Available at
http://pvamu.edu/pages/398.asp
ISSN: 1932-9466
Vol. 1, Issue 1 (June 2006) pp. 11 – 24
(Previously Vol. 1, No. 1)

Applications and Applied Mathematics (AAM):
An International Journal

REMARKS ON THE STABILITY OF SOME SIZE-STRUCTURED POPULATION MODELS I: CHANGES IN VITAL RATES DUE TO POPULATION ONLY

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Received July 14, 2005; revised received October 30, 2005; accepted December 7, 2005

Abstract

We consider a size-structured population model that has been studied in Calsina et al. (2003). We propose a different approach that provides direct stability results, and we correct a stability result given therein. In addition, we obtain global stability results that have not been given in Calsina et al. (2003).

Keywords: Size-structure; Stability; Steady states
MSC 2000 No.: 92D25; 35B35; 45M05; 35F30

1. Introduction

In this paper, we study a size-structured population model studied in Calsina et al. (2003). The model is given by

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial t} + V(P(t))\dfrac{\partial p(a,t)}{\partial a} + \mu(P(t))p(a,t) = 0, \quad a \in [0,l), l \le +\infty, t > 0, \\ V(P(t))p(0,t) = \beta(P(t))P(t), \quad t > 0, \\ p(a,0) = p_0(a), \quad a \ge 0, \end{cases} \tag{1.1}$$

where $p(a,t)$ is the density with respect to size $a \in [0,l)$ at time t, and l is the maximum possible size for an individual in the population. $P(t) = \int_0^l p(a,t)da$ is the total population at time t, and V, μ, β are, respectively, the individual growth rate, mortality rate and birth rate; these rates are assumed to be positive and differentiable with bounded derivatives, and the initial size distribution $p_0(a) \in L^1[0,l) \cap L_\infty[0,l)$, which is a non-negative function of a. This model could describe fish or forest population, where in the latter case individuals compete for light or nutrients. Note that in this case $l < +\infty$ corresponds to harvesting at specific size $l < +\infty$. Also, note that problem (1.1) is a quasilinear first order partial differential equation. Models of size-structured populations were first derived by Sinko et al. (1967), further developments in this field can be found in Metz, et al. (1986), and in Calsina et al. (2003) and the references given therein.

In this paper, we determine the steady states of problem (1.1) and examine their stability. We show that, if $l < +\infty$, then the trivial steady state is always a possible steady state, and we determine a condition for its local asymptotic stability, and if this condition is not satisfied, then the trivial steady state is unstable. We also obtain similar results for the case $l = +\infty$.

Furthermore, we show that if a steady state is non-trivial, then there are as many non-trivial steady states as the solutions of a given equation (see section 2). If $l < +\infty$, these steady states satisfy $\mu(P_\infty) < \beta(P_\infty)$, at a steady state P_∞; and if $l = +\infty$, they satisfy $\mu(P_\infty) = \beta(P_\infty)$. We then study the local asymptotic stability of these non-trivial steady states. If $l = +\infty$, then a non-trivial steady state P_∞ is locally asymptotically stable if $\beta'(P_\infty) < \mu'(P_\infty)$, and unstable, if $\beta'(P_\infty) > \mu'(P_\infty)$. We also determine various conditions for the local asymptotic stability or instability of a non-trivial steady state P_∞ when $l < +\infty$.

In addition, we obtain some global stability results. If $l < +\infty$, we show that the solution of problem (1.1) tends to the trivial steady state, and therefore, the trivial steady state is globally stable. If $l = +\infty$, we derive several conditions for the solution to tend to the trivial steady state; further study shows that in some cases the solution tends to infinity.

Further study of size-structured population models, under more general vital rates, will be reported in a subsequent paper.

The organization of this paper as follows: in section 2 we determine the steady states; in section 3 we study the stability of the steady states; in section 4 we conclude our results.

2. The Steady States

In this section, we determine the steady states of problem (1.1). A steady state of problem (1.1) satisfies the following:

$$\begin{cases} V(P_\infty)\dfrac{dp_\infty(a)}{da} + \mu(P_\infty)p_\infty(a) = 0, \quad a > 0, \\ V(P_\infty)p_\infty(0) = \beta(P_\infty)P_\infty, \\ P_\infty = \int_0^\infty p_\infty(a)da. \end{cases} \tag{2.1}$$

From (2.1), by solving the differential equation, we obtain that $p_\infty(a)$ satisfies the following:

$$p_\infty(a) = P_\infty \frac{\beta(P_\infty)}{V(P_\infty)} e^{-\frac{\mu(P_\infty)}{V(P_\infty)}a}. \tag{2.2}$$

And therefore, P_∞ satisfies the following:

$$P_\infty = P_\infty \frac{\beta(P_\infty)}{V(P_\infty)} \int_0^l e^{-\frac{\mu(P_\infty)}{V(P_\infty)}a} da. \tag{2.3}$$

Then, if $P_\infty \neq 0,$ we obtain from (2.3) that

$$1 = \frac{\beta(P_\infty)}{V(P_\infty)} \int_0^l e^{-\frac{\mu(P_\infty)}{V(P_\infty)}a} da. \tag{2.4}$$

The following theorem is straightforward consequence of equations (2.2)-(2.4), and therefore, we omit the proof.

Theorem (2.1).

(1) Problem (1.1) has the trivial steady state $p_\infty(a) \equiv 0.$

(2) If $l < +\infty$ and $P_\infty \neq 0,$ then problem (1.1) has as many steady states as the solutions of equation (2.4) for $P_\infty,$ and in each case $\mu(P_\infty) < \beta(P_\infty),$ and $p_\infty(a)$ satisfies (2.2).

(3) If $l = +\infty$ and $P_\infty \neq 0,$ then problem (1.1) has as many steady states as the solutions of equation (2.4) for $P_\infty,$ and in each case $\mu(P_\infty) = \beta(P_\infty),$ and $p_\infty(a)$ satisfies (2.2).

We note that the non-trivial steady states are interesting for ecological and economical reasons.

3. Stability of the Steady States

In this section, we study the stability of the steady states for problem (1.1) as given by Theorem (2.1). Our method is different from that in Calsina et al. (2003), and we give a correct proof of proposition (3.2) given therein. In addition, we also obtain some global stability results that have not been given in Calsina et al. (2003).

To study the stability of the trivial steady state $p_\infty(a) = 0,$ we linearize problem (1.1) at $P_\infty = 0$ in order to obtain a characteristic equation, which in turn will determine conditions for stability. To that end, we consider a perturbation $p(a,t)$ defined by where ζ is a complex number. Accordingly, by using (1.1), we obtain the following:

$$\begin{cases} \dfrac{dp(a)}{da} + \left[\dfrac{\zeta + \mu(P)}{V(P)}\right] p(a) = 0, \quad a > 0, \\ p(0) = \dfrac{\beta(P)}{V(P)} \displaystyle\int_0^l p(a)da, \quad P = \int_0^l p(a,t)da. \end{cases} \tag{3.1}$$

Now, we linearize (3.1) at $P_\infty = 0$ to obtain the following:

$$\begin{cases} \dfrac{dp(a)}{da} + \left[\dfrac{\zeta + \mu(0)}{V(0)}\right] p(a) = 0, \quad a > 0, \\ p(0) = \dfrac{\beta(0)}{V(0)} \displaystyle\int_0^l p(a)da. \end{cases} \tag{3.2}$$

Then solving (3.2), we obtain the following characteristic equation:

$$1 = \frac{\beta(0)}{V(0)}\int_0^l e^{-\left[\frac{\zeta+\mu(0)}{V(0)}\right]a}da. \tag{3.3}$$

In the following theorem we describe the stability of the trivial steady state $p_\infty(a) \equiv 0.$

Theorem 3.1.

(1) If $l = +\infty,$ then the trivial steady state is locally asymptotically stable if $\beta(0) < \mu(0),$ and unstable if $\beta(0) > \mu(0).$

(2) If $l < +\infty,$ then the trivial steady state is locally asymptotically stable if

$$\frac{\beta(0)}{V(0)}\int_0^l e^{-\frac{\mu(0)}{V(0)}a}da < 1,$$

and unstable if

$$\frac{\beta(0)}{V(0)}\int_0^l e^{-\frac{\mu(0)}{V(0)}a}da > 1.$$

Proof:

To prove (1), we note that if $\beta(0) < \mu(0),$ then equation (3.3) cannot be satisfied for any ζ with $\mathrm{Re}\zeta \geq 0$ since

$$\left|\frac{\beta(0)}{V(0)}\int_0^\infty e^{-\left[\frac{\zeta+\mu(0)}{V(0)}\right]a}da\right| \leq \frac{\beta(0)}{V(0)}\int_0^\infty \left|e^{-\left[\frac{\zeta+\mu(0)}{V(0)}\right]a}\right|da \leq \frac{\beta(0)}{V(0)}\int_0^\infty e^{-\left[\frac{\mathrm{Re}\,\zeta+\mu(0)}{V(0)}\right]a}da$$

$$\leq \frac{\beta(0)}{V(0)}\int_0^\infty e^{-\frac{\mu(0)}{V(0)}a}da = \frac{\beta(0)}{\mu(0)} < 1.$$

Accordingly, the trivial steady state is locally asymptotically stable if $\beta(0) < \mu(0).$

We also note that if $\beta(0) > \mu(0),$ then if we define a function $g(\zeta)$ by

$$g(\zeta) = \frac{\beta(0)}{V(0)}\int_0^\infty e^{-\left[\frac{\zeta+\mu(0)}{V(0)}\right]a}da,$$

and then if we let ζ to be real, we can easily see that $g(\zeta)$ is a decreasing function of $\zeta > 0, g(\zeta) \to 0$ as $\zeta \to +\infty,$ and $g(0) = \dfrac{\beta(0)}{\mu(0)} > 1.$ Therefore, if $\beta(0) > \mu(0),$ then there exists $\zeta^* > 0$ such that $g(\zeta^*) = 1.$ Hence, the trivial steady state is unstable if $\beta(0) > \mu(0).$ This completes the proof of (1).

To prove (2), we note that if $\dfrac{\beta(0)}{V(0)}\displaystyle\int_0^l e^{-\frac{\mu(0)}{V(0)}a}da < 1,$ then equation (3.3) can not be satisfied for any ζ with $\mathrm{Re}\zeta \geq 0$ since

$$\left|\frac{\beta(0)}{V(0)}\int_0^l e^{-\left[\frac{\zeta+\mu(0)}{V(0)}\right]a}da\right| \le \frac{\beta(0)}{V(0)}\int_0^l \left|e^{-\left[\frac{\zeta+\mu(0)}{V(0)}\right]a}\right|da \le \frac{\beta(0)}{V(0)}\int_0^l e^{-\left[\frac{\operatorname{Re}\zeta+\mu(0)}{V(0)}\right]a}da$$

$$\le \frac{\beta(0)}{V(0)}\int_0^l e^{-\frac{\mu(0)}{V(0)}a}da < 1.$$

Accordingly, the trivial steady state is locally asymptotically stable if $\frac{\beta(0)}{V(0)}\int_0^l e^{-\frac{\mu(0)}{V(0)}a}da < 1.$

We also note that if $\frac{\beta(0)}{V(0)}\int_0^l e^{-\frac{\mu(0)}{V(0)}a}da > 1,$ then if we define a function $g(\zeta)$ by

$$g(\zeta) = \frac{\beta(0)}{V(0)}\int_0^l e^{-\left[\frac{\zeta+\mu(0)}{V(0)}\right]a}da,$$

and then if we let ζ to be real, we can easily see that $g(\zeta)$ is a decreasing function of $\zeta > 0, g(\zeta) \to 0$ as $\zeta \to +\infty,$ and $g(0) = \frac{\beta(0)}{V(0)}\int_0^l e^{-\frac{\mu(0)}{V(0)}a}da > 1.$ Therefore, if $\frac{\beta(0)}{V(0)}\int_0^l e^{-\frac{\mu(0)}{V(0)}a}da > 1,$ then there exists $\zeta^* > 0$ such that $g(\zeta^*) = 1.$ Hence, the trivial steady state is unstable if $\frac{\beta(0)}{V(0)}\int_0^l e^{-\frac{\mu(0)}{V(0)}a}da > 1.$ This completes the proof of (2) and therefore, the proof of the theorem is completed.

Remark 3.1. We note that in Proposition (3.2) in Calsina et al. (2003), which corresponds to (2) of Theorem (3.1) of this paper, the characteristic equation (3.3) is rewritten as follows:

$$\zeta - \alpha_1 - \alpha_2 e^{-\frac{l}{V(0)}\zeta} = 0, \tag{3.4}$$

where $\alpha_1 = \beta(0) - \mu(0)$ and $\alpha_2 = -\beta(0)e^{-\frac{\mu(0)}{V(0)}l}$. And then in Appendix A of that same paper, this equation is rewritten as follows:

$$\zeta - \alpha_1 - \alpha_2 e^{-\tau\zeta} = 0, \tag{3.5}$$

where the left hand-side of equation (3.5) is assumed to be an analytic function of τ and ζ. Now, we can see that there is an error in this argument since α_1 and α_2 are regarded as constants whereas α_2 is clearly a function of τ. Therefore, the proof is incorrect. However, their condition for the stability is correct.

To study the stability of a non-trivial steady state $p_\infty(a)$, which is a solution of (2.1) and is given by equation (2.2), we linearize problem (1.1) at $p_\infty(a)$ in order to obtain a characteristic equation, which in turn will determine conditions for the stability. To that end, we consider a perturbation $\omega(a,t)$ defined by $\omega(a,t) = p(a,t) - p_\infty(a),$ where $p(a,t)$ is a solution of (1.1). Accordingly, we obtain that $\omega(a,t)$ satisfies the following:

$$\begin{cases} \dfrac{\partial\omega(a,t)}{\partial t} + V(P)\dfrac{\partial\omega(a,t)}{\partial a} + \mu(P)\omega(a,t) + \left[V(P)\dfrac{\partial p_\infty(a)}{\partial a} + \mu(P)p_\infty(a)\right] = 0, \\ V(P)\left[\omega(0,t) + p_\infty(0)\right] = \beta(P)P, \\ P = P_\infty + \int_0^l \omega(a,t)da. \end{cases} \tag{3.6}$$

Now, we linearize (3.6) to obtain

$$\begin{cases} \dfrac{\partial\omega(a,t)}{\partial t} + V(P_\infty)\dfrac{\partial\omega(a,t)}{\partial a} + \mu(P_\infty)\omega(a,t) + \left[V'(P_\infty)\dfrac{\partial p_\infty(a)}{\partial a} + \mu'(P_\infty)p_\infty(a)\right]W(t) = 0, \\ V(P_\infty)\omega(0,t) = \left[\beta(P_\infty) + P_\infty\beta'(P_\infty) - p_\infty(0)V'(P_\infty)\right]W(t), \\ W(t) = \int_0^l \omega(a,t)da. \end{cases} \tag{3.7}$$

We then assume that the linearized problem (3.7) has solutions of the form:

$$\omega(a,t) = f(a)e^{t\zeta}, \tag{3.8}$$

where ζ is a complex number.

Now, substituting (3.8) in (3.7) and simplifying, we obtain

$$\begin{cases} f'(a) + \left[\dfrac{\zeta + \mu(P_\infty)}{V(P_\infty)}\right]f(a) + \dfrac{1}{V(P_\infty)}\left[V'(P_\infty)\dfrac{\partial p_\infty(a)}{\partial a} + \mu'(P_\infty)p_\infty(a)\right]W^* = 0, \\ f(0) = \dfrac{1}{V(P_\infty)}\left[\beta(P_\infty) + P_\infty\beta'(P_\infty) - p_\infty(0)V'(P_\infty)\right]W^*, \\ W^* = \int_0^l f(a)da. \end{cases} \tag{3.9}$$

Using (2.2) and solving the ordinary differential equation (3.9), we obtain

$$f(a) = \frac{\beta(P_\infty)e^{-Ea}}{V(P_\infty)}\left\{\frac{P_\infty\mu(P_\infty)}{V(P_\infty)}\left[\frac{V'(P_\infty)}{V(P_\infty)} - \frac{\mu'(P_\infty)}{\mu(P_\infty)}\right]\int_0^a e^{\frac{\zeta\sigma}{V(P_\infty)}}d\sigma + \left[1 + P_\infty\left(\frac{\beta'(P_\infty)}{\beta(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\right)\right]\right\}W^*, \tag{3.10}$$

where E is defined as

$$E = \frac{\left[\zeta + \mu(P_\infty)\right]}{V(P_\infty)}. \tag{3.11}$$

Integrating (3.10) from 0 to l, and simplifying yields the following characteristic equation:

$$1 = \frac{\beta(P_\infty)}{V(P_\infty)}\left[1 + P_\infty\left(\frac{\beta'(P_\infty)}{\beta(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\right)\right]\int_0^l e^{-Ea}da + P_\infty\frac{\beta(P_\infty)\mu(P_\infty)}{\left[V(P_\infty)\right]^2}\left[\frac{V'(P_\infty)}{V(P_\infty)} - \frac{\mu'(P_\infty)}{\mu(P_\infty)}\right]\int_0^l\int_0^a e^{-Ea + \frac{\zeta\sigma}{V(P_\infty)}}d\sigma da = G(\zeta) \tag{3.12}$$

In the next theorem, we give stability results for problem (1.1) in the case $l = +\infty$.

Theorem 3.2. Suppose that $l = +\infty$, then a non-trivial steady state is locally asymptotically stable if $\beta'(P_\infty) < \mu'(P_\infty)$, and is unstable if $\beta'(P_\infty) > \mu'(P_\infty)$.

Proof:

If $l = +\infty$, then from Theorem (2.1) we know that $\mu(P_\infty) = \beta(P_\infty)$, and therefore straight forward calculations in the characteristic equation (3.12) yields

$$\zeta = P_\infty\left[\beta'(P_\infty) - \mu'(P_\infty)\right]. \tag{3.13}$$

From (3.13) the result is clear. This completes the proof of the theorem.

We note that in a simple example of age-structured population model, given in Gurtin et al. (1974), where a is age instead of size and $\mu = \mu(P), \beta = \beta(P)e^{-\alpha a}, \alpha \geq 0, V(P) = 1$, it is shown that the stability of this example is given exactly by the statement of Theorem (3.2), also see Iannelli (1995) for related examples.

In the next theorem, we give a condition for the instability of a non-trivial steady-state of problem (1.1).

Theorem 3.3. A non-trivial steady state of problem (1.1) is unstable if $G(0) > 1$.

Proof:

We note that for ζ real in the right-hand side of the characteristic equation (3.12), we see that $\lim_{\zeta \to \infty} G(\zeta) = 0$, and by assumption $G(0) > 1$. Accordingly, $\exists \zeta^* > 0$ such that $G(\zeta^*) = 1$, and hence a non-trivial steady state is unstable. This completes the proof of the theorem.

We note that if $l < +\infty$, then $G(0) > 1$, is equivalent to

$$\frac{\mu'(P_\infty)}{\mu(P_\infty)} - \frac{\beta'(P_\infty)}{\beta(P_\infty)} < \left[\frac{\mu'(P_\infty)}{\mu(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\right]\left[\beta(P_\infty) - \mu(P_\infty)\right]\frac{l}{V(P_\infty)}. \tag{3.14}$$

And so, according to Theorem (3.3), if $l < +\infty$, then a non-trivial steady state is unstable if (3.14) is satisfied.

In the next theorem, we give stability results for the special case, $l < +\infty$, and $\dfrac{\mu'(P_\infty)}{\mu(P_\infty)} = \dfrac{V'(P_\infty)}{V(P_\infty)}$.

Theorem 3.4. Suppose that $l < +\infty$, and $\dfrac{\mu'(P_\infty)}{\mu(P_\infty)} = \dfrac{V'(P_\infty)}{V(P_\infty)}$, then:

(1) a non-trivial steady state is locally asymptotically stable if

$$\frac{V'(P_\infty)}{V(P_\infty)} - \frac{2}{P_\infty} < \frac{\beta'(P_\infty)}{\beta(P_\infty)} < \frac{V'(P_\infty)}{V(P_\infty)},$$

(2) a non-trivial steady state is unstable if $\dfrac{\beta'(P_\infty)}{\beta(P_\infty)} > \dfrac{V'(P_\infty)}{V(P_\infty)}.$

Proof:

We observe that in this case the characteristic equation (3.12) becomes

$$1 = \frac{\beta(P_\infty)}{V(P_\infty)}\left[1 + P_\infty\left(\frac{\beta'(P_\infty)}{\beta(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\right)\right]\int_0^l e^{-Ea}da = H(\zeta). \tag{3.15}$$

Then it is easy to see that (2) follows directly from Theorem (3.3) and inequality (3.14). To prove (1), we note that straightforward computation yields

$$H(0) = 1 + P_\infty\left(\frac{\beta'(P_\infty)}{\beta(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\right), \tag{3.16}$$

And it is also easy to see that $H(0) > 1$ would imply instability by the same arguments as in Theorem (3.3), and accordingly again (2) follows.

Now, if we suppose that $\mathrm{Re}\,\zeta \geq 0,$ then

$$\left|\frac{\beta(P_\infty)}{V(P_\infty)}\left[1 + P_\infty\left(\frac{\beta'(P_\infty)}{\beta(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\right)\right]\int_0^l e^{-Ea}da\right| \leq \left|\frac{\beta(P_\infty)}{V(P_\infty)}\left[1 + P_\infty\left(\frac{\beta'(P_\infty)}{\beta(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\right)\right]\right|\int_0^l \left|e^{-Ea}\right|da$$

$$\leq |H(0)|.$$

Accordingly, stability occurs if $|H(0)| < 1,$ and therefore,

$$-1 < 1 + P_\infty\left(\frac{\beta'(P_\infty)}{\beta(P_\infty)} - \frac{V'(P_\infty)}{V(P_\infty)}\right) < 1,$$

and hence (1) follows. This completes the proof of the theorem.

In the next theorem, we give stability results for the special case, $l < +\infty,$ and $1 + P_\infty\left(\dfrac{\beta'(P_\infty)}{\beta(P_\infty)} - \dfrac{V'(P_\infty)}{V(P_\infty)}\right) = 0.$

Theorem 3.5. Suppose that $l < +\infty,$ and $1 + P_\infty\left(\dfrac{\beta'(P_\infty)}{\beta(P_\infty)} - \dfrac{V'(P_\infty)}{V(P_\infty)}\right) = 0,$ then:

(1) a non-trivial steady state is locally asymptotically stable if

$$P_\infty\left|\left[\frac{V'(P_\infty)}{V(P_\infty)} - \frac{\mu'(P_\infty)}{\mu(P_\infty)}\right]\right|\left[1 - \frac{l}{V}\big(\beta(P_\infty) - \mu(P_\infty)\big)\right] < 1,$$

(2) a non- trivial steady state is unstable if

$$P_\infty\left[\frac{V'(P_\infty)}{V(P_\infty)}-\frac{\mu'(P_\infty)}{\mu(P_\infty)}\right]\left[1-\frac{l}{V}\big(\beta(P_\infty)-\mu(P_\infty)\big)\right]>1.$$

Proof:

We observe that in this case the characteristic equation (3.12) takes the following form:

$$1=P_\infty\frac{\beta(P_\infty)\mu(P_\infty)}{[V(P_\infty)]^2}\left[\frac{V'(P_\infty)}{V(P_\infty)}-\frac{\mu'(P_\infty)}{\mu(P_\infty)}\right]\int_0^l\int_0^a e^{-Ea+\frac{\zeta\sigma}{V(P_\infty)}}d\sigma da=M(\zeta). \tag{3.17}$$

To prove (1), we note that

$$M(0)=P_\infty\left[\frac{V'(P_\infty)}{V(P_\infty)}-\frac{\mu'(P_\infty)}{\mu(P_\infty)}\right]\left[1-\frac{l}{V}\big(\beta(P_\infty)-\mu(P_\infty)\big)\right],$$

and so, as in the proof of the previous theorem, stability occurs if the condition in (1) is satisfied. The proof of (2) is clear from that of Theorem (3.3). This completes the proof of the theorem.

In the next theorem, we show that, under suitable conditions, and when $l<+\infty$, a non-trivial steady state is locally asymptotically stable.

Theorem 3.6. Suppose that $l<+\infty$, then a non-trivial steady state is locally asymptotically stable if

$$\frac{-2}{P_\infty}<\left(\frac{\beta'(P_\infty)}{\beta(P_\infty)}-\frac{\mu'(P_\infty)}{\mu(P_\infty)}\right)+\left(\frac{V'(P_\infty)}{V(P_\infty)}-\frac{\mu'(P_\infty)}{\mu(P_\infty)}\right)\big(\mu(P_\infty)-\beta(P_\infty)\big)\frac{l}{V(P_\infty)}<0.$$

Proof:

The proof of this theorem follows the same arguments as in Theorem (3.4)-Theorem (3.5), and therefore, we omit the details. This completes the proof of the theorem.

We notice that using the transformation $\tau=\int_0^t V(P(\sigma))d\sigma,$ we can transform problem (1.1) to the following familiar age-structured type, for example, see Calsina et al. (2003) and the references therein:

$$\begin{cases}\dfrac{\partial p(a,\tau)}{\partial\tau}+\dfrac{\partial p(a,\tau)}{\partial a}+\dfrac{\mu(P(\tau))}{V(P(\tau))}p(a,\tau)=0, \quad a\in[0,l),\tau>0,\\ p(0,\tau)=\dfrac{\beta(P(\tau))}{V(P(\tau))}P(\tau), \quad \tau\geq 0,\\ p(a,0)=p_0(a), \quad a\geq 0.\end{cases} \tag{3.18}$$

Integrating along characteristic lines $\tau-a=$ constant, we obtain the following:

$$p(a,t)=\begin{cases} p_0(a-\tau)e^{-\int_0^{\tau}\frac{\mu(P(\sigma))}{V(P(\sigma))}d\sigma}, & a>\tau, \\ \dfrac{\beta(P(\tau-a))}{V(P(\tau-a))}e^{-\int_0^{a}\frac{\mu(P(\tau-a+\sigma))}{V(P(\tau-a+\sigma))}d\sigma}P(\tau-a), & a<\tau. \end{cases} \quad (3.19)$$

From equation (3.19), we obtain that $P(\tau)$ satisfies the following:

$$P(\tau)=\int_0^{\tau}\frac{\beta(P(\tau-a))}{V(P(\tau-a))}e^{-\int_0^{a}\frac{\mu(P(\tau-a+\sigma))}{V(P(\tau-a+\sigma))}d\sigma}P(\tau-a)da+\int_{\tau}^{l}p_0(a-\tau)e^{-\int_0^{\tau}\frac{\mu(P(\sigma))}{V(P(\sigma))}d\sigma}da. \quad (3.20)$$

Now, observe that if $l<+\infty,$ and for $\tau>l,$ we obtain from equation (3.20) that

$$P(\tau)=\int_0^{l}\frac{\beta(P(\tau-a))}{V(P(\tau-a))}e^{-\int_0^{a}\frac{\mu(P(\tau-a+\sigma))}{V(P(\tau-a+\sigma))}d\sigma}P(\tau-a)da. \quad (3.21)$$

On the other hand, if $l=+\infty,$ then from (3.20), we obtain that

$$P(\tau)=\int_0^{\tau}\frac{\beta(P(\tau-a))}{V(P(\tau-a))}e^{-\int_0^{a}\frac{\mu(P(\tau-a+\sigma))}{V(P(\tau-a+\sigma))}d\sigma}P(\tau-a)da+\left(\int_0^{\infty}p_0(a)da\right)e^{-\int_0^{\tau}\frac{\mu(P(\sigma))}{V(P(\sigma))}d\sigma}da. \quad (3.22)$$

We note that the existence and uniqueness of solution for equations of type (3.21) and (3.22) are dealt with in Gripenberg et al. (1990) and O'Regan et al. (1998).

In the following theorem, we prove that, under suitable conditions, the trivial steady state is global stability.

Theorem 3.7. Suppose that $V,\mu,$ and β are positive constants independent of the total population, and that $\beta<\mu,$ then the trivial steady state is globally stable.

Proof:

Let $P^{\infty}=\limsup_{\tau\to\infty}P(\tau),$ then if $l<+\infty,$ we can use Fatou's Lemma in equation (3.21) to obtain that $P^{\infty}\left[1-\frac{\beta}{\mu}(1-e^{-\frac{\mu}{V}l})\right]\le 0,$ therefore, $P^{\infty}=0,$ and hence the trivial steady state is globally stable. On the other hand, if $l=+\infty,$ then from equation (3.22), we obtain the following limiting equation for $P(\tau)$ (see Miller (1971)):

$$P(\tau)=\frac{\beta}{V}\int_0^{\tau}e^{-\frac{\mu}{V}a}P(\tau-a)da. \quad (3.23)$$

Now, we can apply Fatou's Lemma in equation (3.23) to obtain that $P^{\infty}\left[1-\frac{\beta}{\mu}\right]\le 0,$ therefore, again $P^{\infty}=0,$ and hence the trivial steady state is globally stable regardless of $l=+\infty,$ or $l<+\infty.$ This completes the proof of the theorem.

We note that if $V, \mu,$ and β are positive constants independent of the total population, and $\beta < \mu,$ then Theorem (3.1) implies that the trivial steady state is locally asymptotically stable if $l < +\infty,$ or $l = +\infty.$ Also we note that if we use the result in Londen (1973) we will be able to obtain the same result as above for $l = +\infty$ if we use equation (3.22), at the expense of requiring that the solutions of equation (3.22) are bounded i.e.,

$$\sup_{0 \le \tau < \infty} P(\tau) < \infty. \tag{3.24}$$

In the following result, we show that if $V, \mu,$ and β are positive constants independent of the total population, then if $l < +\infty,$ we obtain that the solution of problem (1.1) tends to the trivial steady state.

Theorem 3.8. Suppose that $V, \mu,$ and β are positive constants independent of the total population, and $l < +\infty,$ then $\lim_{\tau \to \infty} P(\tau) = 0.$

Proof:

From equation (3.21), we obtain that

$$P(\tau) = \frac{\beta}{V}\int_0^l e^{-\frac{\mu}{V}a} P(\tau - a)da \le \frac{\beta}{V}\int_0^\tau P(a)da,$$

and therefore, we obtain that

$$P(\tau) \le \frac{\beta}{V}\int_0^\tau P(a)da. \tag{3.25}$$

Accordingly, from (3.25), by using Gronwall's inequality, we obtain that $\lim_{\tau \to \infty} P(\tau) = 0.$ This completes the proof of the theorem.

We note that Theorem (3.8) can easily be generalized as follows.

Theorem 3.9. Suppose that $l < +\infty,$ then $\lim_{\tau \to \infty} P(\tau) = 0.$

Proof:

The arguments are similar to that in Theorem (3.8), and therefore we omit the proof.

We note that Theorem (3.9) asserts that if $l < +\infty,$ then the solution of problem (1.1) tends to the trivial steady state i.e., the trivial steady state is globally stable.

In the next result, we show that if $V, \mu,$ and β are positive constants independent of the total population, and $l = +\infty,$ then the solution of problem (1.1) tends to the trivial steady state.

Theorem 3.10. Suppose that the following hold: 1- $V, \mu,$ and β are positive constants independent of the total population, 2- $l = +\infty,$ 3- (3.24) is satisfied. Then $\lim_{\tau \to \infty} P(\tau) = 0.$

Proof:

From equation (3.22), and by the result in Miller (1971) for limiting equations, we obtain that

$$P(\tau) = \frac{\beta}{V}\int_0^\tau e^{-\frac{\mu}{V}(\tau-a)}P(a)da. \tag{3.26}$$

Accordingly, we obtain the following inequality:

$$P(\tau) \le \frac{\beta}{V}\int_0^\tau P(a)da.$$

And hence the result follows from Gronwall's inequality. This completes the proof of the theorem.

In the following result, we consider the case when $l = +\infty$, and show that if $\frac{\mu(P)}{V(P)} = k = constant,$ then the solution of problem (1.1) tends to the trivial steady-state i.e., the trivial steady state is globally stable.

Theorem 3.11. Suppose that the following conditions hold:
(1) (3.24) is satisfied,
(2) $l = +\infty$, and
(3) $\frac{\mu(P)}{V(P)} = k = constant.$ Then $\lim_{\tau\to\infty} P(\tau) = 0.$

Proof:

First, we consider the case $k > 0$, then by using equation (3.22), condition (3), and the result in Miller (1971) for limiting equations, we obtain the following:

$$P(\tau) = \int_0^\tau \frac{\beta(P(a))}{V(P(a))}e^{-k(\tau-a)}P(a)da.$$

Therefore, we obtain the following inequality:

$$P(\tau) \le \int_0^\tau \frac{\beta(P(a))}{V(P(a))}P(a)da. \tag{3.27}$$

And therefore from (3.27) and by using Gronwall's inequality, we obtain that $\lim_{\tau\to\infty} P(\tau) = 0.$

Secondly, we consider the case $k = 0$, then by considering equation (3.22) and the result in Londen (1973), we obtain that $\lim_{\tau\to\infty} P(\tau)\frac{\beta(P(\tau))}{V(P(\tau))} = 0$, and now we can use equation (3.19) to obtain that $\lim_{\tau\to\infty} p(a,\tau) = 0$, and accordingly by integration, $\lim_{\tau\to\infty} P(\tau) = 0.$ This completes the proof of the theorem.

In the next result, we consider the case when $\lim_{\tau\to\infty}\int_0^\tau \frac{\mu(P(\sigma))}{V(P(\sigma))}d\sigma = +\infty.$

Theorem 3.12. Suppose that $\lim_{\tau\to\infty}\int_0^\tau \frac{\mu(P(\sigma))}{V(P(\sigma))}d\sigma = +\infty.$

Then either $\lim_{\tau\to\infty} P(\tau) = 0,$ or $\lim_{\tau\to\infty} P(\tau)\left[1-\lim_{\tau\to\infty}\frac{\beta(P(\tau))}{\mu(P(\tau))}\right] = 0.$

Proof:

From equation (3.22), we obtain the following:

$$\lim_{\tau\to\infty} P(\tau) = \lim_{\tau\to\infty}\left[\frac{\int_0^\tau \frac{\beta(P(a))}{V(P(a))}P(a)e^{\int_0^a \frac{\mu(P(\sigma))}{V(P(\sigma))}d\sigma}da}{e^{\int_0^\tau \frac{\mu(P(\sigma))}{V(P(\sigma))}d\sigma}}\right]. \tag{3.28}$$

From equation (3.28), we have two cases, the first is that the numerator is finite and accordingly, from our assumption, we obtain that, $\lim_{\tau\to\infty} P(\tau) = 0.$ And the second case is that the numerator approaches $+\infty$ as $\tau \to +\infty,$ and hence, from our assumption, we obtain that

$$\lim_{\tau\to\infty} P(\tau)\left[1-\lim_{\tau\to\infty}\frac{\beta(P(\tau))}{\mu(P(\tau))}\right] = 0.$$

This completes the proof of the theorem.

We note that the result in Theorem (3.12) shows that if $\lim_{\tau\to\infty}\int_0^\infty \frac{\mu(P(\sigma))}{V(P(\sigma))}d\sigma = +\infty,$ then unless $\lim_{\tau\to\infty}\frac{\beta(P(\tau))}{\mu(P(\tau))} = 1,$ $\lim_{\tau\to\infty} P(\tau) = 0.$ Of course, this result is in conformity with the steady state result described in (3) of Theorem (2.1). Now, if we consider the case where $M = \lim_{\tau\to\infty}\int_0^\tau \frac{\mu(P(\sigma))}{V(P(\sigma))}d\sigma < +\infty,$ and assume that $\| p_0 \|_{L^1[0,\infty)} = 0,$ then from equation (3.22), and by the help of Gronwall's inequality, we obtain that $\lim_{\tau\to\infty} P(\tau) = 0.$ However, if we assume that $\| p_0 \|_{L^1[0,\infty)} \neq 0,$ then if we take $\beta, V,$ to be positive constants independent of the total population in equation (3.22), we obtain that

$$P(\tau) \geq \frac{\beta}{V}e^{-M}\int_0^\tau P(a)da \geq \frac{\beta}{V}\|p_0\|_{L^1[0,\infty)}e^{-2M}\tau \to +\infty \text{ as } \tau \to +\infty.$$

Also, if we let $\frac{\beta(P)}{V(P)} = \frac{\beta}{P},$ then $\lim_{\tau\to\infty} P(\tau) \to +\infty$ as $\tau \to +\infty.$

4. Conclusion

In this paper, we studied a size-structured population model where the vital rates i.e., the birth, death, and individual growth rates depend on the total population only. This model could describe some populations like forest, where individuals compete for light or nutrients or fish population. The size of an individual in the population can be finite or infinite with the finite size corresponding to harvesting at specific size. We used a different approach than that in Calsina, et al. (2003), which provided direct stability results, our main goal; and we corrected a proof given therein, and in addition, we proved global stability results. In our approach, we used integral equations formulation to determine the stability of the steady states, whereas in Calsina, et al. (2003), algebraic equations are used. The stability results that we obtained are important for ecological and economical reasons.

It is worth noting that in a subsequent paper, we will continue our study of size-structured population models under more general vital rates.

Acknowledgements

This work is completed while the author is an Arab Regional Fellow at the Center for Advanced Mathematical Sciences (CAMS), American University of Beirut, Beirut, Lebanon, he is supported by a grant from the Arab Fund for Economic and Social Development, and he would like to thank the Director of CAMS for an invitation and hospitality during his stay in CAMS. Also, the author would like to thank Professor Mimmo Iannelli, Professor Angel Calsina and Professor Manual Sanchón for sending references and two anonymous referees for helpful comments and valuable suggestions on the manuscript.

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