



A Reliable Approach for Higher-order Integro-differential Equations

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Abstract

In this paper, we apply the variational iteration method (VIM) for solving higher-order integro-differential equations by converting the problems into system of integral equations. The proposed technique is applied to the re-formulated system of integro-differential equations. Numerical results show the accuracy and efficiency of the suggested algorithm. The fact that the VIM solves nonlinear problems without calculating Adomian's polynomials is a clear advantage of this technique over the decomposition method.

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1. Introduction

In this paper, we consider the general higher-order integro-differential equation of the type

$$y^{(m)}(x) = f(x) + \int_0^x K(x,t)F(y(t))dt,$$

with boundary conditions

$$y^{(j)}(0) = A_j, \quad j = 0, 1, 2, \dots, (r-1),$$
$$y^{(j)}(b) = C_j, \quad j = r, (r+1), (r+2), \dots, (m-1),$$

where $y^m(x)$ indicates the m th derivative of $y(x)$ and $F(y(x))$ is a nonlinear function. In addition the kernel $k(x, t)$, $f(x)$ are assumed real, differential for $x \in [0, b]$ and A_j , $0 \leq j \leq (r-1)$, C_j , $r \leq j \leq (m-1)$ are real finite constants. The higher-order integro-differential equations occur in multiple diversified physical phenomena and various techniques including decomposition have been applied for solving such problems, (see Agarwal (1986), Morchalo (1975a), Agarwal (1983), Morchalo (1975b), Wazwaz (2000)). He developed the variational iteration method (VIM) for solving linear and nonlinear versatile physical problems, (see He (2006), (2007)).

It is worth mentioning that the VIM was originated by Inokuti et al. (see Inokuti, Sekine and Mura (1978)). The method has been successfully applied on a wide class of initial and boundary value problems by various authors, (see Mohyud-Din, Noor and Noor (2008a), Noor and Mohyud-Din (2007a), Noor and Mohyud-Din (2007b), Noor and Mohyud-Din (2008b), Noor and Mohyud-Din (2008c), Noor and Mohyud-Din (2008d), Noor and Mohyud-Din (2008e)). Inspired and motivated by the ongoing research in this area, we implement the variational iteration method (VIM) for solving higher-order integro-differential equations by converting the problems into systems of integral equations.

The proposed algorithm is applied on the re-formulated systems of integral equations. The application of VIM on a re-formulated system of integral equations was developed and implemented first by Noor and Mohyud-Din, (see Noor and Mohyud-Din (2007a), Noor and Mohyud-Din (2007b)). The method has been used without any perturbation, restrictive assumptions or discretization. Moreover, the proposed technique is free from the complexities arising in calculating Adomian's polynomials.

2. Variational iteration technique

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x), \quad (1)$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the forcing term. According to variational iteration method, (see He (2006), (2007), Inokuti, Sekine and Mura (1978), Noor and Mohyud-Din (2007a), Noor and Mohyud-Din (2007b)), we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + N\tilde{u}_n(s) - g(s)) ds, \quad (2)$$

where λ is a Lagrange multiplier, which can be identified optimally via variational iteration method. The subscripts n denote the n th approximation, \tilde{u}_n is considered as a restricted variation. i.e. $\delta \tilde{u}_n = 0$; (2) is called as a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in He (2006), (2007), Inokuti, Sekine and Mura (1978), Noor and Mohyud-Din (2007a), Noor and Mohyud-Din (2007b), Noor and Mohyud-Din (2008f), Noor and

Mohyud-Din (2008g), Noor and Mohyud-Din (2008h)). For the sake of simplicity and to convey the idea of the technique, we consider the following system of differential equations:

$$x'_i(t) = f_i(t, x_1, x_2, \dots, x_n) \tag{3}$$

$$x'_i(t) = f_i(t, x_i), \quad i = 1, 2, 3, \dots, n,$$

subject to the boundary conditions $x_i(0) = c_i, \quad i = 1, 2, 3, \dots, n.$

To solve the system by means of the variational iteration method, we rewrite the system (3) in the following form:

$$x'_i(t) = f_i(x_i) + g_i(t), \quad i = 1, 2, 3, \dots, n. \tag{4}$$

subject to the boundary conditions $x_i(0) = c_i, \quad i = 1, 2, 3, \dots, n$ and g_i is defined in (1). The correction functional for the nonlinear system (4) can be approximated as:

$$\begin{aligned} x_i^{(k+1)}(t) &= x_i^{(k)}(t) + \int_0^t \lambda_1 \left(x_1^{(k)}(T), f_1(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)) - g_1(T) \right) dT, \\ x_2^{(k+1)}(t) &= x_2^{(k)}(t) + \int_0^t \lambda_2 \left(x_2^{(k)}(T), f_2(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)) - g_2(T) \right) dT, \\ &\vdots \\ x_n^{(k+1)}(t) &= x_n^{(k)}(t) + \int_0^t \lambda_n \left(x_n^{(k)}(T), f_n(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)) - g_n(T) \right) dT. \end{aligned} \tag{5}$$

where $\lambda_i = \pm 1, \quad i = 1, 2, 3, \dots, n$ are Lagrange multipliers, $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ denote the restricted variations.

For $\lambda_i = -1, \quad i = 1, 2, 3, \dots, n$ we have the following iterative schemes:

$$\begin{aligned} \lambda'_i(T) \Big|_{T=t} &= 0, \\ 1 + \lambda'_i(T) \Big|_{T=t} &= 0. \end{aligned}$$

For $i = 1, 2, 3, \dots, n$. Therefore the Lagrange multipliers can be easily identified as:

$$\lambda_i = \pm 1, \quad i = 1, 2, 3, \dots, n. \tag{6}$$

Substituting (6) into the correct functional (5), results the following iteration formula:

$$\begin{aligned}
x_1^{(k+1)}(t) &= x_1^{(k)}(t) - \int_0^t \left(x_1^{(k)}(T), f_1(x_1^{(k)}(T), x_2^{(k)}(T), \dots, x_n^{(k)}(T)) - g_1(T) \right) dT, \\
x_2^{(k+1)}(t) &= x_2^{(k)}(t) - \int_0^t \left(x_2^{(k)}(T), f_2(x_1^{(k)}(T), x_2^{(k)}(T), \dots, x_n^{(k)}(T)) - g_2(T) \right) dT, \\
&\vdots \\
x_n^{(k+1)}(t) &= x_n^{(k)}(t) - \int_0^t \left(x_n^{(k)}(T), f_n(x_1^{(k)}(T), x_2^{(k)}(T), \dots, x_n^{(k)}(T)) - g_n(T) \right) dT.
\end{aligned} \tag{7}$$

If we start with the initial approximations $x_i(0) = c_i$, $i = 1, 2, 3, \dots, n$, then the approximations can be completely determined; finally we approximate the solution $x_i(t) = \lim_{k \rightarrow \infty} x_i^{(k)}(t)$ by the n^{th} term $x_i^{(n)}(t)$ for $i = 1, 2, 3, \dots, n$.

3. Numerical Applications

In this section, we first show that the higher order integro differential equations can be re-written in the form of a system of integral equations by using a suitable transformation. The VIM is used for solving the re-formulated system of integral equations. For the sake of comparison, we take the same examples as discussed in Wazwaz (2000).

Example 3.1.

Consider the linear boundary value problem for the fourth-order integro differential equation:

$$y^{(iv)}(x) = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(t) dt, \quad 0 < x < 1$$

with boundary conditions

$$y(0) = 1, \quad y'(0) = 1, \quad y(1) = 1 + e, \quad y'(1) = 2e.$$

Using the transformation $\frac{dy}{dx} = q(x)$, $\frac{dq}{dx} = f(x)$, $\frac{dt}{dx} = z(x)$, the above boundary value problem

can be transformed as:

$$\begin{cases} \frac{dy}{dx} = q(x), & \frac{dq}{dx} = f(x), \\ \frac{dt}{dx} = z(x), & \frac{dz}{dx} = x(1 + e^x) + 3e^x + y(x) + \int_0^x y(t) dt, \end{cases}$$

with boundary conditions

$$y(0) = 1, \quad q(0) = 1, \quad f(0) = A, \quad z(0) = B.$$

The exact solution of the above boundary value problem is $y(x) = 1 + x e^x$.

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = +1, i = 1, 2, 3, 4$.

$$\begin{cases} y^{(k+1)}(x) = 1 + \int_0^x q^{(k)}(t)dt, & q^{(k+1)}(x) = -1 + \int_0^x f^{(k)}(t)dt, \\ f^{(k+1)}(x) = A + \int_0^x z^{(k)}(t)dt, & z^{(k+1)}(x) = B + \int_0^x \left(t(1 + e^t) + 3e^t + y^{(k)}(x) \right) dx + \int_0^x y^{(k)}(t) dt, \end{cases}$$

where

$$A = y''(0), \quad B = y'''(0).$$

Consequently, following approximants are obtained

$$y^{(0)}(x) = 1, q^{(0)}(x) = 1, f^{(0)}(x) = A, z^{(0)}(x) = B,$$

$$\begin{cases} y^{(1)}(x) = 1 + x, & q^{(1)}(x) = 1 + Ax, \\ f^{(1)}(x) = A + Bx, & z^{(1)}(x) = B - 2 + x + x^2 + 2e^x + xe^x, \end{cases}$$

$$\begin{cases} y^{(2)}(x) = 1 + x + \frac{1}{2} Ax^2, & q^{(2)}(x) = 1 + Ax + \frac{1}{2} Bx^2, \\ f^{(2)}(x) = A + Bx - 1 - 2x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + e^x + xe^x, & z^{(2)}(x) = B - 2 + x + x^2 + 2e^x + xe^x - \frac{1}{2!} x^2 - \frac{1}{3!} x^3 - \frac{1}{4!} x^4 - \dots \end{cases}$$

$$\begin{cases} y^{(3)}(x) = 1 + x + \frac{1}{2} Ax^2 + \frac{3}{3!} Bx^3, & q^{(3)}(x) = 1 + Ax + \frac{1}{2} Bx^2 - x - x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + xe^x, \\ f^{(3)}(x) = A + Bx - 1 - 2x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + e^x + xe^x - \frac{1}{3!} x^3 - \frac{1}{4!} x^4, \\ z^{(3)}(x) = B - 2 + x + x^2 + 2e^x + xe^x - \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \frac{1}{3!} Ax^3 + \frac{1}{4!} Ax^4, \end{cases}$$

$$\left\{ \begin{array}{l}
 y^{(4)}(x) = 1 + x + \frac{1}{2}Ax^2 + \frac{3}{3!}Bx^3 + 1 - \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - e^x + xe^x, \\
 q^{(4)}(x) = 1 + Ax + \frac{1}{2}Bx^2 - x - x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + xe^x - \frac{1}{4!}x^4 - \frac{1}{5!}x^5, \\
 f^{(4)}(x) = A + Bx - 1 - 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + e^x + xe^x - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{1}{4!}Ax^4 + \frac{1}{5!}Ax^5, \\
 z^{(4)}(x) = B - 2 + x + x^2 + 2e^x + xe^x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{3!}Ax^3 + \frac{1}{4!}Ax^4 + \frac{3}{4!}Bx^4 + \frac{3}{5!}Bx^5, \\
 y^{(5)}(x) = 1 + x + \frac{1}{2}Ax^2 + \frac{3}{3!}Bx^3 + 1 - \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - e^x + xe^x - \frac{1}{5!}x^5 - \frac{1}{6!}x^6, \\
 q^{(5)}(x) = 1 + Ax + \frac{1}{2}Bx^2 - x - x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + xe^x - \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6, \\
 f^{(5)}(x) = A + Bx - 1 - 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + e^x + xe^x - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{1}{4!}Ax^4 + \frac{1}{5!}Ax^5 + \frac{3}{5!}Bx^5 + \frac{3}{6!}Bx^6, \\
 z^{(5)}(x) = B - 2 + x + x^2 + 2e^x + xe^x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{3!}Ax^3 + \frac{1}{4!}Ax^4 + \frac{3}{4!}Bx^4 + \frac{3}{5!}Bx^5 + 5 \\
 \quad + 3x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{2}{4!}x^4 + \frac{2}{6!}x^6 + \frac{1}{7!}x^7 - 5e^x + xe^x, \\
 \vdots
 \end{array} \right.$$

The series solution is given as

$$\begin{aligned}
 y(x) = & 1 + x + \frac{1}{2!}Ax^2 + \frac{1}{6}Bx^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \left(\frac{1}{720}A + \frac{1}{180}\right)x^6 + \left(\frac{1}{840} - \frac{1}{5040}A + \frac{1}{5040}B\right)x^7 \\
 & + \left(\frac{11}{40320} - \frac{1}{40320}B\right)x^8 + \frac{1}{40320}x^9 + \left(\frac{1}{453600} + \frac{1}{3628800}A\right)x^{10} \\
 & + \left(-\frac{1}{19958400}A + \frac{1}{39916800}B + \frac{1}{3326400}\right)x^{11} + \left(\frac{1}{479001600}A - \frac{1}{239500800}B + \frac{1}{29937600}\right)x^{12} + \dots
 \end{aligned}$$

Imposing the boundary conditions at $x = 1$, we obtained

$$A = 1.999999953, \quad B = 3.000000151.$$

$$\begin{aligned}
 y(x) = & 1 + x + 0.9999999765x^2 + 0.5000000252x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + 0.008333333269x^6 + 0.001388888928x^7 \\
 & + 0.0001984126946x^8 + \frac{1}{40320}x^9 + 0.27557310 \times 10^{-5}x^{10} + 0.2755731983 \times 10^{-6}x^{11} \\
 & + 0.2505210766 \times 10^{-7}x^{12} + \dots
 \end{aligned}$$

Table 3.1 (Error estimates)

x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.00000
0.1	1.11105170920	1.1105170920	2.0 E-10
0.2	1.2442805520	1.1.2442805510	6.09 E-10
0.3	1.4049576420	1.4049576410	1.4 E-9
0.4	1.5967298790	1.5967298780	1.2 E-9
0.5	1.8243606360	1.8243606320	3.5 E-9
0.6	2.0932712800	2.0932712780	2.0 E-9
0.7	2.4096268950	2.4096268920	3.0 E-9
0.8	2.7804327420	2.7804327410	1.0 E-9
0.9	3.2136428000	3.2136427980	2.0E-9
1.0	3.7182818280	3.7182818290	1.0E-9

*Error=Exact solution-Series solution.

Table 3.1 exhibits the errors obtained by applying the variational iteration method. Higher accuracy can be obtained by using some more terms of the series solution.

Example 3.2.

Consider the nonlinear boundary value problem for the integro differential equation

$$y^{(iv)}(x) = 1 + \int_0^x e^{-x} y^2 dx, \quad 0 < x < 1$$

with boundary conditions

$$y(0) = 1, \quad y'(0) = 1, \quad y(1) = e, \quad y'(1) = e.$$

The exact solution of the above boundary value problem is $y(x) = e^x$.

Using transformation $\frac{dy}{dx} = q(x)$, $\frac{dq}{dx} = f(x)$, $\frac{df}{dx} = z(x)$, the above boundary value problems

can be transformed as the following system of differential equations

$$\begin{cases} \frac{dy}{dx} = q(x), & \frac{dq}{dx} = f(x), \\ \frac{df}{dx} = z(x), & \frac{dz}{dx} = 1 + \int_0^x e^{-x} y^2(x) dx, \end{cases}$$

with boundary conditions

$$y(0) = 1, \quad q(0) = 1, \quad f(0) = A, \quad z(0) = B.$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1$, $i = 1, 2, 3, 4$.

$$\begin{cases} y^{(k+1)}(x) = 1 + \int_0^x q(t) dt, & q^{(k+1)}(x) = 1 + \int_{0,x} f(t) dt, \\ f^{(k+1)}(x) = A + \int_0^x z(t) dt, & z^{(k+1)}(x) = B + \int_0^x \left(1 + \int_0^t e^{-s} (y^{(k)}(s))^2 ds \right) dt. \end{cases}$$

Consequently, following approximants are obtained:

$$y^{(0)}(x) = 1, \quad q^{(0)}(x) = 1, \quad f^{(0)}(x) = A, \quad z^{(0)}(x) = B,$$

$$\begin{cases} y^{(1)}(x) = 1 + x, & q^{(1)}(x) = 1 + Ax, \\ f^{(1)}(x) = A + Bx, & z^{(1)}(x) = B + 4 - 4e^{-x}, \end{cases}$$

$$\begin{cases} y^{(2)}(x) = 1 + x + \frac{1}{2!} Ax^2, & q^{(2)}(x) = 1 + Ax + \frac{1}{3!} Bx^2, \\ f^{(2)}(x) = A + Bx - 4 + 4x + 4e^{-x}, & z^{(2)}(x) = B + 8 - 8e^{-x} - 4xe^{-x}, \end{cases}$$

$$\begin{cases} y^{(3)}(x) = 1 + x + \frac{1}{2!} Ax^2 + \frac{3}{4!} Bx^3, & q^{(3)}(x) = q^{(2)}(x) = 1 + Ax + \frac{1}{3!} Bx^2 + 4 - 4x + \frac{4}{2!} - 4e^{-x}, \\ f^{(3)}(x) = A + (B + 8)x + 4e^{-x} - 4xe^{-x}, \\ z^{(3)}(x) = z^{(2)}(x) = B + 8 - 8e^{-x} - 4xe^{-x} + \frac{5}{2!} A(2 - 2e^{-x} - 2xe^{-x} - x^2e^{-x}), \end{cases}$$

⋮

The series solution is given by:

$$\begin{aligned} y(x) = & 1 + x + \frac{1}{2!} Ax^2 + \frac{1}{3!} Bx^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \frac{1}{6!} x^6 + \left(\frac{1}{2520} A - \frac{1}{1680} \right) x^7 + \left(-\frac{1}{6720} A + \frac{1}{20160} B + \frac{1}{8064} \right) x^8 \\ & + \left(\frac{1}{30240} A - \frac{1}{45360} B + \frac{1}{72576} \right) x^9 + \left(-\frac{1}{181440} A + \frac{1}{181440} B + \frac{1}{10!} \right) x^{10} \\ & + \left(\frac{1}{1330560} A - \frac{1}{997920} B + \frac{1}{7983360} \right) x^{11} + \left(-\frac{1}{11404800} A + \frac{1}{6842880} B + \frac{1}{12!} \right) x^{12} + \dots \end{aligned}$$

Imposing the boundary conditions at $x = 1$, we obtained

$$A = 0.9970859583, \quad B = 1.010994057.$$

The series solution is given as

$$y(x) = 1 + x + 0.4985429792x^2 + 0.1684990095x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + -0.0001995690641x^7 + 0.00002578056453x^8 + 0.00002446285010x^9 + 0.3522271752 \times 10^{-6}x^{10} - 0.1384676012 \times 10^{-6}x^{11} + 0.624047474716 \times 10^{-7}x^{12} + \dots$$

Table 3.2. (Error estimates)

x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.00000
0.1	1.1051581800	1.1051581800	1.27 E-5
0.2	1.2214027580	1.2213591310	4.36 E-5
0.3	1.3498588080	1.3497770620	8.17 E-5
0.4	1.4918246980	1.4917081990	1.16 E-4
0.5	1.6487212710	1.6485829960	1.38 E-4
0.6	1.8221188000	1.8219791520	1.39 E-4
0.7	2.0137527070	2.0136354170	1.17-4
0.8	2.2255409280	2.2254662080	7.47 E-5
0.9	2.4596031110	2.4595771740	2.59 E-5
1.0	2.7182818280	2.7182818280	0.000000

*Error=Exact solution-Series solution.

Table 3.2 exhibits the errors obtained by applying the variational iteration method. Higher accuracy can be obtained by using some more terms of the series solution.

Example 3.3.

Consider the two dimensional nonlinear inhomogeneous boundary value problem for the integro differential equation related to the Blasius problem

$$y''(x) = \alpha - \frac{1}{2} \int_0^x y(t)y''(t)dt, \quad -\infty < x < 0$$

with boundary conditions

$$y(0) = 0, \quad y'(0) = 1.$$

and

$$\lim_{x \rightarrow -\infty} y'(x) = 0,$$

where α is a constant and is given by

$$y''(0) = \alpha, \quad \alpha > 0.$$

Using the transformation $\frac{dy}{dx} = q(x)$, the above boundary value problem can be written as the following system of differential equations

$$\begin{cases} \frac{dy}{dx} = q(x), \\ \frac{dq}{dx} = \alpha - \frac{1}{2} \int_0^x y(t)y''(t)dt, \end{cases}$$

with boundary conditions

$$y(0) = 0, \quad q(0) = 1, \quad q'(0) = \alpha.$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1$, $i = 1, 2$.

$$\begin{cases} y(x) = \int_0^x q^{(k)}(t)dt, \\ q(x) = 1 + \int_0^x \left(\alpha - \frac{1}{2} \int_0^x y^{(k)}(t)q'^{(k)}(t)dt \right) dt. \end{cases}$$

Consequently, the following approximants are obtained

$$y^{(0)}(x) = 1, \quad q^{(0)}(x) = 1,$$

$$\begin{cases} y^{(1)}(x) = x, \\ q^{(1)}(x) = 1 + \alpha x, \end{cases}$$

$$\begin{cases} y^{(2)}(x) = x + \frac{1}{2} \alpha x^2, \\ q^{(2)}(x) = 1 + \alpha x - \frac{1}{12} \alpha x^3, \end{cases}$$

$$\begin{cases} y^{(3)}(x) = x + \frac{1}{2} \alpha x^2 - \frac{1}{48} \alpha x^4, \\ q^{(3)}(x) = 1 + \alpha x - \frac{1}{12} \alpha x^3 - \frac{1}{12} \alpha x^3 - \frac{1}{48} \alpha x^4 + \frac{1}{160} \alpha x^5 + \frac{1}{480} \alpha^2 x^6, \end{cases}$$

\vdots

The series solution is given as:

$$y(x) = x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5 + \frac{1}{960}\alpha x^6 + \frac{11}{20160}\alpha^2 x^7 + \left(\frac{11}{161280}\alpha^3 + \frac{1}{960}\alpha\right)x^8 - \frac{43}{967680}\alpha^2 x^9 + \left(\frac{1}{52960}\alpha - \frac{5}{387072}\alpha^3\right)x^{10} + \left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4\right)x^{11} + \left(-\frac{1}{16220160}\alpha + \frac{1}{7257792}\alpha^3\right)x^{12} + \dots,$$

and consequently

$$y'(x) = 1 + \alpha x - \frac{1}{12}\alpha x^3 - \frac{1}{48}\alpha^2 x^4 + \frac{1}{160}\alpha x^5 + \frac{11}{2880}\alpha^2 x^6 \left(\frac{11}{20160}\alpha^3 - \frac{1}{2688}\alpha\right)x^7 - \frac{43}{107520}\alpha^2 x^8 + 10\left(\frac{1}{552960}\alpha - \frac{5}{387072}\alpha^3\right)x^9 + 11\left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4\right)x^{10} + 12\left(-\frac{1}{16220160}\alpha + \frac{1}{725760}\alpha^3\right)x^{11} + \dots,$$

is obtained. The diagonal Pade' approximants can be applied, consequently, we obtained values of the constant α which are listed in the table 3.3, (see Wazwaz (2000)).

Table 3.3

Pade' approximants	α
[2/2]	0.5778502691
[3/3]	0.5163977793
[4/4]	0.5227030798

Table 3.3 Pade' approximants and numerical value of α .

The above results are in full agreement with the calculations in Wazwaz (2000), (1997).

4. Conclusion

In this paper, we applied the variational iteration method (VIM) for solving the higher order integro differential equations by using the re-formulated system of integral equations. The method is used in a direct way without using linearization, perturbation, discretization or restrictive assumption. The method gives more realistic series solutions that converge very rapidly in physical problems. Thus we conclude that the variational iteration method can be considered as an efficient and effective method for solving linear and nonlinear initial and boundary value problems. The fact that the VIM solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

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