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On a Nonlinear Hyperbolic Partial Differential Equation with Irregular Data

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Abstract

The main purpose of this paper is to study the existence and properties of solutions of a certain nonlinear non-Lipschitz hyperbolic partial differential equation in two independent variables with irregular data. Using regularization techniques, we give a meaning to this problem by replacing it by a tow parameters family of Lipschitz regular problems. We prove existence and uniqueness of the solution in an appropriate algebra of generalized functions and we precise how it depends on the choices made. We study the relationship with the classical solution.

Keywords: Regularization of problems, algebras of generalized functions, nonlinear second order hyperbolic equations, non-Lipschitz problems, Goursat problems

MSC 2010 No.: 35L70, 46F30.

1. Introduction

The following hyperbolic equation

$$u_{xt} + a_0(x,t)u_x + b_0(x,t)u_t = c_0(x,t,u)$$

considered on the demi-strip $0 \leq x \leq l$, $0 \leq t \leq \infty$ with the given characteristic data $u|_{(Ox)} = \varphi$,

$u|_{(Ot)} = \psi$, is important in physics. In the monograph (Corduneanu, 1991, p.20), it is pointed out that this equation by means of suitable substitution takes the form"

$$v_{xt} + a(x,t)v_x = c(x,t,v)$$

in which $a(x,t)$ and $c(x,t,v)$ have same regularizing properties as $a_0(x,t)$ and $c_0(x,t,u)$, see (Pachpatte, 2009).

So the main purpose of this paper is to establish the existence of solutions to the non-Lipschitz nonlinear hyperbolic equation with characteristic data, formally written as

$$(P_{form}) \begin{cases} u_{xy} = F(\cdot,\cdot,u,u_x), \\ u|_{(Ox)} = \varphi,\, u|_{(Oy)} = \psi, \varphi(0) = \psi(0). \end{cases}$$

The notation $F(\cdot,\cdot,u,u_x)$ extends, with a meaning to be defined later, the expression $(x,y) \mapsto F(x,y,u(x,y),u_x(x,y))$ in the case where u is a generalized function of two variables x and y. Here φ and ψ are distributions or one-variable generalized functions. The function F are supposed to be smooth, $F \in \mathrm{C}^\infty(\Delta \times \mathbb{R}^2, \mathbb{R})$ with $\Delta = (\mathbb{R}_+)^2$, and F is a non-Lipschitz function.

We reformulate the problem in the framework of generalized functions extending the ideas developed in (Delcroix et al., 2009; Delcroix et al., 2011; Dévoué, 2007; Dévoué, 2009a; Dévoué, 2009b; Dévoué, 2011; Marti, 1999). The reader will find in (Dévoué, 2007; Dévoué, 2009a; Allaud and Dévoué, 2013), the notations and the concepts used in this paper. But if the generalized framework is the same, here, the technics and estimates are new. The Gronwall lemma is unenforceable to get uniqueness and to obtain that we refer to a Pachpatte lemma, see (Pachpatte, 2009, p.42). A general reference for the $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras can be found in (Marti, 1998; Marti, 1999; Delcroix and Scarpalézos, 2000).

To give a meaning to this problem we use the recent theories of generalized functions, see (Colombeau, 1984b; Colombeau, 1984a; Grosser et al., 2001; Nedeljkov et al., 2005), and particularly the $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras of J.-A. Marti, see (Marti, 1998; Marti, 1999). The $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras give an efficient algebraic framework which permits a precise study of solutions as in (Delcroix et al., 2009; Dévoué, 2007; Dévoué, 2009a; Dévoué, 2009b; Marti and Nuiro, 1999). We investigate solutions with distributions or other generalized functions as initial data, thus we must search for solutions in algebras which are invariant under nonlinear functions and contain the space of distributions.

This ill-posed problem remains unsolvable in classical function spaces. To overcome this difficulty, by means of regularizations, we associate to problem (P_{form}) a generalized one (P_{gen}) well formulated in a convenient algebra $\mathcal{A}(\Delta)$.

The general idea goes as follows. The problem is approached by a tow parameters family of classical smooth problems (P_λ) where $\lambda = (\varepsilon, \rho) \in (0,1]^2$. Then we get a tow parameters family of classical solutions. A generalized solution is defined as the class of this family of smooth functions satisfying some asymptotic growth restrictions, (Nedeljkov et al., 2005).

The outline of this paper is as follows. Section 2 introduces the algebras of generalized functions. In Section 3 we define a well formulated generalized differential problem (P_{gen}) associated to the classical one. It is constructed by means of a family (P_λ) of regularized problems. We give estimates needed in the sequel. We replace F with a family of Lipschitz functions (F_ε) given by suitable cutoff techniques which gives rise to a family of regularized Lipschitz problems. We use a family of mollifiers $(\theta_\rho)_\rho$ to regularize the data in singular case. So parameter ε is used to render the problem Lipschitz, ρ makes it regular. Then we can built a $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra, $\mathcal{A}(\Delta)$, stable under the family (F_ε), adapted to the generalized Goursat problem in which the irregular problem can be solved.

Then we proceed in Section 4 with the proof of the existence of the generalized solution. To prove the existence of solution, a parametric representative $(u_\lambda)_\lambda$, with $\lambda = (\varepsilon, \rho)$, is constructed from the existence of smooth solutions u_λ for each regularized Lipschitz problem (P_λ). The class of $(u_\lambda)_\lambda$ is the expected generalized solution. However, the generalized problem (P_{gen}), and obviously its solutions, depend on the choice of the cutoff functions and, in the case of irregular data, on the family of mollifiers. With regard to the regularization, we show that this solution depends solely on the class of cutoff functions as a generalized function, not on the particular representative. In the case of irregular data, the solution of the problem depends on the family of mollifiers but not on a class of that family. Using the study of (Pachpatte, 2009), we show that this solution is unique in the constructed algebra. Moreover, we show that if the initial problem admits a smooth solution v satisfying appropriate growth estimates on some open subset O of Δ, then this solution and the generalized one are equal in a meaning given in Theorem 5.

In the Appendix we precise the results and estimates obtained in classical problem.

2. Algebras of generalized functions

2.1 The presheaves of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras

2.1.1 Definitions

We refer the reader to (Marti, 1998; Marti, 1999) for more details.

We recall the definition of the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras. Take

(1) Λ a set of indices left-filtering for a given partial order relation $\prec$.

(2) A a solid subring of the ring $\mathbb{K}^\Lambda$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), that is A has the following stability property: whenever $(|s_\lambda|)_\lambda \leq (r_\lambda)_\lambda$ (i.e. for any λ, $|s_\lambda| \leq r_\lambda$) for any pair $((s_\lambda)_\lambda, (r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times |A|$, it follows that $(s_\lambda)_\lambda \in A$, with $|A| = \{(|r_\lambda|)_\lambda : (r_\lambda)_\lambda \in A\}$ and I_A a solid ideal of A with the same property;

(3) $\mathcal{E}$ a sheaf of $\mathbb{K}$-topological algebras on a topological space X, such that for any open set Ω in X, the algebra $\mathcal{E}(\Omega)$ is endowed with a family $\mathcal{P}(\Omega) = (p_i)_{i \in I(\Omega)}$ of seminorms satisfying

$$\forall i \in I(\Omega), \exists (j, k, C) \in (I(\Omega))^2 \times \mathbb{R}_+^*, \forall f, g \in \mathcal{E}(\Omega) : p_i(fg) \leq C p_j(f) p_k(g).$$

Assume that

(4) For any two open subsets Ω_1, Ω_2 of X such that $\Omega_1 \subset \Omega_2$, we have $I(\Omega_1) \subset I(\Omega_2)$ and if ρ_1^2

is the restriction operator $\mathcal{E}(\Omega_2) \to \mathcal{E}(\Omega_1)$, then, for each $p_i \in \mathcal{P}(\Omega_1)$, the seminorm $\widetilde{p_i} = p_i \circ \rho_1^2$ extends p_i to $\mathcal{P}(\Omega_2)$;

(5) For any family $\mathcal{F} = (\Omega_h)_{h\in H}$ of open subsets of X if $\Omega = \cup_{h\in H}\Omega_h$, then, for each $p_i \in \mathcal{P}(\Omega)$, $i \in I(\Omega)$, there exists a finite subfamily $(\Omega_j)_{1\leq j\leq n(i)}$ of $\mathcal{F}$ and corresponding seminorms $p_j \in \mathcal{P}(\Omega_j)$, $1 \leq j \leq n(i)$, such that, for each $u \in \mathcal{E}(\Omega)$, $p_i(u) \leq \sum_{j=1}^{n(i)} p_j(u_{|\Omega_j})$.

Set $\mathcal{C} = A/I_A$ and

$$\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda : \forall i \in I(\Omega),\ ((p_i(u_\lambda))_\lambda \in |A|\},$$
$$\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda : \forall i \in I(\Omega),\ (p_i(u_\lambda))_\lambda \in |I_A|\}.$$

One can prove that $\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}$ is a sheaf of subalgebras of the sheaf $\mathcal{E}^\Lambda$ and $\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$ is a sheaf of ideals of $\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}$, (see (Marti, 1998; Marti, 1999). Moreover, the constant sheaf $\mathcal{X}_{(A,\mathbb{K},|.|)}/\mathcal{N}_{(I_A,\mathbb{K},|.|)}$ is exactly the sheaf $\mathcal{C} = A/I_A$, and if $\mathbb{K} = \mathbb{R}$, $\mathcal{C}$ will be denoted $\overline{\mathbb{R}}$. We call presheaf of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra the factor presheaf of algebras $\mathcal{A} = \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$ over the ring $\mathcal{C} = A/I_A$. We denote by $[u_\lambda]$ the class in $\mathcal{A}(\Omega)$ defined by the representative $(u_\lambda)_{\lambda\in\Lambda} \in \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$.

Remark 1: (Overgenerated rings) Let $B_p = \left\{(r_{n,\lambda})_\lambda \in (\mathbb{R}_+^*)^\Lambda : 1 \leq n \leq p\right\}$ and B be the subset of $(\mathbb{R}_+^*)^\Lambda$ obtained as rational functions with coefficients in $\mathbb{R}_+^*$, of elements in B_p as variables. Define

$$A = \left\{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \exists\, (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\right\},$$

we say that A is *overgenerated* by B_p (and it is easy to see that A is a solid subring of $\mathbb{K}^\Lambda$). If I_A is some solid ideal of A, we also say that $\mathcal{C} = A/I_A$ is *overgenerated* by B_p, (Oberguggenberger, 1992; Delcroix et al., 2011).

Remark 2: (Relationship with distribution theory) Let Ω be an open subset of $\mathbb{R}^n$. The space of distributions $\mathcal{D}'(\Omega)$ can be embedded into $\mathcal{A}(\Omega)$. If $(\varphi_\lambda)_{\lambda\in(0,1]}$ is a family of mollifiers $\varphi_\lambda(x) = \lambda^{-n}\varphi(x/\lambda)$, $x \in \mathbb{R}^n$, $\int \varphi(x)\, dx = 1$ and if $T \in \mathcal{D}'(\mathbb{R}^n)$, the convolution product family $(T * \varphi_\lambda)_\lambda$ is a family of smooth functions slowly increasing in $1/\lambda$. So, for $\Lambda = (0,1]$, we shall choose the subring A overgenerated by some B_p of $(\mathbb{R}_+^*)^\Lambda$ containing the family $(\lambda)_\lambda$, (Delcroix, 2005). We choose a special kind of mollifiers which moments of higher order vanish.

Remark 3: (An association process) Let Ω be an open subset of X, E be a given sheaf of topological $\mathbb{K}$-vector spaces containing $\mathcal{E}$ as a subsheaf, a be a given map from Λ to $\mathbb{K}$ such that $(a(\lambda))_\lambda = (a_\lambda)_\lambda$ is an element of A. We also assume that

$$\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) \subset \left\{(u_\lambda)_\lambda \in \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega) : \lim_{E(\Omega),\Lambda} u_\lambda = 0\right\}.$$

We say that $u = [u_\lambda]$ and $v = [v_\lambda] \in \mathcal{E}(\Omega)$ are a-E associated if $\lim_{E(\Omega),\Lambda} a_\lambda(u_\lambda - v_\lambda) = 0$. *That is to say, for each neighborhood V of* 0 *for the E-topology, there exists* $\lambda_0 \in \Lambda$ *such that* $\lambda \prec \lambda_0 \Longrightarrow a_\lambda(u_\lambda - v_\lambda) \in V$. We write $u \underset{E(\Omega)}{\overset{a}{\sim}} v$. We can also define an association process between $u = [u_\lambda]$ and $T \in E(\Omega)$ by writing simply $u \sim T \iff \lim_{E(\Omega),\Lambda} u_\lambda = T$. Taking $E = \mathcal{D}'$, $\mathcal{E} = \mathrm{C}^\infty$, $\Lambda = (0,1]$, we recover the association process defined in (Colombeau, 1984b; Colombeau, 1984a).

2.2 Algebraic framework

Set $\mathcal{E} = \mathrm{C}^\infty$, $X = \mathbb{R}^d$ for $d = 1, 2$, $E = \mathcal{D}'$ and Λ a set of indices, $\lambda \in \Lambda$. For any open set Ω, in $\mathbb{R}^d$, $\mathcal{E}(\Omega)$ is endowed with the $\mathcal{P}(\Omega)$ topology of uniform convergence of all derivatives on compact subsets of Ω. This topology may be defined by the family of the seminorms $P_{K,l}(u_\lambda) = \sup_{|\alpha|\leq l} P_{K,\alpha}(u_\lambda)$ with $P_{K,\alpha}(u_\lambda) = \sup_{x\in K} |D^\alpha u_\lambda(x)|$, $K \Subset \Omega$, where the notation $K \Subset \mathbb{R}^2$ means that K is a compact subset of $\mathbb{R}^2$ and $l \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$.

Let A be a subring of the ring $\mathbb{R}^\Lambda$ of family of reals with the usual laws. We consider a solid ideal I_A of A. Then we have

$$\begin{aligned}
\mathcal{X}(\Omega) &= \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N},\ (P_{K,l}(u_\lambda))_\lambda \in |A|\},\\
\mathcal{N}(\Omega) &= \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N},\ (P_{K,l}(u_\lambda))_\lambda \in |I_A|\},\\
\mathcal{A}(\Omega) &= \mathcal{X}(\Omega)/\mathcal{N}(\Omega).
\end{aligned}$$

The generalized derivation $D^\alpha : u(= [u_\varepsilon]) \mapsto D^\alpha u = [D^\alpha u_\varepsilon]$ provides $\mathcal{A}(\Omega)$ with a differential algebraic structure, (Scarpalézos, 2000; Scarpalézos, 2004).

We have the analogue of theorem 1.2.3. of (Grosser et al., 2001), for $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras.

Proposition 1: Let B be the set introduced in Remark 1 and assume that there exists $(a_\lambda)_\lambda \in B$ with $\lim_{\lambda\to 0} a_\lambda = 0$. C*onsider* $(u_\lambda)_\lambda \in \mathcal{X}(\mathbb{R}^2)$ such that: $\forall K \Subset \mathbb{R}^2$, $(P_{K,0}(u_\lambda))_\lambda \in |I_A|$. *Then* $(u_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2)$.

We refer the reader to (Delcroix et al., 2011; Delcroix, 2008).

2.2.1 Generalized operator associated to a stability property

Set $\Lambda = \Lambda_1 \times \Lambda_2$, $\Lambda_1 = \Lambda_2 = (0, 1]$, denote by $\lambda = (\varepsilon, \rho)$ an element of Λ.

Definition 1: Let Ω be an open subset of $\mathbb{R}^2$, $\Omega' = \Omega \times \mathbb{R}^2 \subset \mathbb{R}^4$. Let $F_\varepsilon \in \mathrm{C}^\infty(\Omega', \mathbb{R})$. *We say that the algebra $\mathcal{A}(\Omega)$ is stable under the family $(F_\varepsilon)_\lambda$ if for all $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$ and $(i_\lambda)_\lambda \in \mathcal{N}(\Omega)$, we have* $(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x))_\lambda \in \mathcal{X}(\Omega)$ and

$$(F_\varepsilon(\cdot,\cdot,u_\lambda + i_\lambda,(u_\lambda + i_\lambda)_x) - F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x))_\lambda \in \mathcal{N}(\Omega).$$

If $\mathcal{A}(\Omega)$ if stable under $(F_\varepsilon)_\lambda$, for $u = [u_\lambda] \in \mathcal{A}(\Omega)$, $[F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)]$ is a well defined element of $\mathcal{A}(\Omega)$ (i.e. not depending on the representative $(u_\lambda)_\lambda$ of u).

Definition 2: Let Ω be an open subset of $\mathbb{R}^2$ *and* $F \in \mathrm{C}^\infty(\Omega\times\mathbb{R}^2, \mathbb{R})$. *We say that F* is *smoothly tempered if the following two conditions are satisfied:*
(i) *For each $K \Subset \Omega$, $l \in \mathbb{N}$* and $u \in \mathrm{C}^\infty(\Omega, \mathbb{R})$, *there is a positive finite sequence C_0,..., C_l, such that*

$$P_{K,l}(F(\cdot,\cdot,u,u_x)) \leq \sum_{i=0}^{l} C_i P^i_{K,l+1}(u).$$

(ii) *For each $K \Subset \Omega$, $l \in \mathbb{N}$, u* and $v \in \mathrm{C}^\infty(\Omega, \mathbb{R})$, *there is a positive finite sequence D_1,..., D_l such that*

$$P_{K,l}(F(\cdot,\cdot,v,v_x) - F(\cdot,\cdot,u,u_x)) \leq \sum_{j=1}^{l} D_j P^j_{K,l+1}(v-u).$$

Proposition 2: For any ε assume that F_ε is smoothly tempered then $\mathcal{A}(\Omega)$ is stable under $(F_\varepsilon)_\lambda$.

Set $f \in \mathrm{C}^\infty(\Omega)$, we define $\mathrm{C}^\infty(\Omega) \to \mathrm{C}^\infty(\Omega)$, $f \mapsto H_\lambda(f) = F_\varepsilon(\cdot,\cdot,f,f_x)$.

$$H_\lambda(f) = F_\varepsilon(\cdot,\cdot,f,f_x) : (x,y) \mapsto F_\varepsilon(x,y,f(x,y),f_x(x,y)).$$

Clearly, the family $(H_\lambda)_\lambda$ maps $(\mathrm{C}^\infty(\Omega))^\Lambda$ into $(\mathrm{C}^\infty(\Omega))^\Lambda$ and allows to define a map from $\mathcal{A}(\Omega)$ into $\mathcal{A}(\Omega)$. For $u = [u_\lambda] \in \mathcal{A}(\Omega)$, $([F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)])$ is a well defined element of $\mathcal{A}(\Omega)$. This leads to the following definition, (Delcroix et al., 2011).

Definition 3: If $\mathcal{A}(\Omega)$ if stable under $(F_\varepsilon)_\lambda$, the operator

$$\mathcal{F} : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega), \quad u = [u_\lambda] \mapsto [F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)]$$

is called the generalized operator associated to the family $(F_\varepsilon)_\lambda$.

Definition 4: Let $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$ and $(g_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$, we define

$$F_\varepsilon(x,y,z,p) = F(x,y,zg_\varepsilon(z),pg_\varepsilon(p)).$$

The family $(F_\varepsilon)_\lambda$ is called the family associated to F *via* the family $(g_\varepsilon)_\varepsilon$. If $\mathcal{A}(\Omega)$ is stable under $(F_\varepsilon)_\lambda$, the operator

$$\mathcal{F} : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega), \quad u = [u_\lambda] \mapsto [F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)]$$

is called the generalized operator associated to F *via* the family $(g_\varepsilon)_\varepsilon$.

2.3 $\mathcal{D}'$-*singular support*

Assume that

$$\mathcal{N}_{\mathcal{D}'}^{\mathcal{A}}(\Omega) = \left\{(u_\lambda)_\lambda \in \mathcal{X}(\Omega) : \lim_{\lambda\to 0} u_\lambda = 0 \text{ in } \mathcal{D}'(\Omega)\right\} \supset \mathcal{N}(\Omega).$$

Set

$$\mathcal{D}'_{\mathcal{A}}(\Omega) = \left\{[u_\lambda] \in \mathcal{A}(\Omega) : \exists T \in \mathcal{D}'(\Omega),\ \lim_{\lambda\to 0}(u_\lambda) = T \text{ in } \mathcal{D}'(\Omega)\right\}.$$

$\mathcal{D}'_{\mathcal{A}}(\Omega)$ is clearly well defined because the limit is independent of the chosen representative; indeed, if $(i_\lambda)_\lambda \in \mathcal{N}(\Omega)$ we have $\lim_{\substack{\lambda\to 0\\ \mathcal{D}'(\mathbb{R})}} i_\lambda = 0$. $\mathcal{D}'_{\mathcal{A}}(\Omega)$ is an $\mathbb{R}$-vector subspace of $\mathcal{A}(\Omega)$. Therefore we can consider the set $\mathcal{O}_{D'_{\mathcal{A}}}$ of all x having a neighborhood V on which u is associated to a distribution:

$$\mathcal{O}_{D'_{\mathcal{A}}}(u) = \{x \in \Omega : \exists V \in \mathcal{V}(x),\ u|_V \in \mathcal{D}'_{\mathcal{A}}(V)\},$$

$\mathcal{V}(x)$ being the set of all neighborhoods of x.

Definition 5: The $\mathcal{D}'$*-singular support of* $u \in \mathcal{A}(\Omega)$, *denoted* $\operatorname{singsupp}_{\mathcal{D}'}(u) = S_{\mathcal{D}'_{\mathcal{A}}}^{\mathcal{A}}(u)$, is the set $S_{\mathcal{D}'_{\mathcal{A}}}^{\mathcal{A}}(u) = \Omega \backslash \mathcal{O}_{D'_{\mathcal{A}}}(u)$.

3. A non Lipschitz Goursat problem

We study the differential problem formally written as

$$(P) : \frac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u,u_x),\ u|_{(Ox)} = \varphi,\ u|_{(Oy)} = \psi, \varphi(0) = \psi(0),$$

where F, a nonlinear function of its arguments, may be non Lipschitz, the data φ, ψ may be as irregular as distributions. We don't have a classical surrounding in which we can pose (and a fortiori solve) the problem.

3.1 Cut off procedure

Let ε be a parameter belonging to the interval $(0,1]$. Let $(r_\varepsilon)_\varepsilon$ be in $\left(\mathbb{R}_+^*\right)^{(0,1]}$ such that $r_\varepsilon > 0$ and $\lim\limits_{\varepsilon\to 0} r_\varepsilon = +\infty$. Set $L_\varepsilon = [0, r_\varepsilon]$. Consider a family of smooth one-variable functions $(g_\varepsilon)_\varepsilon$ such that

$$\sup_{z\in L_\varepsilon} |g_\varepsilon(z)| = 1,\ g_\varepsilon(z) = \begin{cases} 0, \text{ if } z \geq r_\varepsilon \\ 1, \text{ if } 0 \leq z \leq r_\varepsilon - 1 \end{cases}$$

and $\dfrac{\partial^n g_\varepsilon}{\partial z^n}$ is bounded on L_ε for any integer n, $n > 0$. Set

$$\sup_{z\in L_\varepsilon} \left| \frac{\partial^n g_\varepsilon}{\partial z^n}(z) \right| = M_n.$$

Let $\phi_\varepsilon(z) = z g_\varepsilon(z)$. We approximate the function F by the family of functions $(F_\varepsilon)_\varepsilon$ defined by $F_\varepsilon \in \mathrm{C}^\infty(\Delta \times \mathbb{R}^2, \mathbb{R})$ with $\Delta = (\mathbb{R}_+)^2$ and

$$F_\varepsilon(x, y, z, p) = F(x, y, \phi_\varepsilon(z), \phi_\varepsilon(p)).$$

3.2 Estimates for a parametrized regular problem

Assume that, for any ε, we shall let some positive number M_ε such that, for any $K \Subset \Delta$,

$$\sup_{(x,y)\in K;(z,p)\in\mathbb{R}^2} |\partial_z F_\varepsilon(x,y,z,p)| < M_\varepsilon, \quad \sup_{(x,y)\in K;(z,p)\in\mathbb{R}^2} |\partial_p F_\varepsilon(x,y,z,p)| < M_\varepsilon, \tag{H}$$

where the notation $K \Subset \Delta$ means that K is a compact subset of Δ. We shall require that F_ε satisfies the following Lipschitz condition

$$|F_\varepsilon(x,y,z,p) - F_\varepsilon(x',y',z',p')| \leq M_\varepsilon \left(|z - z'| + |p - p'|\right)$$

for all (x, y, z, p), $(x', y', z', p') \in \Delta \times \mathbb{R}^2$.

We recall that $\lambda = (\varepsilon, \rho) \in \Lambda_1 \times \Lambda_2 = \Lambda$, $\Lambda_1 = \Lambda_2 = (0,1]$, where the parameter ρ is used to regularize the data. We denote by (P_λ) the problem which consists in searching for a function $u_\lambda \in \mathrm{C}^2(\Delta)$ satisfying

$$\frac{\partial^2 u_\lambda}{\partial x \partial y}(x,y) = F_\varepsilon(x, y, u_\lambda(x,y), (u_\lambda)_x(x,y)), \tag{1}$$

$$u_\lambda(x,0) = \varphi_\rho(x), u_\lambda(0,y) = \psi_\rho(y), \varphi_\rho(0) = \psi_\rho(0), \tag{2}$$

where φ_ρ, $\psi_\rho : \mathbb{R}_+ \to \mathbb{R}$ are some smooth one-variable functions and F_ε is a smooth function of all its arguments. According to the Appendix, we can say that (P_λ) is equivalent to the integral formulation

$$u_\lambda(x,y) = u_{0,\lambda}(x,y) + \iint_{D(x,y)} F_\varepsilon(\xi, \eta, u_\lambda(\xi,\eta), (u_\lambda)_x(\xi,\eta))\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{Int}$$

where $u_{0,\lambda}(x,y)=\varphi_\rho(x)+\psi_\rho(y)-\psi_\rho(0)$ with $D(x,y)=\{(\xi,\eta):\ 0\le\xi\le x,\ 0\le\eta\le y\}$.

First we are going to prove that (P_λ) has a unique smooth solution under the following assumption

$$F_\varepsilon\in \mathrm{C}^\infty(\Delta\times\mathbb{R}^2,\mathbb{R}),\ \varphi_\rho \text{ and } \psi_\rho\in \mathrm{C}^\infty(\mathbb{R}_+). \tag{H$_\lambda$}$$

Each compact $K\Subset\Delta$ is contained in some compact $K_a=[0,a]^2$ and $\forall(x,y)\in K, D(x,y)\subset K_a$.

Theorem 1: Problem (P_λ) has a unique solution in $\mathrm{C}^\infty(\Delta,\mathbb{R})$.

Corollary 1: For any $K\Subset(\mathbb{R}_+)^2$ there exists $a>0$ such that $K\subset[0,a]^2=K_a$. With the previous notations, we have

$$\Phi_{a,\varepsilon}=\|F_\varepsilon(\cdot,\cdot,0,0)\|_{\infty,K_a}+\left(\|u_{0,\lambda}\|_{\infty,K_a}+\left\|(u_{0,\lambda})_x\right\|_{\infty,K_a}\right)M_\varepsilon$$

and moreover

$$\|u_\lambda\|_{\infty,K}\le\|u_{0,\lambda}\|_{\infty,K_a}+\tfrac{a\Phi_{a,\varepsilon}}{M_\varepsilon(a+1)}e^{M_\varepsilon a(a+1)};\ \|(u_\lambda)_x\|_{\infty,K}\le\left\|(u_{0,\lambda})_x\right\|_{\infty,K_a}+\tfrac{\Phi_{a,\varepsilon}}{M_\varepsilon(a+1)}e^{M_\varepsilon a(a+1)}.$$

These results are proved in Appendix.

3.3 Construction of $\mathcal{A}(\Delta)$

In the following we refer to the Pachpatte lemma, (Pachpatte, 2009, p.42), as an important tool. This lemma needs to have positive values for the variables x and y. This leads to the next framework.

Thanks to the results of (Biagioni, 1990; Aragona, 2006; Aragona et al., 2009) and J-A. Marti (private communication: Generalized functions on the closure of an open set) and using the study of (Dévoué et al., 2013), we can define Colombeau spaces on the closure $\overline{\Omega}$ of an open set $\Omega\subset\mathbb{R}^n$, such that $O\subset\Omega\subset\overline{O}$, where O is an open subset of $\mathbb{R}^n$ and $\overline{O}$ the closure of O.

We can easily define $\mathrm{C}^\infty(\overline{\Omega})$ as the space of restrictions to $\overline{\Omega}$ of functions in $\mathrm{C}^\infty(O)$ for any open set $O\supset\overline{\Omega}$. C^∞ being a sheaf, the definition is independent of the choice of O. The usual topology of $\mathrm{C}^\infty(\overline{\Omega})$ involves the family of compact set $K\Subset\overline{\Omega}$.

Set $\overline{\Omega}=\Delta$. Consider the previous family $(r_\varepsilon)_\varepsilon$. We make the following assumptions to generate a convenient $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra adapted to our problem

$$\begin{aligned}&\exists d>0,\forall n\in\mathbb{N},\exists c_n>0,\forall\varepsilon\in\Lambda_1,\\&\forall K\Subset\Delta,\forall\alpha\in\mathbb{N}^4,\sup_{(x,y)\in K;(z,p)\in\mathbb{R}^2,|\alpha|=n}|D^\alpha F_\varepsilon(x,y,z,p)|\le c_n r_\varepsilon^d,\end{aligned}$$

in particular $M_\varepsilon\le c_1r_\varepsilon^d$.

$$\mathcal{C}=A/I_A \text{ is overgenerated by the following elements of } \left(\mathbb{R}_+^*\right)^\Lambda:\ (\varepsilon)_\lambda,(\rho)_\lambda,(r_\varepsilon)_\lambda,(e^{r_\varepsilon})_\lambda. \tag{3}$$

Then $\mathcal{A}(\Delta)=\mathcal{X}(\Delta)/\mathcal{N}(\Delta)$ is built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E},\mathcal{P})=\left(\mathrm{C}^\infty(\Delta),(P_{K,l})_{K\Subset\Omega,l\in\mathbb{N}}\right)$ and $\mathcal{A}(\mathbb{R}_+)=\mathcal{X}(\mathbb{R}_+)/\mathcal{N}(\mathbb{R}_+)$ are built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E},\mathcal{P})=\left(\mathrm{C}^\infty(\mathbb{R}_+),(P_{K,l})_{K\Subset\mathbb{R},l\in\mathbb{N}}\right)$, φ, $\psi\in\mathcal{A}(\mathbb{R}_+)$.

As the data φ and ψ are as irregular, we set $\varphi_\rho = r * \theta_\rho$ and $\varphi = [\varphi_\rho]$, $\psi_\rho = s * \theta_\rho$ and $\psi = [\psi_\rho]$ where $(\theta_\rho)_\rho$ is a chosen family of mollifiers. Then the data φ, ψ belong to $\mathcal{A}(\mathbb{R}_+)$ and u is searched in the algebra $\mathcal{A}(\Delta)$.

3.4Stability of $\mathcal{A}(\Delta)$

Proposition 3: Set $S_n = \{\alpha \in \mathbb{N}^4 : |\alpha| = n\}$ when $n \in \mathbb{N}^*$. Let $F \in \mathrm{C}^\infty(\Delta \times \mathbb{R}^2, \mathbb{R})$, F_ε defined as above in Section . Assume that

$$\forall \varepsilon \in (0,1], \forall (x,y) \in \Delta, F_\varepsilon(x,y,0,0) = 0\,, \tag{4}$$

$$\exists d > 0, \forall n \in \mathbb{N}, \exists c_n > 0, \forall \varepsilon \in \Lambda_1, \forall K \Subset \Delta, \sup_{\substack{(x,y)\in K;(z,p)\in\mathbb{R}^2,\\ \alpha\in Sn}} |D^\alpha F_\varepsilon(x,y,z,p)| \le c_n r_\varepsilon^d, \tag{5}$$

then $\mathcal{A}(\Delta)$ is stable under the family $(F_\varepsilon)_\varepsilon$.

We refer the reader to (Dévoué, 2009a; Dévoué, 2009b) for a similar proof.

3.5 A generalized differential problem associated to the formal one

Our goal is to give a meaning to the differential Goursat problem formally written as (P_{form}).

Let $(g_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}_+))^{\Lambda_1}$ and $\mathcal{F}$ the generalized operator associated to F *via* the family $(g_\varepsilon)_\varepsilon$ in Definition 4.

The problem associated problem to (P_{form}) can be written as the well formulated one

$$(P_{gen}) : \frac{\partial^2 u}{\partial x \partial t} = \mathcal{F}(u), u|_{(Ox)} = \varphi, u|_{(Oy)} = \psi, \psi(0) = \varphi(0),$$

where u is in the algebra $\mathcal{A}(\Delta)$.

In terms of representatives, and thanks to the stability and restriction hypothesis, solving (P_{gen}) amounts to find a family $(u_\lambda)_\lambda \in \mathcal{X}(\Delta)$ such that

$$\begin{cases} \dfrac{\partial^2 u_\lambda}{\partial x \partial y}(x,y) - F_\varepsilon(x,y,u_\lambda(x,y),(u_\lambda)_x(x,y)) = i_\lambda(x,y)\,, \\ u_\lambda(x,0) - \varphi_\rho(x) = \alpha_\rho(x)\,, u_\lambda(0,y) - \psi_\rho(y) = \beta_\rho(y)\,, \\ \qquad \psi_\rho(0) = \varphi_\rho(0), \alpha_\rho(0) = \beta_\rho(0)\,, \end{cases}$$

where $(i_\lambda)_\lambda \in \mathcal{N}(\Delta)$, $(\alpha_\rho)_\lambda$, $(\beta_\rho)_\lambda \in \mathcal{N}(\mathbb{R}_+)$.

Suppose we can find $u_\lambda \in \mathrm{C}^\infty(\Delta)$ verifying

$$(P_\lambda) \begin{cases} \dfrac{\partial^2 u_\lambda}{\partial x \partial y}(x,y) = F_\varepsilon(x,y,u_\lambda(x,y),(u_\lambda)_x(x,y)), \\ u_\lambda(x,0) = \varphi_\rho(x)\,, u_\lambda(0,y) = \psi_\rho(y)\,, \psi_\rho(0) = \varphi_\rho(0). \end{cases}$$

If we can prove that $(u_\lambda)_\lambda \in \mathcal{X}(\Delta)$ then $u = [u_\lambda]$ is a solution of (P_{gen}).

Remark 4: Uniqueness in the algebra $\mathcal{A}(\Delta)$. Let $v = [v_\lambda]$ another solution to (P_{gen}). There are $(i_\lambda)_\lambda \in \mathcal{N}(\Delta)$, $(\alpha_\rho)_\lambda$, $(\beta_\rho)_\lambda \in \mathcal{N}(\mathbb{R}_+)$, such that

$$\begin{cases} \dfrac{\partial^2 v_\lambda}{\partial x \partial y}(x,y) - F_\varepsilon(x,y,v_\lambda(x,y),(v_\lambda)_x(x,y)) = i_\lambda(x,y)\,, \\ v_\lambda(x,0) = \varphi_\rho(x) + \alpha_\rho(x), v_\lambda(0,y) = \psi_\rho(y) + \beta_\rho(y), \\ \psi_\rho(0) = \varphi_\rho(0), \alpha_\rho(0) = \beta_\rho(0)\,. \end{cases}$$

The uniqueness of the solution to (P_{gen}) will be the consequence of $(w_\lambda)_\lambda = (v_\lambda - u_\lambda)_\lambda \in \mathcal{N}(\Delta)$.

Remark 5: Dependence on some regularizing family. The problem (P_{gen}) itself, so a solution of it, a priori depends on the family of cutoff functions and, in the case of irregular data, on the family of mollifiers. If $(\theta_\rho)_{\rho\in\Lambda_3}$ and $(\tau_\rho)_{\rho\in\Lambda_3}$ are families of mollifiers in $\mathcal{D}(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$, it is well known that generally $[T * \theta_\rho] \neq [T * \tau_\rho]$ in the Colombeau simplified algebra even if $[\theta_\rho] = [\tau_\rho]$. Therefore, in the case of irregular data, the solution of Problem (P_{gen}) in some Colombeau algebra depends on the family of mollifiers $(\theta_\rho)_\rho$ but not on a class of that family.

4. Solving the non Lipschitz Goursat problem

4.1 Solution to (P_{gen})

Theorem 2: If u_λ is the solution to problem (P_λ), then problem (P_{gen}) admits $[u_\lambda]_{\mathcal{A}(\Delta)}$ as solution.

Proof: We have

$$u_\lambda(x,y) = u_{0,\lambda}(x,y) + \int_0^x \int_0^y F_\varepsilon(\xi,\eta,u_\lambda(\xi,\eta),(u_\lambda)_x(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

where $u_{0,\lambda}(x,y) = \varphi_\rho(x) - \psi_\rho(y) - \psi_\rho(0)$. Then $(u_{0,\lambda})_x(x,y) = \varphi'_\rho(x)$. We will actually prove that $\forall K \Subset \Delta$, $(P_{K,n}(u_\lambda))_\lambda \in |A|$ for all n in $\mathbb{N}$.

Moreover as $\varphi, \psi \in \mathcal{A}(\mathbb{R}_+)$, we have

$$\forall l \in \mathbb{N}, (P_{K,l}(u_{0,\lambda}))_\lambda \in |A|\,, \left(P_{K,l}\left((u_{0,\lambda})_x\right)\right)_\lambda \in |A|\,.$$

We have $\forall K \Subset \Delta$, $\exists K_a = [0,a]^2 \subset \Delta$ and thanks to Corollary 1,

$$\|u_\lambda\|_{\infty,K} \le \|u_\lambda\|_{\infty,K_a} \le \|u_{0,\lambda}\|_{\infty,K_a} + \tfrac{a\Phi_{a,\varepsilon}}{M_\varepsilon(a+1)} e^{M_\varepsilon a(a+1)}.$$

We have $(\|u_{0,\lambda}\|_{\infty,K_a})_\lambda \in A$. Then, A being stable, we have $(\|u_\lambda\|_{\infty,K_a})_\lambda \in |A|$ and $(\|u_\lambda\|_{\infty,K})_\lambda \in |A|$, that is $(P_{K,0}(u_\lambda))_\lambda \in |A|$ then the 0th order estimate is verified.

Let us show that $(P_{K,1}(u_\lambda))_\lambda \in |A|$. Thanks to Corollary 1, we have

$$\|(u_\lambda)_x\|_{\infty,K} \le \|(u_\lambda)_x\|_{\infty,K_a} \le \left\|(u_{0,\lambda})_x\right\|_{\infty,K_a} + \tfrac{\Phi_{a,\varepsilon}}{M_\varepsilon(a+1)} e^{M_\varepsilon a(a+1)}.$$

Moreover $(\left\|(u_{0,\lambda})_x\right\|_{\infty,K_a})_\lambda \in |A|$, then we get $(P_{K,(1,0)}(u_\lambda))_\lambda \in |A|$. We have

$$(u_\lambda)_y(x,y) = (u_{0,\lambda})_y(x,y) + \int_0^x F_\varepsilon(\xi,y,u_\lambda(\xi,y),(u_\lambda)_x(\xi,y))\,\mathrm{d}\xi,$$

thus

$$P_{K,(0,1)}(u_\lambda) \le \sup_{(x,y)\in K} \left|(u_{0,\lambda})_y(x,y)\right| + a \sup_{(x,y)\in K_a} \left|\left(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)\right)(x,y)\right|.$$

We obtain $P_{K,(0,1)}(u_\lambda) \le \left\|(u_{0,\lambda})_y\right\|_{\infty,K} + ac_0 r_\varepsilon^d$ and then $(\left\|(u_\lambda)_y\right\|_{\infty,K_a})_\lambda \in |A|$. Finally

$$(P_{K,1}(u_\lambda))_\lambda \in A.$$

Now we proceed by induction. Suppose that $(P_{K,l}(u_\lambda))_\lambda \in |A|$ for every $l \leq n$ and let us show that implies $(P_{K,n+1}(u_\lambda))_\lambda \in |A|$.

In fact we have $P_{K,n+1} = \max(P_{K,n}, P_{1,n}, P_{2,n}, P_{3,n}, P_{4,n})$ with

$$P_{1,n} = P_{K,(n+1,0)},\ P_{2,n} = P_{K,(0,n+1)},$$
$$P_{3,n} = \sup_{\alpha+\beta=n;\ \beta\geq 1} P_{K,(\alpha+1,\beta)},\ P_{4,n} = \sup_{\alpha+\beta=n;\ \alpha\geq 1} P_{K,(\alpha,\beta+1)}.$$

First let us show that $(P_{1,n}(u_\lambda))_\lambda, (P_{2,n}(u_\lambda))_\lambda \in |A|$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geq 1$,

$$\frac{\partial^{n+1} u_\lambda}{\partial x^{n+1}}(x,y) = \frac{\partial^{n+1} u_{0,\lambda}}{\partial x^{n+1}}(x,y) + \int_0^y \frac{\partial^n}{\partial x^n} F_\varepsilon(x,\eta,u_\lambda(x,\eta),(u_\lambda)_x(x,\eta))d\eta.$$

As $K \subset K_a$, we can write

$$\sup_{(x,y)\in K}\left|\frac{\partial^{n+1} u_\lambda}{\partial x^{n+1}}(x,y)\right| \leq \left\|\frac{\partial^{n+1} u_{0,\lambda}}{\partial x^{n+1}}\right\|_{\infty,K} + a\sup_{(x,y)\in K_a}\left|\left(\frac{\partial^n}{\partial x^n}F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)\right)(x,y)\right|.$$

Moreover

$$\sup_{(x,y)\in K_a}\left|\left(\frac{\partial^n}{\partial x^n}F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)\right)(x,y)\right| \leq P_{K_a,n}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)) \leq c_n r_\varepsilon^d$$

and $(\|\partial^{n+1}/\partial x^{n+1}\ u_{0,\lambda}\|_{\infty,K})_\lambda \in |A|$. According to the stability hypothesis, a simple calculation shows that, for every $K \Subset \Omega$, $(P_{K,(n+1,0)}(u_\lambda))_\lambda \in |A|$.

Let us show that $(P_{2,n}(u_\lambda))_\lambda \in A$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geq 1$,

$$\frac{\partial^{n+1} u_\lambda}{\partial y^{n+1}}(x,y) = \frac{\partial^{n+1} u_{0,\lambda}}{\partial y^{n+1}}(x,y) + \int_0^x \frac{\partial^n}{\partial y^n} F_\varepsilon(\xi,y,u_\lambda(\xi,y),(u_\lambda)_x(\xi,y))d\xi.$$

As $K \subset K_a$, we can write

$$\sup_{(x,y)\in K}\left|\frac{\partial^{n+1} u_\lambda}{\partial y^{n+1}}(x,y)\right| \leq P_{K,(0,n+1)}(u_{0,\lambda}) + a\sup_{(x,y)\in K_a}\left|\frac{\partial^n}{\partial y^n}F_\varepsilon(x,y,u_\lambda(x,y),(u_\lambda)_x(x,y))\right|,$$

we have

$$\sup_{(x,y)\in K_a}\left|\frac{\partial^i}{\partial y^i}F(x,y,u_\lambda(x,y),(u_\lambda)_x(x,y))\right| \leq P_{K_a,n}(F(\cdot,\cdot,u_\lambda,(u_\lambda)_x) \leq c_n r_\varepsilon^d.$$

Then, for any $n \in \mathbb{N}$, $\left(P_{K,(0,n+1)}(u_\lambda)\right)_\lambda \in A$. We deduce that $(P_{2,n}(u_\lambda))_\lambda \in A$.

For $\alpha + \beta = n$ and $\beta \geq 1$, we now have

$$\begin{aligned} P_{K,(\alpha+1,\beta)}(u_\lambda) &= \sup_{(x,y)\in K}\left|\left(D^{(\alpha,\beta-1)}D^{(1,1)}u_\lambda\right)(x,y)\right| \\ &= \sup_{(x,y)\in K}\left|D^{(\alpha,\beta-1)}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x))(x,y)\right| \\ &= P_{K,(\alpha,\beta-1)}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)) \leq P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)) \leq c_n r_\varepsilon^d. \end{aligned}$$

Then we finally have $P_{3,n}(u_\lambda) = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)}(u_\lambda) \leq c_n r_\varepsilon^d$ and the stability hypothesis ensures that $(P_{3,n}(u_\lambda))_\lambda \in |A|$.

In the same way, for $\alpha+\beta=n$ and $\alpha\geq 1$, we have

$$\begin{aligned}P_{K,(\alpha,\beta+1)}(u_\lambda) &= \sup_{(x,y)\in K}\left|\left(D^{(\alpha-1,\beta)}D^{(1,1)}u_\lambda\right)(x,y)\right| \\ &= \sup_{(x,y)\in K}\left|\left(D^{(\alpha-1,\beta)}F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)\right)(x,y)\right| \\ &= P_{K,(\alpha-1,\beta)}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)) \leq P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)) \leq c_n r_\varepsilon^d.\end{aligned}$$

Thus we have $P_{4,n}(u_\lambda) = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}(u_\lambda) \leq c_n r_\varepsilon^d$ and the stability hypothesis ensures that $(P_{4,n}(u_\lambda))_\lambda \in |A|$. Finally, we clearly have $(P_{K,n+1}(u_\lambda))_\lambda \in |A|$, consequently $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$. □

4.2 Independence of the generalized solution from the class of cut off functions

See (Dévoué, 2009a; Dévoué, 2009b). Recall that $\Lambda_1 = (0,1]$, set

$$\begin{aligned}&\mathcal{X}_1(\mathbb{R}_+) = \{(g_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\mathbb{R}_+)]^{\Lambda_1} : \forall K \Subset \mathbb{R}_+, \forall l\in\mathbb{N},\ (P_{K,l}(g_\varepsilon))_\varepsilon \in |A|\},\\ &\mathcal{N}_1(\mathbb{R}_+) = \{(g_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\mathbb{R}_+)]^{\Lambda_1} : \forall K \Subset \mathbb{R}_+, \forall l\in\mathbb{N},\ (P_{K,l}(g_\varepsilon))_\varepsilon \in |I_A|\},\\ &\mathcal{A}_1(\mathbb{R}_+) = \mathcal{X}_1(\mathbb{R}_+)/\mathcal{N}_1(\mathbb{R}_+).\end{aligned}$$

Consider $\mathcal{T}(\mathbb{R}_+)$ the set of families of smooth one-variable functions $(h_\varepsilon)_{\varepsilon\in\Lambda_1} \in \mathcal{X}_1(\mathbb{R}_+)$, verifying the following assumptions

$$\exists\,(s_\varepsilon)_\varepsilon \in \left(\mathbb{R}_+^*\right)^{(0,1]} : \sup_{z\in[0,s_\varepsilon]}|h_\varepsilon(z)| = 1,\ h_\varepsilon(z) = \begin{cases} 0, \text{ if } z\geq s_\varepsilon \\ 1, \text{ if } 0\leq z\leq s_\varepsilon - 1.\end{cases}$$

$$\exists l\in\mathbb{N}^*, \forall\,(h_\varepsilon)_\varepsilon \in \mathcal{T}(\mathbb{R}_+), \forall\varepsilon, s_\varepsilon \leq r_\varepsilon^l. \tag{6}$$

Moreover assume that $\dfrac{\partial^n h_\varepsilon}{\partial z^n}$, $\dfrac{\partial^n h_\varepsilon}{\partial p^n}$ are bounded on $J_\varepsilon = [0,s_\varepsilon]$ for any integer n, $n>0$.

We have $(g_\varepsilon)_{\varepsilon\in\Lambda_1} \in \mathcal{T}(\mathbb{R}_+)$. Recall that $\phi_\varepsilon(z) = zg_\varepsilon(z)$ for $z\in\mathbb{R}_+$, $F_\varepsilon(x,y,z,p) = F(x,y,\phi_\varepsilon(z),\phi_\varepsilon(p))$ for $(x,y,z,p)\in\Delta\times\mathbb{R}^2$ and

$$\sup_{z\in[0,r_\varepsilon]}\left|\frac{\partial^n g_\varepsilon}{\partial z^n}(z)\right| \leq M_n;\ \sup_{p\in[0,r_\varepsilon]}\left|\frac{\partial^n g_\varepsilon}{\partial p^n}(z)\right| \leq M_n.$$

Let $g\in\mathcal{T}(\mathbb{R}_+)/\mathcal{N}_1(\mathbb{R}_+)$ be the class of $(g_\varepsilon)_\varepsilon$. Take $(h_\varepsilon)_\varepsilon$ another representative of g, that is to say $(h_\varepsilon)_\varepsilon \in \mathcal{T}(\mathbb{R}_+)$ and $(g_\varepsilon - h_\varepsilon)_\varepsilon \in \mathcal{N}_1(\mathbb{R}_+)$.

Set $\sigma_\varepsilon(z) = zh_\varepsilon(z)$ for $z\in\mathbb{R}$, $H_\varepsilon(x,y,z,p) = F(x,y,\sigma_\varepsilon(z),\sigma_\varepsilon(p))$ for $(x,y,z,p)\in\Delta\times\mathbb{R}^2$ and

$$\sup_{z\in[0,s_\varepsilon]}\left|\frac{\partial^n h_\varepsilon}{\partial z}(z)\right| \leq M'_n;\ \sup_{p\in[0,s_\varepsilon]}\left|\frac{\partial^n h_\varepsilon}{\partial p^n}(z)\right| \leq M'_n.$$

Our choice is made such that $(\operatorname{supp}(h_\varepsilon))_\varepsilon$ have the same growth as $(\operatorname{supp}(g_\varepsilon))_\varepsilon$ with respect to the scale $\left(r_\varepsilon^l\right)_\varepsilon$, in this way the corresponding solutions are lying in the same algebra $\mathcal{A}(\Omega)$.

Proposition 4: Set $S_n = \{\alpha\in\mathbb{N}^4 : |\alpha| = n\}$ when $n\in\mathbb{N}^*$. Let $F\in\mathrm{C}^\infty(\mathbb{R}^4,\mathbb{R})$, H_ε defined by

$$H_\varepsilon(x,y,z,p) = F(x,y,\sigma_\varepsilon(z),\sigma_\varepsilon(p)).$$

where σ_ε is defined previously. Assume that

$$\forall (x,y) \in \Omega, F(x,y,0,0) = 0 \,,$$

$$\exists d_0 > 0, \forall \alpha \in \mathbb{N}^4, |\alpha| = n > d_0, D^\alpha F(x,y,z,p) = 0,$$

$$\forall n \in \mathbb{N}, n \leq d_0, \exists k_n > 0, \forall \varepsilon \in \Lambda_1, \forall K \Subset \Omega, \sup_{\substack{(x,y)\in K;(z,p)\in J_\varepsilon^2,\\ \alpha\in Sn}} |D^\alpha F(x,y,z,p)| \leq k_n r_\varepsilon^{d_0},$$

then

$$\forall n \in \mathbb{N}, n \leq d_0, \exists c_n > 0, \forall \varepsilon \in \Lambda_1, \forall K \Subset \Omega, \sup_{\substack{(x,y)\in K;(z,p)\in J_\varepsilon^2,\\ \alpha\in Sn}} |D^\alpha H_\varepsilon(x,y,z,p)| \leq c_n r_\varepsilon^{d_0(1+l)}$$

and $\mathcal{A}(\Omega)$ is stable under the family $(H_\varepsilon)_{(\varepsilon,\rho)}$.

We refer the reader to (Dévoué, 2009a; Dévoué, 2009b) for a similar detailed proof.

Theorem 3: Assume that $d = d_0(1+l)$ and the hypotheses of Proposition 4 are verified. Let $\mathcal{F}$ the generalized operator associated to F *via* the family $(g_\varepsilon)_\varepsilon$. Let $(h_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$ be another family representative of the class $[g_\varepsilon] = g$ and leading to another generalized operator $\mathcal{H}$ associated to F. Then we have $\mathcal{H} = \mathcal{F}$, that is to say $\mathcal{H}(u) = \mathcal{F}(u)$ for any $u \in \mathcal{A}(\Delta)$. In terms of representatives, that is to say, if $(u_\lambda)_\lambda$, $(v_\lambda)_\lambda \in \mathcal{X}(\Delta)$ and $(w_\lambda)_\lambda = (v_\lambda - u_\lambda)_\lambda \in \mathcal{N}(\Delta)$, if

$$F(\cdot,\cdot,\sigma_\varepsilon(v_\lambda),\sigma_\varepsilon((v_\lambda)_x)) - F(\cdot,\cdot,\phi_\varepsilon(v_\lambda),\phi_\varepsilon((v_\lambda)_x)) = T(\sigma_\varepsilon(v_\lambda),\phi_\varepsilon(v_\lambda))$$

then $(T(\sigma_\varepsilon(v_\lambda),\phi_\varepsilon(v_\lambda)))_\lambda \in \mathcal{N}(\Delta)$.

We refer the reader to (Dévoué, 2009a; Dévoué, 2009b) for a similar detailed proof.

Corollary 2: Problem (P_{gen}), a fortiori its solutions, does not depend of the choice of the representative $(g_\varepsilon)_\varepsilon$ of the class $g \in \mathcal{T}(\mathbb{R}_+)/\mathcal{N}_1(\mathbb{R}_+)$.

Proof: $(w_\lambda)_\lambda = (v_\lambda - u_\lambda)_\lambda \in \mathcal{N}(\Delta)$ then $((w_\lambda)_x)_\lambda \in \mathcal{N}(\Delta)$. We deduce that

$$(T(\sigma_\varepsilon(v_\lambda),\phi_\varepsilon(u_\lambda)))_\lambda \in \mathcal{N}(\Delta),$$

that is to say $\mathcal{H}(u) = \mathcal{F}(u)$ for any $u \in \mathcal{A}(\Delta)$. □

4.3 Uniqueness of the solution

Using the Pachpatte lemma, (Pachpatte, 2009, p.42), we can prove the main result.

Theorem 4: The solution to Problem (P_{gen}) *is unique* in the algebra $\mathcal{A}(\Delta)$.

Proof: Let $[u_\lambda]_{\mathcal{A}(\Delta)}$ be the solution to (P_{gen}) obtain in Theorem 2. Let $v = [v_\varepsilon]$ be another solution to (P_{gen}). There are $(i_\lambda)_\lambda \in \mathcal{N}(\Delta)$ and $(\alpha_\rho)_\rho$, $(\beta_\rho)_\rho \in \mathcal{N}(\mathbb{R}_+)$, such that

$$\begin{cases} \dfrac{\partial^2 v_\lambda}{\partial x \partial y}(x,y) = F_\varepsilon(x,y,v_\lambda(x,y),(v_\lambda)_x(x,y)) + i_\lambda(x,y)\,, \\ v_\lambda(x,0) = \varphi_\rho(x) + \alpha_\rho(x), \quad v_\lambda(0,y) = \psi_\rho(y) + \beta_\rho(y), \\ \psi_\rho(0) = \varphi_\rho(0), \alpha_\rho(0) = \beta_\rho(0)\,. \end{cases}$$

The uniqueness of the solution to (P_G) will be consequence of $(v_\lambda - u_\lambda)_\lambda \in \mathcal{N}(\Delta)$.

$$\iint_{D(x,y)} i_\lambda(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = xy\left(i_\lambda(x_\varepsilon,y_\varepsilon)\right)$$

where $i_\lambda(x_\varepsilon,y_\varepsilon)$ is the average value of i_λ on $D(x,y)$. Then

$$(j_\lambda : (x,y) \mapsto \iint_{D(x,y)} i_\lambda(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta)_\lambda \in \mathcal{N}\left(\Delta\right).$$

So $(j_\lambda)_\lambda \in \mathcal{N}\left(\Delta\right)$ and

$$v_\lambda(x,y) = v_{0,\lambda}(x,y) + \int_0^x \int_0^y F_\varepsilon(\xi,\eta,v_\lambda(\xi,\eta),\left(v_\lambda\right)_x(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta + j_\lambda(x,y),$$

with $v_{0,\lambda}(x,y) = u_{0,\lambda}(x,y)+\theta_\rho(x,y)$, where $\theta_\rho(x,y) = \alpha_\rho(x)+\beta_\rho(y)-\beta_\rho(0)$. So $(\theta_\rho)_{\varepsilon,\rho}$ belongs to $\mathcal{N}(\Delta)$. Hence there is $(\sigma_\lambda)_\lambda \in \mathcal{N}(\Delta)$ such that

$$v_\lambda(x,y) = u_{0,\lambda}(x,y) + \sigma_\lambda(x,y) + \int_0^x \int_0^y \left(F_\varepsilon(\cdot,\cdot,v_\lambda,\left(v_\lambda\right)_x)\right)(\xi,\eta)\mathrm{d}\xi\,\mathrm{d}\eta.$$

Let us put $w_\lambda = v_\lambda - u_\lambda$ and show that $(w_\lambda)_{(\lambda)} \in \mathcal{N}(\Delta)$. Take K a compact of Δ, K is contained in some compact $[0,a]^2 = K_a$. We have to prove that $(P_{K,0}(w_\lambda))_\lambda \in I_A$. Let $(x,y) \in K$, we have

$$w_\lambda(x,y) = \sigma_\lambda(x,y) + \int_0^x \int_0^y \left(F_\varepsilon(\cdot,\cdot,v_\lambda,\left(v_\lambda\right)_x) - F_\varepsilon(\cdot,\cdot,u_\lambda,\left(u_\lambda\right)_x)\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

then

$$\left(w_\lambda\right)_x(x,y) = \left(\sigma_\lambda\right)_x(x,y) + \int_0^y \left(F_\varepsilon(\cdot,\cdot,v_\lambda,\left(v_\lambda\right)_x) - F_\varepsilon(\cdot,\cdot,u_\lambda,\left(u_\lambda\right)_x)\right)(x,\eta)\,\mathrm{d}\eta.$$

So

$$|w_\lambda(x,y)| = |\sigma_\lambda(x,y)| + \int_0^x \int_0^y \left|F_\varepsilon(\cdot,\cdot,v_\lambda,\left(v_\lambda\right)_x) - F_\varepsilon(\cdot,\cdot,u_\lambda,\left(u_\lambda\right)_x)\right|(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{7}$$

and

$$\left|\left(w_\lambda\right)_x(x,y)\right| = \left|\left(\sigma_\lambda\right)_x(x,y)\right| + \int_0^y \left|F_\varepsilon(\cdot,\cdot,v_\lambda,\left(v_\lambda\right)_x) - F_\varepsilon(\cdot,\cdot,u_\lambda,\left(u_\lambda\right)_x)\right|(x,\eta)\,\mathrm{d}\eta. \tag{8}$$

But

$$\left|\left(F_\varepsilon(\cdot,\cdot,v_\lambda,\left(v_\lambda\right)_x) - F_\varepsilon(\cdot,\cdot,u_\lambda,\left(u_\lambda\right)_x)\right)(\xi,\eta)\right| \leq M_\varepsilon\left(|w_\lambda(\xi,\eta)| + \left|\left(w_\lambda\right)_x(\xi,\eta)\right|\right).$$

Since $D(x,y) \subset K_a$, according to 7, we have

$$|w_\lambda(x,y)| \leq \|\sigma_\lambda\|_{\infty,K_a} + \int_0^x \int_0^y M_\varepsilon\left(|w_\lambda(\xi,\eta)| + \left|\left(w_\lambda\right)_x(\xi,\eta)\right|\right)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

and, according to 8, we have

$$\left|(w_\lambda)_x(x,y)\right| \leq \left\|(\sigma_\lambda)_x\right\|_{\infty,K_a} + \int\limits_0^y M_\varepsilon\left(|w_\lambda(x,\eta)| + \left|(w_\lambda)_x(x,\eta)\right|\right)\mathrm{d}\eta.$$

Set

$$E(\xi,\eta) = \left(|w_\lambda(\xi,\eta)| + \left|(w_\lambda)_x(\xi,\eta)\right|\right). \tag{9}$$

We obtain

$$E(x,y) \leq k + \int\limits_0^y M_\varepsilon E(x,\eta)\,\mathrm{d}\eta + \int\limits_0^x\int\limits_0^y M_\varepsilon E(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta.$$

with $k = \|\sigma_\lambda\|_{\infty,K_a} + \left\|(\sigma_\lambda)_x\right\|_{\infty,K_a}$. Then, according to Pachpatte lemma we have

$$E(x,y) \leq kH(x,y)\exp(\int\limits_0^x\int\limits_0^y M_\varepsilon H(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta)$$

where $H(x,y) = \exp(\int\limits_0^y M_\varepsilon\,\mathrm{d}\eta) = e^{yM_\varepsilon}$. So

$$\exp(\int\limits_0^x\int\limits_0^y M_\varepsilon H(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta) = \exp(\int\limits_0^x\int\limits_0^y M_\varepsilon e^{\eta M_\varepsilon}\,\mathrm{d}\xi\mathrm{d}\eta) = \exp((e^{yM_\varepsilon}-1)x).$$

We deduce that

$$E(x,y) \leq ke^{yM_\varepsilon}\exp((e^{yM_\varepsilon}-1)x).$$

Thus

$$|w_\lambda(x,y)| \leq (\|\sigma_\lambda\|_{\infty,K_a} + \left\|(\sigma_\lambda)_x\right\|_{\infty,K_a})e^{yM_\varepsilon}\exp((e^{yM_\varepsilon}-1)x).$$

Since $(\sigma_\lambda)_\lambda \in \mathcal{N}(\Delta)$, we have

$$(\|\sigma_\lambda\|_{\infty,K_a})_\lambda \in I_A, (\left\|(\sigma_\lambda)_x\right\|_{\infty,K_a})_\lambda \in I_A,$$

then $(\|w_\lambda\|_{\infty,K_a})_\lambda \in I_A$. This implies the 0th order estimate. According to Proposition 1, $(w_\lambda)_\lambda \in \mathcal{N}(\Delta)$ and consequently u is the unique solution to (P_G). □

Remark 6: Case of regular data. If the data φ and ψ are smooth, we take $\varepsilon \in \Lambda = (0,1]$. Let $(r_\varepsilon)_\varepsilon$ be in $(\mathbb{R}_+^*)^{(0,1]}$ such that $\lim\limits_{\varepsilon\to 0} r_\varepsilon = +\infty$. We take $\mathcal{C} = A/I_A$ the ring overgenerated by $(\varepsilon)_\varepsilon$, $(r_\varepsilon)_\varepsilon$, $(e^{r_\varepsilon})_\varepsilon$, elements of $(\mathbb{R}_+^*)^{(0,1]}$. Then $\mathcal{A}(\Delta) = \mathcal{X}(\Delta)/\mathcal{N}(\Delta)$ is built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E},\mathcal{P}) = \left(\mathrm{C}^\infty(\Delta), (P_{K,l})_{K\Subset\Omega, l\in\mathbb{N}}\right)$ and $\mathcal{A}(\mathbb{R}_+) = \mathcal{X}(\mathbb{R}_+)/\mathcal{N}(\mathbb{R}_+)$ is built on the ring $\mathcal{C}$ of generalized constants with $(\mathcal{E},\mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}_+), (P_{K,l})_{K\Subset\mathbb{R}, l\in\mathbb{N}}\right)$. Nonetheless, the algebra $\mathcal{A}(\Delta)$ is not the same in the two cases, regular data and irregular data. We get similar results replacing φ_ρ by φ and ψ_ρ by ψ. As previously, we can prove that Problem (P_{gen}) has a generalized solution $u = [u_\varepsilon]$ in the algebra $\mathcal{A}(\Delta)$.

4.4 Comparison with classical solutions

Even if the data are as irregular as distributions, it may happen that the initial formal ill-posed problem (P_{form}) has nonetheless a local smooth solution. We are going to prove that this solution is exactly the restriction (according to the sheaf theory sense) of the generalized one.

The generalized solution to Problem (P_{gen}) is defined from the integral representation (Int). Thus, we are going to study the relationship between this generalized function and the classical solutions to (P_{form}) (when they exist) on a domain O such that $\forall\,(x,y) \in O$, $D(x,y) \subset O$. This justified to choose $O = \left]0,a\right[\times \left]0,b\right[$ with $0 < a$ and $0 < b$.

Remark 7: If the non regularized problem (P_{form}) has a smooth solution v on O then, necessarily we have $O \subset \mathbb{R}^2 \backslash \mathrm{singsupp}\,(u)$.

Recall that there exists a canonical sheaf embedding of $\mathrm{C}^\infty(\cdot)$ into $\mathcal{A}\,(\cdot)$, through the morphism of algebra

$$\sigma_O : \mathrm{C}^\infty\,(O) \to \mathcal{A}\,(O)\,, \quad f \mapsto [f_\lambda]\,,$$

where O is any open subset of Ω and $f_\lambda = f$. The presheaf $\mathcal{A}$ allows to restriction and as usually we denote by $u|_O$ the restriction on O of $u \in \mathcal{A}\,(\Delta)$.

We take φ, $\psi \in \mathrm{C}^\infty(\mathbb{R}_+)$, $\varphi_\rho = \varphi$, $\psi_\rho = \psi$.

Theorem 5: Let $u = [u_\lambda]$ be the solution to Problem (P_{gen}). Let O be an open subset of Δ such that $O \subset \mathbb{R}^2 \backslash \mathrm{singsupp}\,(u)$. Assume that $O = \bigcup\limits_{\varepsilon \in \Lambda_1} O_\varepsilon$ with $(O_\varepsilon)_\varepsilon$ is an increasing family of open subsets of Δ such that $O_\varepsilon = \left]0,a_\varepsilon\right[\times \left]0,b_\varepsilon\right[$ with $0 < a_\varepsilon$, $0 < b_\varepsilon$. Assume that problem (P_{form}) has a smooth solution v on O such that $\sup\limits_{(x,y)\in O_\varepsilon} |v(x,y)| < r_\varepsilon - 1$, $\sup\limits_{(x,y)\in O_\varepsilon} |v_x(x,y)| < r_\varepsilon - 1$ for any ε. Then v (element of $\mathrm{C}^\infty\,(O)$ canonically embedded in $\mathcal{A}(O)$), is the restriction (according to the sheaf theory sense) of u to O, $v = \sigma_O\,(v) = u|_O$.

Proof: We clearly have $\forall\,(x,y) \in O, \exists \varepsilon_0, \forall \varepsilon \le \varepsilon_0, (x,y) \in O_\varepsilon$. Then $D(x,y) \subset O_\varepsilon \subset O$; we have

$$v(x,y) = v_0(x,y) + \iint_{D(x,y)} F(\xi,\eta,v(\xi,\eta),v_x(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

We take has representative of u the family $(u_\lambda)_\lambda$ with $\lambda = (\varepsilon,\rho)$. We have, for any $(x,y) \in O$,

$$u_\lambda(x,y) = u_{0,\lambda}(x,y) + \iint_{D(x,y)} F_\varepsilon(\xi,\eta,u_\lambda(\xi,\eta),(u_\lambda)_x\,(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

where $u_{0,\lambda}(x,y) = \varphi_\rho(x) + \psi_\rho(y) - \psi_\rho(0)$. Moreover we have $v_0(x,y) = u_{0,\lambda}(x,y)$ and

$$(u_\lambda)_x\,(x,y) = (u_{0,\lambda})_x\,(x,y) + \int\limits_0^y \left(F(\cdot,\cdot,u_\lambda,(u_\lambda)_x)\right)(x,\eta)\,\mathrm{d}\eta.$$

Set $(w_\lambda)_\lambda = (u_\lambda|_O - v)_\lambda$ and take $K \Subset O$. There exists ε_1 such that, for all $\varepsilon < \varepsilon_1$, $K \Subset O_\varepsilon$. According to the definition of O_ε, there exist a_ε, b_ε such that $K \subset \left]0,a_\varepsilon\right[\times \left]0,b_\varepsilon\right[= O_\varepsilon$. Take

$(x,y)\in K$, then $D(x,y)\subset O_\varepsilon$, we have

$$w_\lambda(x,y)=\iint_{D(x,y)}\left(F(\cdot,\cdot,v,v_x)-F_\varepsilon(\cdot,\cdot,v,v_x)\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ +\iint_{D(x,y)}\left(F_\varepsilon(\cdot,\cdot,v,v_x)-F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

and

$$(w_\lambda)_x(x,y)=\int_0^y\left(F(\cdot,\cdot,v,v_x)-F_\varepsilon(\cdot,\cdot,v,v_x)\right)(x,\eta)\mathrm{d}\eta \\ +\int_0^y\left(F_\varepsilon(\cdot,\cdot,v,v_x)-F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)\right)(x,\eta)\mathrm{d}\eta.$$

Note that, for $(\xi,\eta,z,p)\in O_\varepsilon\times(]0,r_\varepsilon-1[)^2$, we have $F(\xi,\eta,z,p)=F_\varepsilon(\xi,\eta,z,p)$ by construction of F_ε. As values of v, v_x are in $]0,r_\varepsilon-1[$ we have $F(\cdot,\cdot,v,v_x)-F_\varepsilon(\cdot,\cdot,v,v_x)=0$. Thus

$$w_\lambda(x,y)=\iint_{D(x,y)}\left(F_\varepsilon(\cdot,\cdot,v,v_x)-F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

and

$$(w_\lambda)_x(x,y)=\int_0^y\left(F_\varepsilon(\cdot,\cdot,v,v_x)-F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)\right)(x,\eta)\mathrm{d}\eta.$$

But

$$\left|\left(F_\varepsilon(\cdot,\cdot,v,v_x)-F_\varepsilon(\cdot,\cdot,u_\lambda,(u_\lambda)_x)\right)(\xi,\eta)\right|\le M_\varepsilon\left(|w_\lambda(\xi,\eta)|+|(w_\lambda)_x(\xi,\eta)|\right).$$

Since $D(x,y)\subset O_\varepsilon$, we have

$$|w_\lambda(x,y)|\le\int_0^x\int_0^y M_\varepsilon\left(|w_\lambda(\xi,\eta)|+|(w_\lambda)_x(\xi,\eta)|\right)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Set

$$E(\xi,\eta)=\left(|w_\lambda(\xi,\eta)|+|(w_\lambda)_x(\xi,\eta)|\right).$$

So

$$|w_\lambda(x,y)|\le\int_0^x\int_0^y M_\varepsilon\left(|w_\lambda|+|(w_\lambda)_x|\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\le\int_0^x\int_0^y M_\varepsilon E(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta.$$

Moreover

$$|(w_\lambda)_x(x,y)|\le\int_0^y M_\varepsilon\left(|w_\lambda|+|(w_\lambda)_x|\right)(x,\eta)\,\mathrm{d}\eta\le\int_0^y M_\varepsilon E(x,\eta)\,\mathrm{d}\eta.$$

So

$$E(x,y)\le\int_0^y M_\varepsilon E(x,\eta)\,\mathrm{d}\eta+\int_0^x\int_0^y M_\varepsilon E(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta.$$

According to Pachpatte lemma, (Pachpatte, 2009, p.42), $E(x,y)=0$, so $w_\lambda=0$. Thus v and u_λ are solutions of the same integral equation, which admits a unique solution since F_ε is a smooth function of its arguments. Thus, for all $\varepsilon \leq \varepsilon_1$, v and u_λ, v_x and $(u_\lambda)_x$ are equal on O_ε. We deduce that v and u_λ are solutions of the same integral equation, which admits a unique solution. Thus $(P_{K,n}(v))_\lambda \in |A|$ for any $K \Subset O$ and $n \in \mathbb{N}$. Then v (identified with $[v_\lambda]$) belongs to $\mathcal{A}(O)$. Moreover, for all $\varepsilon \leq \varepsilon_1$, $\sup_{(x,y)\in O_\varepsilon} |w_\lambda(x,y)| = 0$, hence $(P_{K,l}(w_\lambda\ ))_\lambda \in |I_A|$ for any $l \in \mathbb{N}$ as w_λ vanishes on K. Thus $(w_\lambda)_\lambda \in \mathcal{N}(O)$ and $v = u|_O$ as claimed. □

Example 1: Assume that $\lambda = (\varepsilon, \rho) \in \Lambda_1 \times \Lambda_2 = \Lambda$, $\Lambda_1 = \Lambda_2 = (0,1]$. Consider the problem

$$(P_{form}) : u_{xy} = F(\cdot,\cdot,u,u_x),\, u|_{(Ox)} = \varphi,\, u|_{(Oy)} = vp(\frac{1}{1-y})\Big|_{(Oy)}.$$

where $F(x,y,u(x,y),u_x(x,y)) = (\exp(-x^2))\, u(x,y)u_x(x,y)$ and $\varphi(x) = \exp(x^2)$. We take $\phi = vp(\frac{1}{1-y})$ and we have

$$\phi|_{(Oy)} = \left(y \mapsto \frac{1}{1-y}\right) = \psi.$$

This problem is classically ill-posed. According to our previous notations, let (P_{gen}) be the generalized associated problem

$$(P_{gen}) : \frac{\partial^2 u}{\partial x \partial y} = \mathcal{F}(u),\, u|_{(Ox)} = \varphi,\, u|_{(Oy)} = \psi,$$

where φ and ψ, elements of $\mathrm{C}^\infty(\mathbb{R}_+)$ canonically embedded in $\mathcal{A}(\mathbb{R}_+)$, are respectively the generalized functions $\sigma_{\mathbb{R}_+}(\varphi)$, $\sigma_{\mathbb{R}_+}(\psi)$ and $\phi = [\phi_\rho] \in \mathcal{A}(\mathbb{R}_+)$ with

$$\phi_\rho(y) = \left(\theta_\rho * vp(\frac{1}{1-\cdot})\right)(y) = \langle vp(\frac{1}{1-z}), z \mapsto \theta_\rho(y-z)\rangle = \lim_{\varepsilon\to 0}\int_{|1-z|>\varepsilon} \frac{\theta_\rho(y-z)}{1-z}dz,$$

where $(\theta_\rho)_\rho$ is a chosen family of mollifiers. To solve the Problem (P_{gen}) associate to (P_{form}) we can consider the family of problems

$$(P_\lambda) \begin{cases} \dfrac{\partial^2 u_\lambda}{\partial x \partial y}(x,y) = F_\varepsilon(x,y,u_\lambda(x,y),(u_\lambda)_x(x,y)), \\ u_\lambda(x,0) = \varphi_\rho(x),\, u_\lambda(0,y) = \psi_\rho(y),\, \psi_\rho(0) = \varphi_\rho(0), \end{cases}$$

If u_λ is a solution to (P_λ) then $u = [u_\lambda]$ is solution to (P_{gen}). Moreover (P_{form}) has the classical solution v in $\mathrm{C}^\infty(O)$, where $O =]0,+\infty[\times]0,1[$, defined by $v(x,y) = (\exp x^2)\frac{1}{(1-y)}$. Theorem 5 shows that the restriction of $u \in \mathcal{A}(\Delta)$ to O is precisely v. The local classical solution v which blows-up for $y=1$, extends to a global generalized solution u which absorbs this blow-up.

5. Conclusions

Given a classical ill posed problem, we define a well-posed associated problem by means of suitable regularizations. We remark that, in the same way of Biagoni, we can study the problem on the closure of an open set. A Pachpatte inequality permits us to solve the problem. This inequality plays a vital role in studying the solution. So we extend some results of our previous papers to this particular problem.

If the initial problem admits a smooth solution v on some subset O of $(\mathbb{R}_+)^2$, then this solution and the generalized one coincide on O. So the theory of generalized functions appears as the continuation of the classical theory of functions and distributions. Moreover, it is an efficient tool to solve nonlinear problems.

Appendix

The Appendix is devoted to the construction of global smooth solutions to the Goursat problem when the data are smooth. This is achieved by rewriting the differential equation as an integral equation and making a thorough investigation on the method of successive approximations, (Garabedian, 1964). Several improvements to classical methods and results are needed to obtain precise estimates used in the previous sections. Namely, the growth in the parameter ε of the families of solutions has to be known to choose the algebraic structure to solve the regularized problems. So the results of the Appendix form an essential basis for the construction of generalized solutions.

A.1 Smooth solutions to the Goursat problem

Solution of the Goursat problem for the semi-linear wave equation whose nonlinearity satisfies a global Lipschitz condition, by means of successive approximation techniques, is well known, (Garabedian, 1964). However, for the study of generalized problem, we will need precise estimates for the case of smooth data, which is not sufficiently detailed in the available literature.

A.1.1 Formulation of the problem

We shall be interested in the equation $\dfrac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u,u_x)$ where the function F on the right must satisfy smoothness requirements in its dependence on the arguments x, y, z, p which will be specified later.

We shall establish that the problem is well posed for the hyperbolic partial differential equation. For that we prove that the solution of the equivalent integro-differential equation exists, is unique and depends continuously on the data. Thus we consider the problem

$$(P) : \frac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u,u_x),\, u|_{(Ox)} = \varphi,\, u|_{(Oy)} = \psi, \psi\,(0) = \varphi(0),$$

where $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}$ are some smooth one-variable functions and $F \in \mathrm{C}^\infty(\Delta \times \mathbb{R}^2, \mathbb{R})$ with $\Delta = (\mathbb{R}_+)^2$. Assume that it exists some positive number M such that

$$\sup_{\Delta\times\mathbb{R}^2} |\partial_z F(x,y,z,p)| < M,\, \sup_{\Delta\times\mathbb{R}^2} |\partial_p F(x,y,z,p)| < M \tag{10}$$

and we shall require that F satisfies the following Lipschitz condition

$$|F(x,y,z,p) - F(x',y',z',p')| \leq M\left(|z-z'| + |p-p'|\right) \tag{11}$$

for all (x,y,z,p), $(x',y',z',p') \in \Delta \times \mathbb{R}^2$.

We denote by (P_∞) the problem which consists in searching for a function $u \in \mathrm{C}^2(\Delta)$ satisfying

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = F(x,y,u(x,y),u_x(x,y)), \tag{12}$$

$$u\,(x,0) = \varphi(x), u(0,y) = \psi(y), \psi\,(0) = \varphi(0). \tag{13}$$

We denote by (P_i) the problem which consists in searching for a function $u \in \mathrm{C}^0(\Delta)$ satisfying

$$u(x,y) = u_0(x,y) + \iint\limits_{D(x,y)} F(\xi,\eta,u(\xi,\eta),u_x(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{14}$$

where $u_0(x,y) = \varphi(x) + \psi\,(y) - \psi\,(0)$ with $D(x,y) = \{(\xi,\eta) :\ 0 \le \xi \le x,\ 0 \le \eta \le y\}$.

Theorem 6: Let $u \in \mathrm{C}^0(\Delta)$. The function u is a solution to (P_∞) if and only if u is a solution to (P_i).

Corollary 3: If u is a solution to (P_i) (or to (P_∞)), then u belongs to $\mathrm{C}^\infty(\Delta)$.

We refer the reader to (Dévoué, 2007) for a similar detailed proof.

A.1.2 Existence and uniqueness of solutions

Theorem 7: From assumptions (10), (11) it follows that problem (P_∞) has a unique solution in $\mathrm{C}^\infty(\Delta)$.

Proof: According to Theorem 6, solving problem (P_∞) amounts to solving problem (P_i), that is searching for $u \in \mathrm{C}^0(\Delta)$ satisfying (14). Picard's procedure for solving (P_i) is to set up a sequence of successive approximations u_n defined by the formula for any $n \in \mathbb{N}^*$,

$$u_{n+1}(x,y) = u_0(x,y) + \int_0^x \int_0^y F(\xi,\eta,u_n(\xi,\eta),(u_n)_x\,(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{15}$$

Our purpose is to establish that the limit $u = \lim u_n = \sum_{n=0}^{+\infty} (u_{n+1} - u_n)$ of the successive approximations u_n exists and satisfies the integro-differential equation.

For all $(\xi,\eta) \in D(x,y)$, according to assumption (11), we can write

$$|F(\xi,\eta,z,p) - F(\xi,\eta,z',p')| \le |z - z'|\,M + |p - p'|\,M.$$

By differentiating (15) with respect to x we obtain the formulas

$$(u_{n+1})_x\,(x,y) = (u_0)_x\,(x,y) + \int_0^y F(x,\eta,u_n(x,\eta),(u_n)_x\,(x,\eta))\,\mathrm{d}\eta,$$

For any $K \Subset (\mathbb{R}_+)^2$ we can find a, long enough, such that $K \subset [0,a]^2$. Moreover,

$$|F(\xi,\eta,u_0(\xi,\eta),(u_0)_x\,(\xi,\eta)) - F(\xi,\eta,0,0)| \le |u_0(\xi,\eta)|\,M + |(u_0)_x\,(\xi,\eta)|\,M.$$

Then

$$|F(\xi,\eta,u_0(\xi,\eta),(u_0)_x\,(\xi,\eta))| \le |F(\xi,\eta,0,0)| + \left(\|u_0\|_{\infty,K_a} + \|(u_0)_x\|_{\infty,K_a}\right) M.$$

Put $\Phi_a = \|F(\cdot,\cdot,0,0)\|_{\infty,K_a} + \Big(\|u_0\|_{\infty,K_a} + \|(u_0)_x\|_{\infty,K_a}\Big) M$ and, for any n, $n \in \mathbb{N}^*$, $V_n = u_n - u_{n-1}$. In particular

$$|V_1(x,y)| \leq \int_0^x \int_0^y F(\xi,\eta,u_0(\xi,\eta),(u_0)_x(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta \leq \int_0^x \int_0^y \Phi_a\,\mathrm{d}\xi\,\mathrm{d}\eta \leq \Phi_a xy \leq \Phi_a ay.$$

We have

$$\begin{aligned}|V_{n+1}(x,y| &\leq \int_0^x \int_0^y \left(F(\cdot,\cdot,u_n,(u_n)_x) - F(\cdot,\cdot,u_{n-1},(u_{n-1})_x)\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\leq M \int_0^x \int_0^y \left(|u_n - u_{n-1}| + |(u_n)_x - (u_{n-1})_x|\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

Thus

$$|V_{n+1}(x,y| \leq M \int_0^x \int_0^y \left(|V_n| + |(V_n)_x|\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Furthermore, in a similar way we have the inequalities

$$|(V_{n+1})_x(x,y)| \leq M \int_0^y \left(|V_n| + |(V_n)_x|\right)(x,\eta)\,\mathrm{d}\eta.$$

To exploit the similarity of the integrands, it is convenient to set

$$E_n(\xi,\eta) = \left(|V_n| + |(V_n)_x|\right)(\xi,\eta).$$

We have

$$|V_{n+1}(x,y)| \leq M \int_0^x \int_0^y \left(|V_n| + |(V_n)_x|\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \leq M \int_0^x \int_0^y E_n(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta.$$

Moreover

$$|(V_{n+1})_x(x,y)| \leq M \int_0^y \left(|V_n| + |(V_n)_x|\right)(x,\eta)\,\mathrm{d}\eta \leq M \int_0^y E_n(x,\eta)\,\mathrm{d}\eta.$$

Then we obtain

$$E_{n+1}(x,y) \leq M \left(\int_0^x \int_0^y E_n(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta + \int_0^y E_n(x,\eta)\,\mathrm{d}\eta \right). \tag{16}$$

Moreover $|V_1(x,y)| \leq \Phi_a xy \leq \Phi_a ay$ and

$$|(V_1)_x(x,y)| \leq \left| \int_0^y F(x,\eta,u_0(x,\eta),(u_0)_x(x,\eta))\,\mathrm{d}\eta \right| \leq \Phi_a y. \tag{17}$$

So we have

$$E_1(x,y) = |V_1(\xi,\eta)| + |(V_1)_x(\xi,\eta)| \leq \Phi_a a y + \Phi_a y \leq \Phi_a(a+1)y.$$

From (16) it may be deduced that

$$E_2(x,y) \leq M\left(\int_0^x\int_0^y E_1(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta + \int_0^y E_1(x,\eta)\,\mathrm{d}\eta\right) \leq M\Phi_a\left(y^2/2\right)(a+1)^2.$$

By mathematical induction we have

$$E_n(x,y) \leq M^{n-1}\Phi_a\frac{y^n}{n!}(a+1)^n$$

and

$$\begin{aligned}|V_{n+1}(x,y)| &\leq M\int_0^x\int_0^y E_n(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta \leq M^n\Phi_a\frac{y^{n+1}}{(n+1)!}a(a+1)^n \\ &\leq \frac{a\Phi_a}{M(a+1)}\frac{M^{n+1}y^{n+1}(a+1)^{n+1}}{(n+1)!} \leq \frac{a\Phi_a}{M(a+1)}\frac{M^{n+1}(a^2+a)^{n+1}}{(n+1)!},\end{aligned}$$

moreover

$$\begin{aligned}|(V_{n+1})_x(x,y)| &\leq M\int_0^y E_n(x,\eta)\,\mathrm{d}\eta \leq M^n\Phi_a(a+1)^n\frac{y^{n+1}}{(n+1)!} \\ &\leq \frac{\Phi_a}{M(a+1)}\frac{M^{n+1}y^{n+1}(a+1)^{n+1}}{(n+1)!} \leq \frac{\Phi_a}{M(a+1)}\frac{M^{n+1}(a^2+a)^{n+1}}{(n+1)!}.\end{aligned}$$

Then the exponential series $\sum_{n=0}^{\infty}\frac{a\Phi_a}{M(a+1)}\left(\frac{M^{n+1}(a^2+a)^{n+1}}{(n+1)!}\right) = \frac{a\Phi_a}{M(a+1)}e^{Ma(a+1)}$ is a majorant for the infinite series $\sum_{n=0}^{+\infty}(u_{n+1}-u_n)$ which ensures the uniform convergence of the series $\sum_{n\geq1}V_n$ on K. From the equality $\sum_{k=1}^{n}V_k = u_n - u_0$ we deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ converges uniformly on K to a function u.

The exponential series $\sum_{n=0}^{\infty}\frac{\Phi_a}{M(a+1)}\frac{M^{n+1}(a^2+a)^{n+1}}{(n+1)!} = \frac{\Phi_a}{M(a+1)}e^{Ma(a+1)}$ is a majorant for the infinite series $\sum_{n=0}^{+\infty}((u_{n+1})_x - (u_n)_x)$ which ensures the uniform convergence of the series $\sum_{n\geq1}(V_n)_x$ on K.

As every u_n is derivable with respect to x, from the equality $\sum_{k=1}^{n}(V_k)_x = (u_n)_x - (u_0)_x$, we deduce that the uniform limit u is derivable with respect to x on K and the sequence $((u_n)_x)_{n\in\mathbb{N}}$ converges uniformly on K to the function $(u)_x$.

Let us put $d_n(x,y) = u(x,y) - u_n(x,y)$. Then

$$u(x,y) - u_0(x,y) - \int_0^x \int_0^y F(\xi,\eta,u(\xi,\eta),u_x(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$= u(x,y) - u_n(x,y) + (u_n(x,y) - u_0(x,y) - \int_0^x \int_0^y F(\cdot,\cdot,u,u_x)\,\mathrm{d}\xi\,\mathrm{d}\eta)$$

$$= d_n(x,y) + \int_0^x \int_0^y \left(F(\cdot,\cdot,u_n,(u_n)_x) - F(\cdot,\cdot,u,u_x)\,\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

As for all $(\xi,\eta) \in D(x,y)$,

$$\left|\left(F(\cdot,\cdot,u_n,(u_n)_x) - F(\cdot,\cdot,u,u_x)\right)(\xi,\eta)\right| \le M\left(\left(|u-u_n| + |u_x - (u_n)_x|\right)(\xi,\eta)\right),$$

the limit of the second member is 0 when n tends to $+\infty$. It follows that, for $(x,y) \in K$,

$$u(x,y) = u_0(x,y) + \int_0^x \int_0^y F(\xi,\eta,u(\xi,\eta),u_x(\xi,\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Let us show the uniqueness of the solution. Let w be another solution to (P). Putting $\Theta = u - w$, we obtain

$$\Theta(x,y) = \int_0^x \int_0^y \left(F(\cdot,\cdot,u,u_x)\; - F(\cdot,\cdot,w,w_x)\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Let $(x,y) \in K$, we have

$$|\Theta(x,y)| \le \int_0^x \int_0^y M\left(|(u-w)(\xi,\eta)| + |(u_x - (w)_x)(\xi,\eta)|\right)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Set $L(\xi,\eta) = \left(|\Theta| + |(\Theta)_x|\right)(\xi,\eta)$. We have

$$|\Theta(x,y)| \le \int_0^x \int_0^y M\left(|\Theta| + |(\Theta)_x|\right)(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \le \int_0^x \int_0^y ML(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta.$$

Moreover

$$|(\Theta)_x(x,y)| \le \int_0^y M\left(|\Theta| + |(\Theta)_x|\right)(x,\eta)\,\mathrm{d}\eta \le \int_0^y ML(x,\eta)\,\mathrm{d}\eta.$$

So

$$|\Theta(x,y)| + |(\Theta)_x(x,y)| \le \int_0^y ML(x,\eta)\,\mathrm{d}\eta + \int_0^x \int_0^y ML(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta$$

then

$$L(x,y) \le \int_0^y ML(x,\eta)\,\mathrm{d}\eta + \int_0^x \int_0^y ML(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta.$$

According to Pachpatte lemma, (Pachpatte, 2009, p.42), $L(x,y)=0$. The conclusion to be drawn is that u and w are identical. This completes our proof that the solution u of the problem is unique on Δ. □

Corollary 4: For any $K \Subset (\mathbb{R}_+)^2$ we can find a such that $K \subset [0,a]^2$. With the previous notations, we have

$$\Phi_a = \|F(\cdot,\cdot,0,0)\|_{\infty,K_a} + \left(\|u_0\|_{\infty,K_a} + \|(u_0)_x\|_{\infty,K_a}\right) M$$

and

$$\|u\|_{\infty,K} \leq \|u_0\|_{\infty,K_a} + \tfrac{a\Phi_a}{M(a+1)} e^{Ma(a+1)}; \, \|u_x\|_{\infty,K} \leq \|(u_0)_x\|_{\infty,K_a} + \tfrac{\Phi_a}{M(a+1)} e^{Ma(a+1)}.$$

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