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Solutions of Tenth and Ninth-order Boundary Value Problems by Modified Variational Iteration Method

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Abstract

In this paper, we apply the modified variational iteration method (MVIM) for solving the ninth and tenth-order boundary value problems. The proposed modification is made by introducing He's polynomials in the correction functional. The suggested algorithm is quite efficient and is practically well suited for use in these problems. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions. Several examples are given to verify the reliability and efficiency of the method. The fact that the proposed technique solves nonlinear problems without using the Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method.

Key words: Modified variational iteration method, He's polynomials, nonlinear problems, initial value problems, boundary value problems, error estimates.

MSC (2000) No: 65 N 10

1. Introduction

The higher order boundary value problems are known to arise in hydrodynamic, hydro magnetic stability and applied sciences. In addition, it is well known that when a layer of fluid is heated from below and is subject to the action of rotation, instability may set in as ordinary convection which may be modeled by a tenth-order boundary value problem, Djidjeli, Twizell and Boutayeb (1993), Noor and Mohyud-Din (2008), Mohyud-Din, Noor and Noor (2009), Mohyud-Din (2009), Wazwaz (2000, 2008). The boundary value problems of higher order have been investigated due to their mathematical importance and the potential for applications in hydrodynamic and hydro magnetic stability. Several techniques including finite-difference, polynomial and non polynomial spline, homotopy perturbation and decomposition have been employed for solving such problems, see Djidjeli, Twizell and Boutayeb (1993), Noor and Mohyud-Din (2008), Mohyud-Din, Noor and Noor (2009), Mohyud-Din (2009), Wazwaz (2000, 2008). Most of these techniques have their inbuilt deficiencies, like divergence of the results at the points adjacent to the boundary and calculation of the so-called Adomian's polynomials. Moreover, the performance of most of the methods used so far is well known that they provide the solution at grid points only.

He (2006, 2008) developed the variational iteration and homotopy perturbation methods for solving linear, nonlinear, initial and boundary value problems. Moreover, He realized the physical significance of the variational iteration method, its compatibility with the physical problems and applied this promising technique to a wide class of linear and nonlinear, ordinary, partial, deterministic or stochastic differential equation. The homotopy perturbation method Abbasbandy (2005), Abbasbandy (2007), Abdou and soliman (2005), Ghorbani and Nadjfi (2007), He (2006, 2008), Momani and Odibat (2006), Mohyud-Din and Noor (2007), Mohyud-Din, Noor and Noor (2009), Noor and Mohyud-Din (2008) was also developed by He by merging two techniques, the standard homotopy and the perturbation. The homotopy perturbation method was formulated by taking full advantage of the standard homotopy and perturbation methods.

The variational iteration and homotopy perturbation methods have been applied to a wide class of functional equations; see Abbasbandy (2005), Abbasbandy (2007), Abdou and soliman (2005), Ghorbani and Nadjfi (2007), He (2006, 2008), Momani and Odibat (2006), Mohyud-Din and Noor (2007), Mohyud-Din, Noor and Noor (2009), Noor and Mohyud-Din (2008). In these methods the solution is given in an infinite series usually converging to an accurate solution, in a later work Ghorbani et al., Ghorbani and Nadjfi (2007) split the nonlinear term into a series of polynomials calling them as the He's polynomials. Most recently, Noor and Mohyud-Din made the elegant coupling of He's polynomials and the correction functional for solving diversified nonlinear problems of physical nature Mohyud-Din and Noor (2007), Mohyud-Din, Noor and Noor (2009), Noor and Mohyud-Din (2008).

In this paper, we use the modified variational iteration method (MVIM) which is formulated by the coupling of correction functional of variational iteration method and He's polynomials for solving the ninth and tenth-order boundary value problems. The proposed MVIM provides the solution in a rapid convergent series which may lead the solution in a closed form. In this

technique, the correction functional is developed and the Lagrange multipliers are calculated optimally via variational theory.

The use of Lagrange multipliers reduces the successive application of the integral operator and the cumbersome of huge computational work while still maintaining a very high level of accuracy. Finally, the He's polynomials are introduced in the correction functional and the comparison of like powers of p gives solutions of various orders.

The proposed iterative scheme takes full advantage of variational iteration and the homotopy perturbation methods. It is worth mentioning that the suggested method is applied without any discretization, restrictive assumption or transformation and is free from round off errors. Unlike the method of separation of variables that require initial and boundary conditions, the method provides an analytical solution by using the initial conditions only. The proposed method work efficiently and the results so far are very encouraging and reliable. The fact that the proposed MVIM solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method. Several examples are given to verify the reliability and efficiency of the algorithm.

2. Variational Iteration Method (VIM)

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to variational iteration method Abbasbandy (2005), Abbasbandy (2007), Abdou and soliman (2005), Ghorbani and Nadjfi (2007), He (2006, 2008), Momani and Odibat (2006), Mohyud-Din and Noor (2007), Mohyud-Din, Noor and Noor (2009), Noor and Mohyud-Din (2008), we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + N\tilde{u}_n(s) - g(s))\, ds, \tag{2}$$

where λ is a Lagrange multiplier He (2006, 2008), which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta \tilde{u}_n = 0$; (2) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in He (2006, 2008). In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

3. Homotopy Perturbation Method and He's Polynomials

To explain the homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0, \tag{3}$$

where L is any integral or differential operator. We define a convex homotopy H (u, p) by

$$H(u, p) = (1 - p)F(u) + pL(u), \tag{4}$$

where F (u) is a functional operator with known solutions v_0, which can be obtained easily. It is clear that for

$$H(u, p) = 0, \tag{5}$$

we have

$$H(u,0) = F(u), \quad H(u,1) = L(u).$$

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(f, 1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem F (u) = 0 is continuously deforms the original problem L (u) = 0. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter Ghorbani and Nadjfi (2007), Mohyud-Din and Noor (2007), Mohyud-Din, Noor and Noor (2009), Noor and Mohyud-Din (2008). The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{6}$$

if $p \to 1$, then (6) corresponds to (4) and becomes the approximate solution of the form,

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \tag{7}$$

It is well known that series (7) is convergent for most of the cases and also the rate of convergence is dependent on L(u); Ghorbani and Nadjfi (2007). We assume that (7) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to Ghorbani and Nadjfi (2007), He's HPM considers the nonlinear term $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + p H_1 + p^2 H_2 + \ldots,$$

where H_n's are the so-called He's polynomials Ghorbani and Nadjfi (2007), which can be calculated by using the formula

$$H_n(u_0,\ldots,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N(\sum_{i=0}^{n}p^i u_i)\right)_{p=0}, \qquad n=0,\,1,\,2,\cdots.$$

4. Modified Variational Iteration Method (MVIM)

To illustrate the basic concept of the modified variational iteration method, we consider the following general differential equation

$$Lu+Nu=g(x), \tag{8}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the forcing term. According to variational iteration method, we can construct a correction functional as follows

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(\xi)\,\left(Lu_n(\xi)+N\tilde{u}_n(\xi)-g(\xi)\right)d\xi, \tag{9}$$

where λ is a Lagrange multiplier He (2006, 2008), which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n=0$; (9) is called a correction functional. Now, we apply the homotopy perturbation method

$$\sum_{n=0}^{\infty} p^{(n)}u_n=u_0(x)+p\int_0^x \lambda(\xi)\left(\sum_{n=0}^{\infty}p^{(n)}L(u_n)+\sum_{n=0}^{\infty}p^{(n)}N(\tilde{u}_n)\right)d\xi-\int_0^x \lambda(\xi)\,g(\xi)\,d\xi, \tag{10}$$

which is the modified variational iteration method (MVIM) Mohyud-Din and Noor (2007), Mohyud-Din, Noor and Noor (2009), Noor and Mohyud-Din (2008) and is formulated by the coupling of variational iteration method and He's polynomials. The comparison of like powers of p gives solutions of various orders.

5. Numerical Applications

In this section, we apply the MVIM for solving the ninth and tenth-order boundary value problems.

Example 5.1.

Consider the following nonlinear boundary value problems of tenth-order

$$y^{(x)}(x) = e^{-x} y^2(x), \qquad 0 < x < 1,$$

with boundary condition

$$\begin{aligned} &y(0)=1, \qquad y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) = 1, \\ &y(1)=e, \qquad y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = y^{(viii)}(1) = e. \end{aligned}$$

The exact solution of the problem is

$$y(x) = e^x.$$

The correct functional for the above boundary value problem is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left(\frac{d^{10} y_n}{d\,x^{10}} - e^{-x} \tilde{y}_n^{\,2}(x) \right) ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = \frac{1}{9!}(s-x)^9$, we get the following iterative formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{9!}(s-x)^9 \left(\frac{d^{10} y_n}{d\,x^{10}} - e^{-x} y_n^{\,2}(x) \right) ds.$$

Using the initial conditions

$$\begin{aligned} y_{n+1}(x) = {} & 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}D\,x^7 + \frac{1}{8!}x^8 + \frac{1}{9!}E\,x^9 \\ & + \int_0^x \frac{1}{9!}(s-x)^9 \left(\frac{d^{10} y_n}{d\,x^{10}} - e^{-x} y_n^{\,2}(x) \right) ds, \end{aligned}$$

where

$$A = y'(0), \qquad B = y^{(3)}(0), \qquad C = y^{(5)}(0), \qquad D = y^{(7)}(0), \quad E = y^{(9)}(0).$$

Applying the modified variational iteration method

$$y_0 + p\,y_1 + p^2\,y_2 + \cdots = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}D\,x^7 + \frac{1}{8!}x^8$$
$$+\frac{1}{9!}E\,x^9 + p\int_0^x \frac{1}{9!}(s-x)^9\left(\frac{d^{10}y_0}{d\,x^{10}} + p\frac{d^{10}y_1}{d\,x^{10}} + p^2\frac{d^{10}y_2}{d\,x^{10}} + p^3\frac{d^{10}y_3}{d\,x^{10}} + \cdots\right)ds$$
$$-p\int_0^x \frac{1}{9!}(s-x)^9\left(e^{-x}\left(y_0 + p\,y_1 + p^2\,y_2 + p^3\,y_3 + \cdots\right)^2\right)ds.$$

Comparing the co-efficient of like powers of p, the following approximants are obtained

$$p^{(0)}: y_0(x) = 1,$$

$$p^{(1)}: y_1(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}D\,x^7 + \frac{1}{8!}x^8 + \frac{1}{9!}E\,x^9 + \frac{1}{10!}x^{10}$$
$$+\frac{1}{11!}x^{11} + \frac{1}{12!}x^{12} + \cdots,$$

$$p^{(2)}: y_2(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}D\,x^7 + \frac{1}{8!}x^8 + \frac{1}{9!}E\,x^9$$
$$+\frac{1}{10!}x^{10} + \frac{1}{11!}x^{11} + \frac{1}{12!}x^{12} + \frac{2}{11!}A\,x^{11} + \left(-\frac{4}{12!}A + \frac{1}{239500800}\right)x^{12} + \cdots,$$
$$\vdots$$

The series solution is given as:

$$y(x) = 1 + A\,x + \frac{1}{2!}x^2 + \frac{1}{3!}B\,x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}C\,x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}D\,x^7 + \frac{1}{8!}x^8 + \frac{1}{9!}E\,x^9 + \frac{1}{10!}D\,x^{10}$$
$$+\left(-\frac{1}{19958400}A + \frac{1}{39916800}\right)x^{11} + \left(-\frac{1}{119750400}A + \frac{1}{159667200}\right)x^{12} + \cdots.$$

Imposing the boundary conditions at $x = 1$ yield

$$A = 1.00001436, \qquad B = 0.999858964, \qquad C = 1.001365775,$$
$$D = 0.987457318, \qquad E = 1.0932797434.$$

The series solution is given as

$$y(x) = 1 + 1.00001436\,x + \frac{1}{2!}x^2 + 0.1666431607\,x^3 + \frac{1}{4!}x^4 + 0.008344714791\,x^5 + \frac{1}{6!}x^6$$
$$+0.00019524071\,x^7 + \frac{1}{8!}x^8 + 3.013\times10^{-6}\,x^9 + \frac{1}{10\,!}x^{10} + 2.51\times10^{-8}\,x^{11} - 2.087\times10^{-9}\,x^{12} + \cdots.$$

Table 5.1. (Error estimates)

x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.00000
0.1	1.105170918	1.10517233	-1.41 E-6
0.2	1.221402758	1.221405446	-2.69 E-6
0.3	1.349858808	1.349862509	-3.70 E-6
0.4	1.491824698	1.49182905	-4.35 E-6
0.5	1.648721271	1.648725849	-4.58 E-6
0.6	1.822118800	1.822123158	-4.36 E-6
0.7	2.013752707	2.013756415	-3.71 E-6
0.8	2.225540928	2.225543623	-2.69 E-6
0.9	2.459603111	2.459604528	-1.42 E-6
1.0	2.718281828	2.7182830	2.00E-9

*Error = Exact solution-Series solution.

Table 5.1 exhibits the exact solution and the series solution along with the errors obtained by using the modified variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $y(x)$.

Example 5.2.

Consider the following linear boundary value problem of tenth-order

$$y^{(x)}(x) = -8e^x + y''(x), \qquad 0 < x < 1,$$

with boundary conditions

$$\begin{aligned} &y(0) = 1, && y'(0) = 0, && y''(0) = -1, && y'''(0) = -2, && y^{(iv)}(0) = -3, \\ &y(1) = 0, && y'(1) = -e, && y''(1) = -2e, && y'''(1) = -3e, && y^{(iv)}(1) = -4e. \end{aligned}$$

The exact solution of the problem is

$$y(x) = (1-x)e^x.$$

The correction functional for the above boundary value problem is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left(\frac{d^{10}y_n}{dx^{10}} - \left(-8e^x + \tilde{y}_n''(x)\right)\right)ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = \frac{1}{9!}(s-x)^9$, we get the following iterative formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{9!}(s-x)^9 \left(\frac{d^{10} y_n}{d\,x^{10}} - \left(-8e^x + \tilde{y}_n''(x)\right)\right) ds.$$

Using the initial conditions

$$y_{n+1}(x) = 1 - \frac{1}{2!}x^2 - \frac{1}{3}x^3 - \frac{1}{8!}x^4 + \frac{1}{5!}Ax^5 + \frac{1}{6!}Bx^6 + \frac{1}{7!}Cx^7 + \frac{1}{8!}Dx^8 + \frac{1}{9!}Ex^9$$
$$+ \int_0^x \frac{1}{9!}(s-x)^9 \left(\frac{d^{10} y_n}{d\,x^{10}} - \left(-8e^x + y_n^{//}(x)\right)\right) ds,$$

where

$$A = y^{(5)}(0), \quad B = y^{(6)}(0), \quad C = y^{(7)}(0), \quad D = y^{(8)}(0), \quad E = y^{(9)}(0).$$

Applying the modified variational iteration method

$$y_0 + p\,y_1 + p^2\,y_2 + \cdots = 1 - \frac{1}{2!}x^2 - \frac{1}{3}x^3 - \frac{1}{8!}x^4 + \frac{1}{5!}Ax^5 + \frac{1}{6!}Bx^6 + \frac{1}{7!}Cx^7 + \frac{1}{8!}Dx^8$$
$$+ \frac{1}{9!}Ex^9$$
$$+ p\int_0^x \frac{1}{9!}(s-x)^9 \left(\frac{d^{10} y_0}{d\,x^{10}} + p\frac{d^{10} y_1}{d\,x^{10}} + p^2\frac{d^{10} y_2}{d\,x^{10}} + p^3\frac{d^{10} y_3}{d\,x^{10}} + \cdots \right) ds$$
$$- p\int_0^x \frac{1}{9!}(s-x)^9 \left(-8e^x + \left(y_0'' + p\,y_1'' + p^2\,y_2'' + \cdots\right)\right) ds.$$

Comparing the co-efficient of like powers of p, consequently, we obtain the following approximants:

$$p^{(0)} : y_0(x) = 1,$$
$$p^{(1)} : y_1(x) = 1 - 8e^x + 8 + 8x + \frac{7}{2!}x^2 + x^3 + \frac{5}{4!}x^4 + \left(\frac{1}{15} + \frac{1}{5!}A\right)x^5 + \left(\frac{1}{90} + \frac{1}{6!}B\right)x^6$$
$$+ \left(\frac{1}{630} + \frac{1}{7!}C\right)x^7 + \left(\frac{1}{7!} + \frac{1}{8!}D\right)x^8 + \left(\frac{1}{45360} + \frac{1}{9!}E\right)x^9,$$

$$p^{(2)}: y_2(x) = 1 - 16e^x + 16 + 16x + \frac{15}{2!}x^2 + \frac{7}{3}x^3 + \frac{13}{4!}x^4 + \left(\frac{2}{15} + \frac{1}{5!}A\right)x^5 + \left(\frac{1}{45} + \frac{1}{6!}B\right)x^6$$
$$+ \left(\frac{1}{315} + \frac{1}{7!}C\right)x^7 + \left(\frac{2}{7!} + \frac{1}{8!}D\right)x^8 + \left(\frac{2}{45360} + \frac{1}{9!}E\right)x^9 + \frac{1}{518400}x^{10} + \frac{1}{6652800}x^{11}$$
$$+ \frac{1}{95800320}x^{12} + \left(\frac{1}{778377600} + \frac{1}{6227020800}A\right)x^{13} + \cdots,$$

$$p^{(3)}: y_3(x) = 1 - 24e^x + 24 + 24x + \frac{23}{2!}x^2 + \frac{11}{3}x^3 + \frac{21}{4!}x^4 + \left(\frac{1}{5} + \frac{1}{5!}A\right)x^5 + \left(\frac{1}{30} + \frac{1}{6!}B\right)x^6$$
$$+ \left(\frac{1}{210} + \frac{1}{7!}C\right)x^7 + \left(\frac{3}{7!} + \frac{1}{8!}D\right)x^8 + \left(\frac{3}{45360} + \frac{1}{9!}E\right)x^9 + \left(\frac{1}{453600} + \frac{1}{518400}\right)x^{10}$$
$$+ \left(\frac{1}{4989600} + \frac{1}{6652800}\right)x^{11} + \left(\frac{1}{59875200} + \frac{1}{95800320}\right)x^{12}$$
$$+ \left(\frac{2}{778377600} + \frac{1}{6227020800}A\right)x^{13} + \cdots,$$

$$\vdots$$

The series solution is given by

$$y(x) = 17 - 16e^x + 16x + \frac{15}{2!}x^2 + \frac{7}{3}x^3 + \frac{13}{4!}x^4 + \left(\frac{2}{15} + \frac{1}{15!}\right)x^5 + \left(\frac{1}{45} + \frac{1}{6!}B\right)x^6 + \left(\frac{1}{315} + \frac{1}{7!}C\right)x^7$$
$$+ \left(\frac{2}{7!} + \frac{1}{8!}D\right)x^8 + \frac{1}{9!}(8 + E)x^9 + \frac{7}{10!}x^{10} + \frac{6}{11!}x^{11} + \frac{5}{12!}x^{12} + \cdots.$$

Imposing the boundary conditions at x = 1 yields

$$A = -4.00002, \qquad B = -4.99999999, \quad C = -6.00100, \quad D = -7.00000, \quad E = -8.010000.$$

The series solution is given by

$$y(x) = 17 - 16e^x + 16x + \frac{15}{2!}x^2 + \frac{7}{3}x^3 + \frac{13}{4!}x^4 + 0.99999999999997x^5 + 0.15277791666666666x^6$$
$$+ 0.00198392857142857x^7 + 0.000233214285714286x^8 - 2.75573192239859 \times 10^{-8}x^9$$
$$+ \frac{7}{518400}x^{10} + \frac{1}{6652800}x^{11} + \frac{1}{9580320}x^{12} + \cdots.$$

Table 5.2. (Error estimates)

x	Exact solution	Series solution	*Errors
0.00	1.0000000000	1.0000000000	0.0000000000
0.10	0.9946538263	0.9946538263	0.0000000000
0.20	0.9771222065	0.9771222065	0.0000000000
0.30	0.9449011653	0.9449011648	0.0000000005
0.40	0.8950948186	0.8950948125	00.0000000061
0.50	0.8243606354	0.8243605909	0.0000000444
0.60	0.7288475202	0.7288472928	0.0000002274
0.70	0.6041258122	0.6041249068	0.0000009055
0.80	0.4451081857	0.4451051850	0.0000030006
0.90	0.2459603111	0.2459516717	0.0000086394
1.00	0.0000000000	-.0000222586	0.0000222586

*Error =Exact solution- series solution.

Table 5.2 exhibits the exact solution and the series solution along with the errors obtained by using the modified variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of y(x).

Example 5.3.

Consider the following ninth order boundary value problem

$$y^{(ix)} = -9e^x + y(x), \qquad 0 < x < 1,$$

with boundary conditions

$$y(0) = 1, \quad y^{(1)}(0) = 0, \quad y^{(2)}(0) = -1, \quad y^{(3)}(0) = -2, \quad y^{(4)}(0) = -3,$$
$$y(1) = 0, \quad y^{(1)}(1) = -e, \quad y^{(2)}(1) = -2e, \quad y^{(3)}(1) = -3e.$$

The exact solution of the problem is

$$y(x) = (1-x)e^x.$$

The correct functional is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left(\frac{d^9 y_n}{d\,x^9} + 9e^x - \tilde{y}_n(x)\right)ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = \frac{1}{8!}(s-x)^8$, we get the following iterative formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{8!}\left(s-x\right)^8 \left(\frac{d^9 y_n}{d\,x^9} + 9e^x - y_n(x)\right) ds.$$

Using the initial conditions

$$y_{n+1}(x) = 1 - \frac{1}{2!}x^2 - \frac{2}{3!}x^3 - \frac{3}{4!}x^4 + \frac{1}{5!}A\,x^5 + \frac{1}{6!}B\,x^6 + \frac{1}{7!}C\,x^7 + \frac{1}{8!}D\,x^8$$
$$+ \int_0^x \frac{1}{8!}\left(s-x\right)^8 \left(\frac{d^9 y_n}{d\,x^9} + 9e^x - y_n(x)\right) ds.$$

Applying the modified variational iteration method

$$y_0 + p\,y_1 + p^2\,y_2 + \cdots = 1 - \frac{1}{2!}x^2 - \frac{2}{3!}x^3 - \frac{3}{4!}x^4 + \frac{1}{5!}A\,x^5 + \frac{1}{6!}B\,x^6 + \frac{1}{7!}C\,x^7 + \frac{1}{8!}D\,x^8$$
$$+ p\int_0^x \frac{1}{8!}(s-x)^8 \left(\frac{d^9 y_0}{d\,x^9} + p\frac{d^9 y_1}{d\,x^9} + p^2\frac{d^9 y_2}{d\,x^9} + p^3\frac{d^9 y_3}{d\,x^9} + \cdots\right) ds$$
$$- p\int_0^x \frac{1}{8!}(s-x)^8 \left(\left(y_0 + p\,y_1 + p^2\,y_2 + p^3\,y_3 + \cdots\right)^2 - 9e^x\right) ds.$$

Comparing the co-efficient of like powers of p, the following approximants are obtained

$$p^{(0)} : y_0(x) = 1,$$
$$p^{(1)} : y_1(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{5!}A\,x^5 + \frac{1}{6!}B\,x^6 + \frac{1}{7!}C\,x^7 + \frac{1}{8!}D\,x^8 - \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11}$$
$$- \frac{11}{12!}x^{12} + \cdots,$$
$$\vdots$$

The series solution is given by

$$y(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{5!}A\,x^5 + \frac{1}{6!}B\,x^6 + \frac{1}{7!}C\,x^7 + \frac{1}{8!}D\,x^8 - \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11}$$
$$- \frac{11}{12!}x^{12} + \cdots.$$

Imposing the boundary condition at x=1 gives

A = -3.999992, B = -5.00017, C = -5.9985, D = -7.005.

The series solution is given as

$$y(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - 0.03333326667x^5 - 0.006944680556x^6 - 0.001190178571x^7$$
$$- 0.000173735119x^8 - \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11} - \frac{11}{12!}x^{12} + \cdots.$$

Table 5.3. (Error estimates)

x x	Exact solution	Series solution	*Errors
0.0	1.00000000	1.0000000000	0.000000
0.1	0.99465383	0.9946538264	-2.0E-10
0.2	0.97712221	0.9771222066	-2.0E-10
0.3	0.94490117	0.9449011654	-2.0E-10
0.4	0.89509482	0.8950948186	-2.0E-10
0.5	0.82436064	0.8243606355	-2.0E-10
0.6	0.72884752	0.7288475206	-6.0E-10
0.7	0.60412581	0.6041258131	-1.0E-9
0.8	0.44510819	0.4451081876	-2.0E-9
0.9	0.24596031	0.2459603145	-3.4E-9
1.0	0.00000000	0.0000000000	0.000000

*Error=Exact solution-Series solution.

Table 5.3 exhibits the exact solution and the series solution along with the errors obtained by using the modified variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of y(x).

5. Conclusion

In this paper, we applied the modified variational iteration method (MVIM) for finding the solution of ninth and tenth-order boundary value problems. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. It may be concluded that MVIM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result. The fact that the MVIM solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

REFERENCES

Abbasbandy, S. (2007). Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method, Internat. J. Numer. Mech. Engg., **70**, 876-881.

Abbasbandy, S. (2007). A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials, J. Comput. Appl. Math., **207,** 59-63.

Abdou, M. A. and Soliman A. A. (2005). New applications of variational iteration method, Phys. D **211**, 1-8.

Djidjeli, K., Twizell, E. H. and Boutayeb, A. (1993). Numerical methods for special nonlinear boundary value problems of order 2m, J. Comput. Appl. Math., **47**, 35-45.

Ghorbani, A. and Nadjfi, J. S. (2007). He's homotopy perturbation method for calculating Adomian's polynomials, Int. J. Nonlin. Sci. Num. Simul., **8**, 229-332.

He, J. H. (2008). An elementary introduction of recently developed asymptotic methods and nanomechanics in textile engineering, Int. J. Mod. Phys., B **22,** 3487-4578.

He, J. H. (2004). Comparison of homotopy perturbation method and homotopy analysis method, Appl. Math. Comput., **156**, 527-539.

He, J. H. (2008). Recent developments of the homotopy perturbation method, Top. Meth. Nonlin. Anal., **31**, 205-209.

He, J. H. (2006). Some asymptotic methods for strongly nonlinear equation, Int. J. Mod. Phys., **20,** 1144-1199.

He, J. H., (2007). Variational iteration method- Some recent results and new interpretations, J. Comput. Appl. Math., **207**, 3-17.

Momani, S. and Odibat, Z. (2006). Application of He's variational iteration method to Helmholtz equations, Chaos, Solitons & Fractals, **27,** 1119-1123.

Mohyud-Din, S. T. and Noor, M. A. (2007). Homotopy perturbation method for solving fourth-order boundary value problems, Math. Prob. Engg., 1-15, Article ID 98602, doi:10.1155/2007/98602.

Mohyud-Din S. T., Noor, M. A. and Noor, K. I. (2009). Travelling wave solutions of seventh-order generalized KdV equations using He's polynomials, Int. J. Nonlin. Sci. Num. Sim., **10**, 223-229.

Mohyud-Din, S. T. and Noor, M. A. (2009). Homotopy perturbation method for solving partial differential equations, Zeitschrift für Naturforschung A, **64a**, 157-170.

Mohyud-Din, S. T., Noor, M. A. and Noor, K. I. (2009). Parameter-expansion techniques for strongly nonlinear oscillators, Int. J. Nonlin. Sci. Num. Sim., **10**, 581-583.

Mohyud-Din, S. T., Noor, M. A. and Noor, K. I. (2009). Some relatively new techniques for nonlinear problems, Math. Prob. Eng., Article ID 234849, 25 pages, doi:10.1155/2009/234849.

Mohyud-Din, S. T., Noor, M. A. and Noor, K. I. (2009). Solving second-order singular problems using He's polynomials, Wd. Appl. Sci. J., **6**, 769-775.

Mohyud-Din, S. T. (2009). Solution of nonlinear differential equations by exp-function method, *World Applied Sciences Journal,* **7,** 116-147.

Mohyud-Din, S. T., Noor, M. A and Noor, K. I. (2009). On the coupling of polynomials with correction functional, Int. J. Mod. Phys., B.

Mohyud-Din, S. T., Noor, M. A. and Noor, K. I. (2009). Variational iteration method for solving telegraph equations, *Applications and Applied Mathematics,* **4**, 114-121.

Noor, M. A. and Mohyud-Din, S. T. (2008). Homotopy perturbation method for nonlinear higher-order boundary value problems, Int. J. Nonlin. Sci. Num. Simul., **9,** 395-408.

Noor, M. A. and Mohyud-Din, S. T. (2008). Variational iteration method for solving fifth-order boundary value problems using He's polynomials, Math. Prob. Engg., Article ID 954794, doi: 10:1155/2008/954794.

Noor, M. A. and Mohyud-Din, S. T. (2008). Variational homotopy perturbation method for solving higher dimensional initial boundary value problems, Math. Prob. Engg., Article ID 696734, doi:10.1155/2008/696734.

Noor, M. A. and Mohyud-Din, S. T. (2008). Modified variational iteration method for heat and wave-like equations, Acta Applnda. Mathmtce., DOI: 10.1007/s10440-008-9255-x.

Wazwaz, A. M. (2000). Approximate solutions to boundary value problems of higher-order by the modified decomposition method, Comput. Math. Appl., **40**, 679-691.

Wazwaz, A. M. (2000). The modified decomposition method for solving linear and nonlinear boundary value problems of tenth-order and twelfth-order, Int. J. Nonlin. Sci. Num. Sim., **1**, 17-24.