conclude that $\dfrac{B}{D} < \dfrac{1}{p_\infty(0)V(0)\mu'(P_\infty)}$ except when $(x,y)=(0,0)$. And therefore, the result follows from equation (17) via the same arguments as in Lemma 6. This completes the proof of the lemma. ■

In the next lemma, we obtain stability results for the special case

$$\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da > 0 = \mu'(P_\infty).$$

Lemma 8. *Suppose that* $\displaystyle\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da > 0 = \mu'(P_\infty)$. *Then a nontrivial steady state is locally asymptotically stable if,* $R'(P_\infty) < 0$, *or equivalently,* $\displaystyle\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da < \frac{C\mu(P_\infty)}{p_\infty(0)V(0)}$.

Proof: We note that, in this case, by assuming that $x \geq 0$, and using equations (13)-(14), and (16), we obtain

$$\begin{aligned} 1 &= \frac{(x+\mu(P_\infty))}{[(x+\mu(P_\infty))^2+y^2]}\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da - \frac{C}{p_\infty(0)V(0)} \\ &\quad + \int_0^\infty \frac{\beta(a,P_\infty)}{V(a)}e^{-\mu(P_\infty)\int_0^a \frac{d\tau}{V(\tau)}}\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}\Big)e^{-x\int_0^a \frac{d\tau}{V(\tau)}}da + \frac{C}{p_\infty(0)V(0)} \\ &\leq \frac{1}{\mu(P_\infty)}\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da - \frac{C}{p_\infty(0)V(0)} \\ &\quad + \int_0^\infty \frac{\beta(a,P_\infty)}{V(a)}e^{-\mu(P_\infty)\int_0^a \frac{d\tau}{V(\tau)}}\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}\Big)e^{-x\int_0^a \frac{d\tau}{V(\tau)}}da + \frac{C}{p_\infty(0)V(0)}, \\ &\leq 1 + \frac{1}{\mu(P_\infty)}\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da - \frac{C}{p_\infty(0)V(0)} \end{aligned}$$

$$< \quad 1.$$

Accordingly, the characteristic equation (12) can not be satisfied for any ξ with, $Re\xi \geq 0$, and hence a nontrivial steady state is locally asymptotically stable. This completes the proof of the lemma. ∎

In the next result, we apply our three Lemmas 6-8 to obtain stability result that generalizes our result in El-Doma (2008), for the special case when, $C = 0$.

Theorem 9. *A nontrivial steady state is locally asymptotically stable if,* $\mu'(P_\infty) \geq 0$, *and,* $\int_0^\infty \beta_P(a, P_\infty)p_\infty(a)da \leq 0$.

Proof: We assume that $x \geq 0$. By the three Lemmas 6-8, we only need to prove the theorem with strict inequalities. Now, using equations (13) and (5), we obtain

$$\begin{aligned}
1 &= \frac{A}{D}\int_0^\infty \beta_P(a, P_\infty)p_\infty(a)da + C\mu'(P_\infty)\frac{B}{D} - \frac{C}{p_\infty(0)V(0)} \\
&\quad + \int_0^\infty \frac{\beta(a, P_\infty)}{V(a)} e^{-\mu(P_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}\Big) e^{-x\int_0^a \frac{d\tau}{V(\tau)}} da + \frac{C}{p_\infty(0)V(0)} \\
&\leq 1 + \frac{A}{D}\int_0^\infty \beta_P(a, P_\infty)p_\infty(a)da + C\mu'(P_\infty)\frac{B}{D} - \frac{C}{p_\infty(0)V(0)}. \qquad (19)
\end{aligned}$$

We note that, in this case, $R'(P_\infty) < 0$, and therefore, by Theorem 3, $(x, y) = (0, 0)$ is not a root of the characteristic equation (9). Since $D > 0$, and so, since we assumed that $x \geq 0$, then $A > 0$ except when $(x, y) = (0, 0)$. Also, similar arguments as in Lemma 7 show that either $B \leq 0$ or $\frac{B}{D} < \frac{1}{p_\infty(0)V(0)\mu'(P_\infty)}$ except when $(x, y) = (0, 0)$.

Accordingly, from (19), the characteristic equation (12) can not be satisfied for any ξ with, $Re\xi \geq 0$, and hence a nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem. ∎

The case: $\underline{l = +\infty, V(a, P) = V(a), \mu(a, P) = \mu(a), \int_0^\infty \frac{\mu(\tau)}{V(\tau)}d\tau = +\infty.}$

We note that, in this case, the form of the characteristic equation resembles that of cannibalism, for example, see Iannelli (1995), Bekkal-Brikci, et al. (2007), El-Doma (2008), and El-Doma (2007).

Theorem 10. *Suppose that* $\int_0^\infty \beta_P(a, P_\infty)p_\infty(a)da \leq 0$, *and,* $V(a)\mu'(a) \leq \mu^2(a)$. *Then a nontrivial steady state is locally asymptotically stable.*

Proof: We note that if we set $\xi = x + iy$ in the characteristic equation (9), we obtain

$$\begin{aligned} 1 &= \frac{1}{V(0)} \int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)} d\tau} \beta(a, P_\infty) \cos\Big(y \int_0^a \frac{d\tau}{V(\tau)}\Big) da + \qquad (20) \\ & \frac{1}{V(0)} \Big(\int_0^\infty \beta_P(a, P_\infty) p_\infty(a) da \Big) \int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)} d\tau} \cos\Big(y \int_0^a \frac{d\tau}{V(\tau)} d\tau\Big) da, \end{aligned}$$

$$\begin{aligned} 0 &= \frac{1}{V(0)} \int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)} d\tau} \beta(a, P_\infty) \sin\Big(y \int_0^a \frac{d\tau}{V(\tau)}\Big) da + \qquad (21) \\ & \frac{1}{V(0)} \Big(\int_0^\infty \beta_P(a, P_\infty) p_\infty(a) da \Big) \int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)} d\tau} \sin\Big(y \int_0^a \frac{d\tau}{V(\tau)} d\tau\Big) da. \end{aligned}$$

Now, suppose that $x \geq 0$ and $y = 0$, then from equations (5) and (20), we obtain

$$\begin{aligned} 1 &= \frac{1}{V(0)} \int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)} d\tau} \beta(a, P_\infty) da + \\ & \Big(\int_0^\infty \beta_P(a, P_\infty) p_\infty(a) da \Big) \int_0^\infty \frac{1}{V(a)} e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)} d\tau} da \\ &\leq \frac{1}{V(0)} \int_0^\infty e^{-\int_0^a \frac{[V'(\tau)+\mu(\tau)]}{V(\tau)} d\tau} \beta(a, P_\infty) da + \\ & \Big(\int_0^\infty \beta_P(a, P_\infty) p_\infty(a) da \Big) \int_0^\infty \frac{1}{V(a)} e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)} d\tau} da \\ &= 1 - \frac{C}{p_0(0)V(0)} + \Big(\int_0^\infty \beta_P(a, P_\infty) p_\infty(a) da \Big) \int_0^\infty \frac{1}{V(a)} e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)} d\tau} da \\ &< 1. \end{aligned}$$

We note that the last inequality is obtained by using, $\displaystyle\int_0^\infty \beta_P(a, P_\infty) p_\infty(a) da \leq 0.$

Accordingly, the characteristic equation (9) is not satisfied for any $x \geq 0$ and $y = 0$.

Now, suppose that $x \geq 0$ and $y \neq 0$, and observe that equation (20) can be rewritten in the following form:

$$
\begin{aligned}
1 \quad = \quad & \frac{1}{V(0)}\int_0^\infty e^{-\int_0^a \frac{[x+V'(\tau)+\mu(\tau)]}{V(\tau)}d\tau}\beta(a,P_\infty)\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}\Big)da + \\
& \frac{1}{y^2}\Big(\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da\Big)\times \\
& \int_0^\infty \frac{1}{V(a)}e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)}d\tau}\Big[(x+\mu(a))^2 - V(a)\mu'(a)\Big]\Big(1-\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}d\tau\Big)\Big)da \\
& \leq 1 - \frac{C}{p_0(0)V(0)} + \frac{1}{y^2}\Big(\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da\Big)\times \\
& \int_0^\infty \frac{1}{V(a)}e^{-\int_0^a \frac{[x+\mu(\tau)]}{V(\tau)}d\tau}\Big[(x+\mu(a))^2 - V(a)\mu'(a)\Big]\Big(1-\cos\Big(y\int_0^a \frac{d\tau}{V(\tau)}d\tau\Big)\Big)da \\
& < 1.
\end{aligned}
$$

We note that the last inequality follows because, $\int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da \leq 0$, and, $V(a)\mu'(a) \leq \mu^2(a)$.

Accordingly, the characteristic equation (9) is not satisfied for any ξ with, $Re\xi \geq 0$. Therefore, a nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem. ■

We note that if we only assume that $V(a,P_\infty) = V(a), \mu(a,P_\infty) = \mu(a)$, then from Theorem 4, we obtain the following condition for the local asymptotic stability of a nontrivial steady state when $l \leq +\infty$:

$$\int_0^l \frac{\pi(a,P_\infty)}{V(a,P_\infty)}\Big|\Big[\beta(a,P_\infty) + \int_0^l \beta_P(b,P_\infty)p_\infty(b)db\Big]\Big|da < 1.$$

Example 1: In this example, we consider the case when $\beta(a,P) = \beta(P)e^{-\alpha a}$, where $\alpha > 0$ is a constant, $\mu(a,P) = \mu(P), V(a,P) = V(a), l = +\infty, \int_0^\infty \frac{da}{V(a)} = +\infty$. We note that the form of $\beta(a,P)$ allows the concentration of reproduction in the smallest sizes, for example, see Gurtin, et al. (1974).

Now, using equation (5) and integration by parts, we obtain

$$1 = \frac{\beta(P_\infty)}{\mu(P_\infty)}\Big[1 - \alpha\int_0^\infty e^{-\int_0^a[\alpha + \frac{\mu(P_\infty)}{V(\tau)}]d\tau}da\Big] + \frac{C}{\mu(P_\infty)P_\infty}. \tag{22}$$

If we also assume that $\beta(P), \mu(P)$ are defined as follows

$$\beta(P) \;=\; \frac{c_1}{P^n}, \quad n = 1, 2, ..., \tag{23}$$

$$\mu(P) \;=\; c_2 P^m, \quad m = 0, 1, 2, ..., \tag{24}$$

where $c_1, c_2,$ are positive constants, then from equation (22), we obtain

$$1 = \frac{c_1}{c_2 P_\infty^{n+m}}\left[1 - \alpha \int_0^\infty e^{-\int_0^a [\alpha + \frac{c_2 P_\infty^m}{V(\tau)}]d\tau} da\right] + \frac{C}{c_2 P_\infty^{m+1}}. \tag{25}$$

So, it is easy to see that equation (25) has a unique positive solution, which by Theorem 1 corresponds to a unique nontrivial steady state of problem (1), and is locally asymptotically stable via Theorem 9.

Example 2: In this example, we consider the case given in Calsina, et al. (1995), where $\beta(a, P) = \beta(P)\Big[1 - e^{-\alpha a}\Big]$, $\alpha > 0$ is a constant, which corresponds to the case when reproduction is concentrated at large-sizes, $\mu(a, P) = \mu(P), V(a, P) = V(a), \displaystyle\int_0^\infty \frac{da}{V(a)} = +\infty.$

By using equation (5) and integration by parts, we obtain

$$1 = \frac{\alpha\beta(P_\infty)}{\mu(P_\infty)} \int_0^\infty e^{-\int_0^a [\alpha + \frac{\mu(P_\infty)}{V(\tau)}]d\tau} da + \frac{C}{\mu(P_\infty) P_\infty}. \tag{26}$$

Now, if we assume that $\beta(P), \mu(P)$ satisfy equations (23)-(24), respectively, we obtain

$$1 = \frac{c_1 \alpha}{c_2 P_\infty^{n+m}} \int_0^\infty e^{-\int_0^a [\alpha + \frac{c_2 P_\infty^m}{V(\tau)}]d\tau} da + \frac{C}{c_2 P_\infty^{m+1}}. \tag{27}$$

It is easy to see that equation (27) has a unique positive solution, which by Theorem 1 corresponds to a unique nontrivial steady state of problem (1), and is locally asymptotically stable via Theorem 9.

Example 3: In this example, we consider the case when $\beta(a, P_\infty) = \dfrac{c_1}{P_\infty^n}, n = 1, 2, ..., \mu(a, P_\infty) = \mu(a), V(a, P_\infty) = V(a).$

By using equation (5) and integration by parts, we obtain

$$1 = \frac{c_1}{P_\infty^n} \int_0^l \frac{e^{-\int_0^a \frac{\mu(\tau)}{V(\tau)} d\tau}}{V(a)} da + \frac{C}{P_\infty} \int_0^l \frac{e^{-\int_0^a \frac{\mu(\tau)}{V(\tau)} d\tau}}{V(a)} da. \tag{28}$$

Now, we assume that $\displaystyle\int_0^l \frac{e^{-\int_0^a \frac{\mu(\tau)}{V(\tau)} d\tau}}{V(a)} da < +\infty$. Accordingly, It is easy to see that equation (28) has a unique positive solution, which by Theorem 1 corresponds to a unique nontrivial steady state of problem (1), and is locally asymptotically stable via Corollary 5, provided that

$\dfrac{c_1(n-1)}{P_\infty^n}\displaystyle\int_0^l \frac{e^{-\int_0^a \frac{\mu(\tau)}{V(\tau)}d\tau}}{V(a)}da < 1$, which is obviously satisfied when, $n = 1$; and via Theorem 10, provided that $l = +\infty$, $\displaystyle\int_0^l \frac{\mu(a)}{V(a)}da = +\infty$ and $\mu'(a)V(a) \le \mu^2(a)$.

Example 4: In this example, we consider the case when $\beta(a, P_\infty) = c_1(1+a)e^{-P_\infty}$, $\mu(a, P_\infty) = (1+a)(2+a+a^2), V(a, P_\infty) = (1+a)$.

By using equation (5), we obtain

$$1 = c_1 e^{-P_\infty}\int_0^\infty e^{-(2a+\frac{a^2}{2}+\frac{a^3}{3})}da + \frac{C}{P_\infty}\int_0^\infty \frac{e^{-(2a+\frac{a^2}{2}+\frac{a^3}{3})}}{(1+a)}da. \tag{29}$$

It is easy to see that equation (29) has a unique positive solution, which by Theorem 1 corresponds to a unique nontrivial steady state of problem (1), and is locally asymptotically stable via Theorem 10.

Example 5: In this example, we look at the case when $\beta(a, P), \mu(a, P), V(a, P)$ are given as follows, for example, see Weinstock, et al. (1987) for the special case, $V \equiv 1, C = 0$,

$$\begin{aligned}
\beta(a, P) &= c_1 a^n e^{-c_2 P^k a}, \quad n = 1, 2, ..., \quad k = 1, 2, ..., \\
\mu(a, P) &= c_3 P^m, \quad m = 0, 1, 2, ..., \\
V(a, P) &= V(a),
\end{aligned}$$

where c_1, c_2, c_3 are positive constants, and we assume that $\displaystyle\int_0^\infty \frac{d\tau}{V(\tau)} = +\infty$.

By using equation (5), we obtain

$$\begin{aligned}
1 &= c_1\int_0^\infty a^n e^{-c_2 P_\infty^k a}\frac{e^{-c_3 P_\infty^m \int_0^a \frac{d\tau}{V(\tau)}}}{V(a)}da + \frac{C}{P_\infty}\int_0^\infty \frac{e^{-c_3 P_\infty^m \int_0^a \frac{d\tau}{V(\tau)}}}{V(a)}da \\
&= c_1\int_0^\infty a^n e^{-c_2 P_\infty^k a}\frac{e^{-c_3 P_\infty^m \int_0^a \frac{d\tau}{V(\tau)}}}{V(a)}da + \frac{C}{c_3 P_\infty^{m+1}}.
\end{aligned} \tag{30}$$

From equation (30), it is easy to see that the right-hand side of equation (30) approaches $+\infty$ as $P_\infty \longrightarrow 0$; and it approaches zero as $P_\infty \longrightarrow +\infty$ since by integration by parts we only need to establish the following fact: $c_1\displaystyle\int_0^\infty a^n e^{-c_2 P_\infty^k a} e^{-c_3 P_\infty^m \int_0^a \frac{d\tau}{V(a)}}da \le \frac{c_1 n!}{(c_2 P_\infty^k)^{n+1}} \longrightarrow 0$ as $P_\infty \longrightarrow +\infty$.

Accordingly, it is easy to see that equation (30) has a unique positive solution, which by Theorem 1 corresponds to a unique nontrivial steady state of problem (1), and is locally asymptotically stable via Theorem 9.

4. Conclusion

In this paper, we studied a size-structured population dynamics model where the maximum size is either finite or infinite, and we assumed that there is an inflow of newborns from an external source, for example, seeds, when carried by winds and, eggs from fish, when carried by water. We determined the steady states of the model and examined their stability. We proved that the trivial steady state is not a steady state and that there are as many nontrivial steady states, P_∞, as the positive solutions of the equation, $R(P_\infty) = 1$, where $R(P)$ is given by equation (6).

We studied the stability of a nontrivial steady state, and we proved a theorem that provided a sufficient condition for the local asymptotic stability of a nontrivial steady state of the general model, and then we proved a corollary to that theorem for the special case when, $\beta(a, P) = \beta(P)$. We also studied two other special cases, the first was when, $V(a, P) = V(a)$, and, $\mu(a, P) = \mu(P)$, and the second was when, $V(a, P) = V(a)$, and, $\mu(a, P) = \mu(a)$. We note that the first special case, when, $C = 0$, linked our study of the stability of our size-structured population dynamics model to the study of the classical Gurtin-MacCamy's age-structured population dynamics model given in Gurtin, et al. (1974), specifically, the studies for the stability given in Gurney, et al. (1980), and Weinstock, et al. (1987), in fact, when, $C = 0$, the characteristic equation for this special case has the same qualitative properties as the characteristic equation of the Gurtin-MacCamy's age-structured population dynamics model, this result is proved in El-Doma (2008). Also similarly, the second special case linked our study to studies related to cannibalism, for example, see Iannelli (1995), Bekkal-Brikci, et al. (2007) and El-Doma (2007). We also showed that if, $R'(P_\infty) > 0$, then a nontrivial steady state is unstable. And we illustrated our stability results by several examples.

We note that our model in this paper generalized that given in El-Doma (2008), where, $C = 0$. We retained all the stability results given therein, and showed that if a nontrivial steady state of problem (1) with, $C = 0$, is locally asymptotically stable, then a nontrivial steady state of problem (1) with, $C > 0$, is locally asymptotically stable.

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References

Ackleh, A. S. and Banks, H. T. and Deng, K. and Hu, S. (2005). Parameter estimation in a coupled system of nonlinear size-structured populations. *Mathematical Biosciences and Engineering*, Vol. 2, pp. 289–315.

Bekkal-Brikci, F. and Boushaba, K. and Arino, O. (2007). Nonlinear age structured model with cannibalism. *Discrete and Continuous Dynamical Systems-Series B*, Vol. 7, pp. 201–218.

Calsina, A. and Sanchón, M. (2003). Stability and instability of equilibria of an equation of size structured population dynamics. *J. Math. Anal. Appl.*, Vol. 286, pp. 435–452.

Calsina, A. and Saldaña, J. (1995). A model of physiologically structured population dynamics with a nonlinear individual growth rate. *J. Math. Biol.*, Vol. 33, pp. 335–364.

El-Doma, M. (2006). Remarks on the stability of some size-structured population models I: changes in vital rates due to population only. *AAM: Intern. J.*, Vol. 1, No. 1, pp. 11–24.

El-Doma, M. (2007). Age-structured population model with cannibalism. *AAM: Intern. J.*, Vol. 2, No. 2, pp. 92–106.

El-Doma, M. (2008). Remarks on the stability of some size-structured population models II: changes in vital rates due to Size and population size. *AAM: Intern. J.*, Vol. 3, No. 1, pp. 113–127.

Farkas, J. Z. and Hagen, T. (2008). Asymptotic behavior of size structured populations via juvenile-adults interaction. *Discrete Contin. Dyn. Syst. Ser. B*, Vol. 9, pp. 249–266.

Farkas, J. Z. and Hagen, T. (2007). Stability and regularity results for a size-structured population model. *J. Math. Anal. Appl.*, Vol. 86, pp. 1087-1103.

Gurney, W. S. C. and Nisbet, R. M. (1980). Age-and density-dependent population dynamics in static and variable environments. *Theoret. Population Biol.*, Vol. 17, pp. 321–344.

Gurtin, M. E. and MacCamy, R. C. (1974). Non-linear age-dependent Population Dynamics. *Arch. Rational Mech. Anal.*, Vol. 54, pp. 281–300.

Huyer, W. (1994). A size-structured population model with dispersion. *J. Math. Anal. Appl.*, Vol. 181, pp. 716–754.

Iannelli, M. (1995). Mathematical theory of age-structured population dynamics. *Applied mathematics monographs*, Vol. 7, CNR, Giardini-Pisa.

Metz, J.A.J. and Diekmann, O. (1986). The dynamics of physiologically structured populations. *Lecture notes in biomathematics*, 68, (Eds.), Springer-Verlag, (1986).

Mimura, M. and Takigawa, S. (1988). A size-distribution model with density-dependent growth rates. *Japan J. Appl. Math.*, Vol. 5, pp. 33–51.

Sinko, J. W. and Streifer, W. (1967). A new model for age-size structure of a population. *Ecology*, Vol. 48, pp. 910–918.

Weinstock, E. and Rorres, C. (1987). Local stability of an Age-structured Population with density-dependent fertility and mortality. *SIAM J. Appl. Math.*, Vol. 47, pp. 589–604.