



Bidimensional PR QMF with FIR Filters

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Abstract

Multidimensional perfect reconstruction (PR) quadrature mirror filter (QMF) banks with finite impulse response (FIR) filters induced from systems of biorthogonal multivariate scaling functions and wavelets are investigated. In particular, bivariate scaling functions and wavelets with dilation as an expansive integer matrix whose determinant is two in absolute value are considered. Demonstrative quincunxial examples are explicitly given and new FIR filters are constructed.

Key Words: Biorthogonality; Filter banks; Perfect reconstruction; Quadrature mirror filter

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1. Introduction

TENSOR product wavelets have been used in a variety of successful applications due to their simplicity and easy implementation. However, they have several drawbacks, for example, there is no design freedom, and their separable property may be the main reason for high image compression ratios when applied to image processing. Indeed, tensor product scaling functions are scaling functions with matrix dilation $A = 2I_s$, where I_s is the identity matrix of order s . The number of corresponding wavelet generators is as large as $2^s - 1$ for a multivariate scaling function in $P \in \mathbb{R}^s$. In general, the number of multivariate multiwavelet generators corresponding to a multivariate scaling function vector $\phi = [\phi_1, \dots, \phi_r]^T$ with matrix dilation A is $(|\det A| - 1)r$. Hence, to reduce the number of wavelet generators, it is natural to consider dilation matrices with small determinants in modulus such as two or three.

Some tedious but necessary notations and definitions in Section 2. Simple properties of dilation matrices will be listed in Section 3. Polynomial preservation order will be considered in Section 4. Bivariate scaling functions as the four-directional box splines will constitute Section 5. Finally, we investigate bivariate biorthogonal quincunx scaling functions and wavelets in Section 6 while the orthonormality will be considered in Section 7.

2. Preliminaries

To facilitate our presentation and to be more precise, we list some fairly standard but necessary notations and definitions in this section. A multivariate *refinable function vector* $\phi(\mathbf{x}) = [\phi_1(\mathbf{x}), \dots, \phi_r(\mathbf{x})]^\top$ is a vector-valued function satisfying the two-scale relation

$$\phi(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^s} P_{\mathbf{k}} \phi(A\mathbf{x} - \mathbf{k}), \quad x \in \mathbb{R}^s, \quad (1)$$

where A is an expansive integer *dilation matrix*, meaning that it has integer entries and all its eigenvalues are greater than 1 in modulus, $\{P_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^s}$, a set of square matrices of order r and with only finitely many nonzero, is the *two-scale sequence* of ϕ . Such a refinable function vector ϕ is called an *scaling function vector*, if $\{\phi_1(\cdot - \mathbf{k}_1), \dots, \phi_r(\cdot - \mathbf{k}_r) : \mathbf{k}_1, \dots, \mathbf{k}_r \in \mathbb{Z}^s\}$ forms a Riesz basis of $V_0 \subset L^2(\mathbb{R}^s)$, where

$$V_0 = \text{span}_{L^2} \{\phi_1(\cdot - \mathbf{k}_1), \dots, \phi_r(\cdot - \mathbf{k}_r) : \mathbf{k}_1, \dots, \mathbf{k}_r \in \mathbb{Z}^s\}.$$

It is well-known that, corresponding to such a scaling function ϕ , there are $a - 1$ multiwavelet vectors, denoted by

$$\psi^\ell(\mathbf{x}) = [\psi_1^\ell(\mathbf{x}), \dots, \psi_r^\ell(\mathbf{x})]^\top, \quad \ell = 1, \dots, a - 1, \quad (2)$$

where $a := |\det(A)|$. With the Fourier transform \hat{f} of f defined by $\hat{f}(\omega) = \int_{\mathbb{R}^s} f(\mathbf{x}) e^{-j\omega \cdot \mathbf{x}} d\mathbf{x}$, it follows from (1) and with notations in (2), that

$$\hat{\phi}(\omega) = P(e^{-jA^{-\top}\omega}) \hat{\phi}(A^{-\top}\omega), \quad (3)$$

$$\hat{\psi}^\ell(\omega) = Q^\ell(e^{-jA^{-\top}\omega}) \hat{\phi}(A^{-\top}\omega), \quad \ell = 1, \dots, a - 1, \quad (4)$$

where P and Q^ℓ are the *two-scale symbols* of ϕ and ψ^ℓ given by

$$P(\mathbf{z}) = \frac{1}{a} \sum_{\mathbf{k} \in \mathbb{Z}^s} P_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}, \quad (5)$$

$$Q^\ell(\mathbf{z}) = \frac{1}{a} \sum_{\mathbf{k} \in \mathbb{Z}^s} Q_{\mathbf{k}}^\ell \mathbf{z}^{\mathbf{k}}, \quad \ell = 1, \dots, a - 1, \quad (6)$$

respectively. Here the notion $\mathbf{z}^{\mathbf{k}} := \prod_{j=1}^s z_j^{k_j}$ has been used for $\mathbf{z} = [z_1, \dots, z_s]^\top$ and $\mathbf{k} = [k_1, \dots, k_s]^\top \in \mathbb{Z}^s$. Hence, a multivariate scaling function vector ϕ and a set of $a - 1$ multiwavelets corresponding to ϕ can be well-defined via their Fourier transforms, namely,

$$\hat{\phi}(\omega) = \left[\prod_{n=1}^{\infty} P(e^{-jA^{-n\top}\omega}) \right] \hat{\phi}(\mathbf{0}), \quad (7)$$

$$\hat{\psi}^\ell(\omega) = Q^\ell(e^{-jA^{-\top}\omega}) \left[\prod_{n=2}^{\infty} P(e^{-jA^{-n\top}\omega}) \right] \hat{\phi}(\mathbf{0}), \quad \ell = 1, \dots, a - 1. \quad (8)$$

If we let $\gamma_0, \dots, \gamma_{a-1}$ be a set of a representors of the coset space $\mathbb{Z}^s/A^\top\mathbb{Z}^s$, with $\gamma_0 = 0$, it follows from (3) that

$$\sum_{\ell=0}^{a-1} (P \Phi P^\star) \left(e^{-jA^{-\top}(\omega+2\pi\gamma_\ell)} \right) = \Phi(e^{-j\omega}), \quad \omega \in [0, 2\pi]^s, \quad (9)$$

where \star denotes the transpose of the complex conjugation, Φ is the *autocorrelation symbol* of ϕ or the *bracket product* of ϕ and itself, namely,

$$\Phi(\mathbf{z}) = [\phi, \phi](\omega) = \sum_{\mathbf{k} \in \mathbb{Z}^s} \langle \phi(\cdot), \phi(\cdot - \mathbf{k}) \rangle \mathbf{z}^{\mathbf{k}}, \quad (10)$$

where the notation $\mathbf{z} = e^{-j\omega} = [e^{-j\omega_1}, \dots, e^{-j\omega_s}]^\top$ has been used. We will also call (10) the multivariate z -transform of the vector-valued function ϕ . The *stability* of ϕ can be simply described by the positive definiteness of $\Phi(\mathbf{z})$ on $|z_1| = \dots = |z_s| = 1$, while the *orthonormality* (o. n.) of ϕ is equivalent to $\Phi(\mathbf{z}) = I_s$.

For $\psi^1, \dots, \psi^{a-1}$ to be a set of $a-1$ multiwavelets corresponding to such a ϕ , their two-scale symbols Q^1, \dots, Q^{a-1} have to satisfy

$$\sum_{\ell=0}^{a-1} \left(P \Phi (Q^k)^\star \right) \left(e^{-jA^{-\top}(\omega+2\pi\gamma_\ell)} \right) = 0, \quad k = 1, \dots, a-1. \quad (11)$$

Analogously, if Ψ^ℓ is the z -transform of ψ^ℓ in (2), namely,

$$\Psi^\ell(\mathbf{z}) = [\psi^\ell, \psi^\ell](\omega) = \sum_{\mathbf{k} \in \mathbb{Z}^s} \langle \psi^\ell(\cdot), \psi^\ell(\cdot - \mathbf{k}) \rangle \mathbf{z}^{\mathbf{k}},$$

it is also easy to see that

$$\Psi^\ell(e^{-j\omega}) = \sum_{\ell=0}^{a-1} \left(Q^\ell \Phi (Q^\ell)^\star \right) \left(e^{-jA^{-\top}(\omega+2\pi\gamma_\ell)} \right), \quad \omega \in [0, 2\pi]^s, \quad (12)$$

for $\ell = 1, \dots, a-1$. If we further require that the wavelet subspaces generated by $\psi^1, \dots, \psi^{a-1}$ be mutually orthogonal, their two-scale symbols Q^1, \dots, Q^{a-1} are further required to satisfy

$$\sum_{\ell=0}^{a-1} \left(Q^p \Phi (Q^q)^\star \right) \left(e^{-jA^{-\top}(\omega+2\pi\gamma_\ell)} \right) = 0, \quad (13)$$

for $p \neq q$ and $p, q = 1, \dots, a-1$. To summarize, similar to the univariate setting, if we introduce

$$M(\mathbf{z}) = \begin{bmatrix} P(e^{-jA^{-\top}2\pi\gamma_0\mathbf{z}}) & \dots & P(e^{-jA^{-\top}2\pi\gamma_{a-1}\mathbf{z}}) \\ Q^1(e^{-jA^{-\top}2\pi\gamma_0\mathbf{z}}) & \dots & Q^1(e^{-jA^{-\top}2\pi\gamma_{a-1}\mathbf{z}}) \\ \dots & \dots & \dots \\ Q^{a-1}(e^{-jA^{-\top}2\pi\gamma_0\mathbf{z}}) & \dots & Q^{a-1}(e^{-jA^{-\top}2\pi\gamma_{a-1}\mathbf{z}}) \end{bmatrix},$$

all identities in (9), (11), (12), and (13) can be rewritten as the following matrix identity

$$\begin{aligned} M(\mathbf{z}) D(\mathbf{z}) M(\mathbf{z})^\star &= \text{diag} \left(\Phi(e^{-j\omega}), \Psi^1(e^{-j\omega}), \dots, \Psi^{a-1}(e^{-j\omega}) \right), \\ D(\mathbf{z}) &= \text{diag} \left(\Phi(e^{-jA^{-\top}2\pi\gamma_0\mathbf{z}}), \dots, \Phi(e^{-jA^{-\top}2\pi\gamma_{a-1}\mathbf{z}}) \right). \end{aligned}$$

In the wavelet literature, there were some studies for the setting when $A = 2I_2$, particularly with box spline prewavelets (cf., e.g., Chui, et al. [10], Riemenschneider & Shen [26], [27]).

Quincunx dilation was also investigated, e.g., bidimensional o. n. quincunx wavelets in Cohen & Daubechies [11] and Lian [24], biorthogonal box spline wavelets in He & Lai [16], quincunx subdivision in Velho & Zorin [29], quincunx biorthogonal wavelets in Han & Jia [15] and Lian [23], and semi-orthogonal bivariate quincunx wavelets in Lian [22]. In Vetterli & Kovačević [30], the authors studied nonseparable orthogonal quincunx bivariate wavelets as well.

We are interested in, in this paper, the setting when

$$a = |\det(A)| = 2, \quad A^s = \pm 2I_s, \quad (14)$$

so that the number of multivariate wavelet generators corresponding to such a scaling function vector is one. In other words, corresponding to such a multivariate scaling function vector ϕ with r components, there are r wavelets as well. By doing so, some of the identities induced can also be significantly simplified. Observe also that if $A^s = 2I_s$, ϕ is also a scaling function vector with dilation matrix $2I_s$, namely,

$$\widehat{\phi}(\omega) = \check{P}(e^{-j\omega/2}) \widehat{\phi}\left(\frac{1}{2}\omega\right), \quad (15)$$

with

$$\check{P}(e^{-j\omega/2}) = \frac{1}{2^s} \sum_{\mathbf{k} \in \mathbb{Z}^s} \check{P}_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} = \prod_{\ell=1}^s P(e^{-jA^{-\ell\top}\omega}), \quad \mathbf{z} = e^{-j\omega/2}. \quad (16)$$

In particular, when $s = 2$,

$$\check{P}(e^{-j\omega/2}) = P(e^{-jA^{-\top}\omega}) P(e^{-j\omega/2}). \quad (17)$$

We would also like to mention that, most recently, there is a related study in Bakić, et al. [1] to the family $E_n^{(2)}$ of expanding matrix dilations A with $|\det A| = 2$.

3. Dilation Matrices

Under the assumption (14), the dilation matrices are plenty. This fact can be demonstrated by the following.

Lemma 1. Let $s = 2$ and λ_1 and λ_2 be the two eigenvalues of an expansive integer dilation matrix $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, i.e., $a_1, b_1, c_1, d_1 \in \mathbb{Z}$, $|\lambda_1|, |\lambda_2| > 1$. Then A satisfies both $|\det A| = 2$ and $A^2 = \pm 2I_2$ if and only if $d_1 = -a_1$ and $|a_1^2 + b_1c_1| = 2$.

Proof. The conclusion follows immediately from $b_1c_1 \neq 0$ and the formulation of A 's eigenvalues λ , namely, $\lambda = (a_1 + d_1 \pm \sqrt{(a_1 + d_1)^2 - 4\det(A)})/2$. \square

A typical and probably the most interesting example of such a dilation matrix is the quincunx dilation matrix, namely,

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (18)$$

Other examples of dilation matrices satisfying $A^2 = 2I_2$ and with entries being ± 1 's are listed in the following,

$$A_2 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = -A_2, \quad \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = -A_1.$$

The dilation matrix $A = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ was considered in Grochenig & Madych [14] while the dilation matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ was considered in Belogay & Wang [2], which correspond respectively to $a_1 = 0, b_1 = 2$, and $c_1 = \mp 1$ in Lemma 1. It is also clear that when $A^2 = -2I_2$, both eigenvalues λ_1 and λ_2 of A are conjugate complex, namely, $\lambda_1, \lambda_2 = \pm\sqrt{2}j$. When $A^2 = 2I_2$, both λ_1 and λ_2 are real, namely, $\lambda_1, \lambda_2 = \pm\sqrt{2}$. In general, it is easy to see from $\lambda^s I_s - A^s = (\lambda^s \pm 2)I_s$ that $\det(\lambda I_s - A) |(\lambda^s \pm 2)^s$. Hence, we have the following.

Lemma 2. Let w_1, \dots, w_s be a group of s roots of $w^s = 2$. Then $A \in \mathbb{R}^{s \times s}$ satisfies both $|\det A| = 2$ and $A^s = \pm 2I_s$ if and only if

$$A = \pm S \operatorname{diag}(w_1, \dots, w_s) S^{-1},$$

for some nonsingular matrix $S \in \mathbb{R}^{s \times s}$.

In particular, when $s = 3$ and $A^3 = 2I_3$, either $w_1 = w_2 = w_3 = \sqrt[3]{2}$ or $w_1 = \sqrt[3]{2}$ and $w_2, w_3 = \sqrt[3]{2}(-1 \pm \sqrt{3}j)/2$. An example of such a dilation matrix, being called *face-centered orthorhombic* (FCO) sampling matrix and considered as a generalization of the quincunx bivariate to trivariate scaling functions and wavelets (Vetterli & Kovačević [30], p. 435), is given by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

4. Polynomial Preservation and Smoothness

Let π_{m-1}^s be the collection of all s -dimensional polynomials of total degree $\leq m-1$. A compactly supported scaling function vector $\phi(\mathbf{x}) = [\phi_1(\mathbf{x}), \dots, \phi_r(\mathbf{x})]^\top \in (\mathbb{R}^s)^r$ is said to have *polynomial preservation* of order m , $\phi \in \mathbb{PP}_m$ for short, if there is a *superfunction* $f(\mathbf{x})$, which is a finite linear combination of integer shifts of $\phi(\mathbf{x})$, such that it satisfies the Fix-Strang conditions in Fix & Strang [13], namely, $D_\omega^\alpha \widehat{f}(2\pi\mathbf{k}) = \delta_{0,\alpha} \delta_{0,\mathbf{k}}$ for all $|\alpha| \leq m-1$ with $\alpha \in \mathbb{Z}_+^s$ and $\mathbf{k} \in \mathbb{Z}^s$, which, in turn, is equivalent to

$$\sum_{\mathbf{k} \in \mathbb{Z}^s} q(\mathbf{k}) f(\mathbf{x} - \mathbf{k}) = q(\mathbf{x}), \quad q \in \pi_{m-1}^s. \quad (19)$$

Here the notations $|\alpha| = \sum_{j=1}^s \alpha_j$ and $D_\omega^\alpha = \prod_{j=1}^s \partial^{\alpha_j} / \partial \omega_j^{\alpha_j}$ for $\alpha = [\alpha_1, \dots, \alpha_s]^\top$ and $\omega = [\omega_1, \dots, \omega_s]^\top$ have been used. Denote by $\mathbb{S}(\phi)$ the *shift-invariant space* generated by ϕ , namely, $\mathbb{S}(\phi) = \operatorname{span}_{\ell(\mathbb{Z}^s)} \{\phi(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^s\}$, (19) is equivalent to the fact that $\pi_{m-1}^s \subset \mathbb{S}(\phi)$ in the distributional sense. To be more precise, there are vectors

$$\{\mathbf{a}_\alpha^j\}_{\mathbf{j} \in \mathbb{Z}^s} \subset \mathbb{R}^s, \quad |\alpha| \leq m-1,$$

such that

$$\mathbf{x}^\alpha = \sum_{\mathbf{j} \in \mathbb{Z}^s} (\mathbf{a}_\alpha^{\mathbf{j}})^\top \phi(\mathbf{x} - \mathbf{j}), \quad |\alpha| \leq m - 1.$$

The two-scale symbol P of ϕ satisfies *sum rule of order m* with respect to A , $P \in \mathbb{SR}_m$ for short, if the two-scale sequence $\{P_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^s}$ of ϕ satisfies

$$\sum_{\mathbf{k} \in \mathbb{Z}^s} q(A\mathbf{k} + \mathbf{j})P_{A\mathbf{k}+\mathbf{j}} = \sum_{\mathbf{k} \in \mathbb{Z}^s} q(A\mathbf{k})P_{A\mathbf{k}}, \quad \mathbf{j} \in \mathbb{Z}^s; \quad q \in \pi_{m-1}^s.$$

We use the Sobolev exponent to describe the smoothness $\nu(\phi)$ of ϕ , namely,

$$\nu(\phi) = \sup \{ \nu : \phi_\ell \in \mathbb{W}^\nu(\mathbb{R}^s), \ell = 1, \dots, s \}, \tag{20}$$

$$\mathbb{W}^\nu(\mathbb{R}^s) = \left\{ f : \int_{\mathbb{R}^s} (1 + |\omega|^2)^\nu |\hat{f}(\omega)|^2 d\omega < \infty \right\}. \tag{21}$$

There were a plenty of papers in the wavelet literature that studied \mathbb{PP}_m , \mathbb{SR}_m , and the Sobolev smoothness of scaling function vectors and wavelets. We are not in the position here to further elaborate along this topic. For more details of \mathbb{PP}_m and \mathbb{SR}_m , the relationship between \mathbb{SR}_m and \mathbb{PP}_m , the similar notion “polynomial reproduction” introduced earlier in the wavelet literature, and the Sobolev (and Hölder) smoothness, the reader is referred to, e.g., de Boor, et al. [3], Jia [17], Jiang [18], Chui & Jiang [9], Lian [21], and the references therein.

Back to assumptions (14) and $r = 1$, a (single and real-valued) scaling function ϕ that has \mathbb{PP}_m can be significantly simplified. Indeed, in terms of its two-scale symbol P in (5), $\phi \in \mathbb{PP}_m$ if

$$P(\mathbf{1}) = 1, \tag{22}$$

$$D_\omega^\alpha \left(P(e^{-j\omega}) \right) \Big|_{\omega=\pi\mathbf{1}} = 0, \quad |\alpha| \leq m - 1, \tag{23}$$

where $\mathbf{1} = [1, \dots, 1]^\top$. The following proposition can be obtained straightforward from (22)–(23) and a direct application of the multivariate Taylor expansion of a polynomial f at $-\mathbf{1}$, namely,

$$f(\mathbf{z}) = \sum_{q \in \mathbb{Z}_+} \frac{1}{q!} \left((\mathbf{1} + \mathbf{z}) D_{\mathbf{z}}^{\mathbf{1}} \right)^q f(-\mathbf{1}),$$

$$\left((\mathbf{1} + \mathbf{z}) D_{\mathbf{z}}^{\mathbf{1}} \right)^q = \left(\prod_{\ell=1}^s (1 + z_\ell) \frac{\partial}{\partial z_\ell} \right)^q.$$

Proposition 3. Let A be a dilation matrix satisfying (14) and ϕ be a scaling function (i.e., $r = 1$) with two-scale symbol P satisfying (22), namely,

$$\hat{\phi}(\omega) = P(e^{-jA^{-\top}\omega}) \hat{\phi}(A^{-\top}\omega),$$

$$P(\mathbf{z}) = \frac{1}{2} \sum_{\mathbf{k} \in \mathbb{Z}^s} p_{\mathbf{k}} \mathbf{z}^{\mathbf{k}},$$

where $\sum_{\mathbf{k} \in \mathbb{Z}^s} p_{\mathbf{k}} = 2$. Then the following are equivalent.

- 1) $\phi \in \mathbb{PP}_m$.
- 2) P satisfies (23).

3) $\{p_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^s}$ satisfies

$$\sum_{\mathbf{k} \in \mathbb{Z}^s} (-1)^{|\mathbf{k}|} \mathbf{k}^\alpha p_{\mathbf{k}} = 0, \quad |\alpha| \leq m - 1.$$

4) $P \in \mathbb{S}\mathbb{R}_m$, i.e., $\{p_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^s}$ satisfies

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbb{Z}^s} (A\mathbf{k})^\alpha p_{A\mathbf{k}} &= \sum_{\mathbf{k} \in \mathbb{Z}^s} (A\mathbf{k} + \gamma_1)^\alpha p_{A\mathbf{k} + \gamma_1}, \\ |\alpha| &\leq m - 1, \quad \alpha \in \mathbb{Z}_+^s, \end{aligned} \quad (24)$$

where γ_1 is a representer of the coset space $\mathbb{Z}^s/A\mathbb{Z}^s$.

5) P has the form

$$P(\mathbf{z}) = \mathbf{z}^\beta \sum_{\alpha \in \mathbb{Z}_+^s, |\alpha| \geq m} s_\alpha \left(\frac{\mathbf{1} + \mathbf{z}}{2} \right)^\alpha, \quad (25)$$

for some $\beta \in \mathbb{Z}^s$, where $s_\alpha, \alpha \in \mathbb{Z}_+^s, |\alpha| \geq m$, are constants satisfying

$$\sum_{\alpha \in \mathbb{Z}_+^s, |\alpha| \geq m} s_\alpha = 1.$$

First, observe that, from (22) and (24), $P \in \mathbb{S}\mathbb{R}_m$ if and only if $\{p_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^s}$ satisfies

$$\sum_{\mathbf{k} \in \mathbb{Z}^s} p_{A\mathbf{k}} = \sum_{\mathbf{k} \in \mathbb{Z}^s} p_{A\mathbf{k} + \gamma_1} = 1, \quad (26)$$

$$\sum_{\mathbf{k} \in \mathbb{Z}^s} (A\mathbf{k})^\alpha p_{A\mathbf{k}} = \sum_{\mathbf{k} \in \mathbb{Z}^s} (A\mathbf{k} + \gamma_1)^\alpha p_{A\mathbf{k} + \gamma_1}, \quad (27)$$

for $1 \leq |\alpha| \leq m - 1, \alpha \in \mathbb{Z}_+^s$, which is completely similar to the univariate single scaling function setting. Secondly, it is clear from (25) that an *immediate family* of scaling functions $\phi \in \mathbb{P}\mathbb{P}_m$ of non-tensor-product type can be determined by the symbols

$$P(\mathbf{z}) = \left(\frac{\mathbf{1} + \mathbf{z}}{2} \right)^\alpha, \quad |\alpha| = m, \quad (28)$$

which include symbols of certain box splines as we will see in the following section.

5. Four Directional Box Splines

Let M_{d_1, d_2, d_3, d_4} be the bivariate box spline (c. f., e.g., Chui [7] and de Boor [4]), generated by the four directions

$$\xi^\ell = [\xi_1^\ell, \xi_2^\ell]^\top, \quad \ell = 1, \dots, 4,$$

with multiplicities d_1, \dots, d_4 , respectively, i.e.,

$$\widehat{M}_{d_1, d_2, d_3, d_4}(\omega) = \prod_{\ell=1}^4 \left(\frac{1 - e^{-j\xi^\ell \cdot \omega}}{j\xi^\ell \cdot \omega} \right)^{d_\ell}. \quad (29)$$

Then it is easy to see that, for an expansive integer dilation matrix A satisfying (14) with $s = 2$,

$$\frac{\widehat{M}_{d_1, d_2, d_3, d_4}(\omega)}{\widehat{M}_{d_1, d_2, d_3, d_4}(A^{-\top}\omega)} = \left(\frac{1 + e^{-j\xi^1 \cdot A^{-\top}\omega}}{2} \right)^{d_1} \left(\frac{1 + e^{-j\xi^2 \cdot A^{-\top}\omega}}{2} \right)^{d_2}, \quad (30)$$

if d_3, d_4 and ξ^3, ξ^4 are given in terms of d_1, d_2 and ξ^1, ξ^2 , namely,

$$d_3 = d_1, \quad d_4 = d_2, \quad \xi^3 = A\xi^1, \quad \xi^4 = A\xi^2. \quad (31)$$

In other words, under the conditions in (31), the box spline in (29) is a refinable function with dilation matrix A and two-scale symbol in (30). In particular, when A is the quincunx dilation matrix in (18), $d_1 = d_2 = 1$, and

$$\xi^1 = [1, 0]^\top, \quad \xi^2 = [0, 1]^\top, \quad (32)$$

$M_{1,1,1,1}$ is the classical *ZP-element*, or Zwart-Powell function (not a scaling function though). For wavelets corresponding to $M_{1,1,1,1}$, see, for instance, Chui, et al. [8]. Under the same assumptions in (32) but with the dilation matrix A as $A = \begin{bmatrix} 2 & -2 & -2 \end{bmatrix}$, we have a new refinable function that looks like a “twisted” ZP-element.

Observe that (29) gives a family of bivariate refinable scaling functions, as the four-directional box splines, with d_j 's and ξ^j 's satisfying (31), two-scale symbols in (30), and arbitrary dilation matrices A satisfying (14) with $s = 2$. It is also easy to see that the simple two-scale symbols in (28) include those for all four-directional box splines' when $s = 2$.

6. Bivariate Biorthogonal Scaling Functions and Wavelets

We will focus on the quincunx setting, namely, A is fixed as A_1 in (18). Then a representer γ_1 of $\mathbb{Z}^2/A^\top\mathbb{Z}^2$ can be simply chosen as $\gamma_1 = [1, 0]^\top$. Hence, the identity in (9) becomes

$$|P(\mathbf{z})|^2 \Phi(\mathbf{z}) + |P(-\mathbf{z})|^2 \Phi(-\mathbf{z}) = \Phi(z_1 z_2, z_1 z_2^{-1}), \quad \mathbf{z} = e^{-jA^{-\top}\omega}. \quad (33)$$

Following Vaidyanathan ([28], p. 553), a multidimensional filter $H(\omega)$ has *linear phase* if $H(\omega) = ce^{-j\mathbf{k}^\top\omega}H_R(\omega)$ for some constant c , real constant vector \mathbf{k} , and real-valued function $H_R(\omega)$. For bivariate setting, we extend the notion of linear phase filter to *bi-linear phase filter*, if both $H(\omega_1, \omega_2)$ and $H(\omega_2, \omega_1)$ have linear phase. It is easy to see that a bi-linear phase filter is indeed *circularly symmetric*. To motivate our further presentation, we start with the following 5-tap diamond-shape bi-linear phase filter

$$p_{\mathbf{k}} = \frac{1}{4} \begin{bmatrix} & 1 & \\ 1 & 4 & 1 \\ & 1 & \end{bmatrix}. \quad (34)$$

It can be obtained straightforward from (26)–(27) with both $m = 2$ and circular symmetry. If, without loss of generality, the origin is at the middle, it is easy to see that the two-scale symbol corresponding to the 5-tap quincunx bi-linear phase filter is given by

$$P(\mathbf{z}) = \frac{1}{2} \left(1 + \frac{1}{4} (z_1 + z_1^{-1} + z_2 + z_2^{-1}) \right), \quad (35)$$

which satisfies (25) with $\beta = -\mathbf{1}$ and $m = 2$, i.e., $P \in \mathbb{S}\mathbb{R}_2$. In fact,

$$P(\mathbf{z}) = \mathbf{z}^{-1} \left[-\frac{1}{2} \begin{pmatrix} \mathbf{1} + \mathbf{z} \\ 2 \end{pmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{pmatrix} \mathbf{1} + \mathbf{z} \\ 2 \end{pmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{pmatrix} \mathbf{1} + \mathbf{z} \\ 2 \end{pmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{pmatrix} \mathbf{1} + \mathbf{z} \\ 2 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]. \quad (36)$$

Furthermore, Φ in (33) can be calculated (by comparing the coefficients of (33) both sides), omitted here, which is indeed positive on $|z_1| = |z_2| = 1$ after appropriate normalization. Hence, there is a unique scaling function ϕ , to be called *2-d hat quincunx-scaling function*, with its two-scale symbol P given by (35) or (36). By applying the *Matlab* routines in Jiang [19], we have $\phi \in \mathbb{W}^{1.5776}$, where \mathbb{W}^μ denotes the Sobolev smoothness class of functions f introduced in (20)–(21). This 2-d hat quincunx-scaling function is a natural extension of the 1-d hat function which is the second order cardinal B -spline $N_2(x) := (1 - |x|)\chi_{[-1,1]}(x)$, where χ_B is the characteristic function of a set B . Observe also that the 2-d hat quincunx-scaling function ϕ determined by the two-scale symbol P in (35) is also a scaling function with dilation matrix $2I_2$ and two-scale symbol \check{P} given by (17). Its two-scale sequence $\{\check{p}_{\mathbf{k}}\}$ is consequently given by

$$\check{p}_{\mathbf{k}} = \frac{1}{16} \begin{bmatrix} 1 & & & & 1 \\ 1 & 4 & 6 & 4 & 1 \\ & 6 & 16 & 6 & \\ 1 & 4 & 6 & 4 & 1 \\ & 1 & & & 1 \end{bmatrix}.$$

As usual, let ϕ, ψ and $\tilde{\phi}, \tilde{\psi}$ be a bivariate quincunx scaling function and wavelet system such that $\phi, \tilde{\phi}$ and $\psi, \tilde{\psi}$ are *biorthogonal* each other. Let P, Q, \check{P} , and \tilde{Q} be their two-scale symbols, respectively. Then, without going into any detail here, we can construct a $\tilde{\phi} \in PP_4$ with non-diamond shaped *bi-linear phase filter* $\{\tilde{p}_{\mathbf{k}}\}$ given by

$$\tilde{p}_{\mathbf{k}} = \frac{1}{256} \begin{bmatrix} & & 3 & & 3 & & \\ & 6 & -12 & -16 & -12 & 6 & \\ 3 & -12 & -38 & 88 & -38 & -12 & 3 \\ & -16 & 88 & 424 & 88 & -16 & \\ 3 & -12 & -38 & 88 & -38 & -12 & 3 \\ & 6 & -12 & -16 & -12 & 6 & \\ & & 3 & & 3 & & \end{bmatrix}. \quad (37)$$

Consequently,

$$Q(\mathbf{z}) = z_1 \overline{\check{P}(-\mathbf{z})}, \quad \tilde{Q}(\mathbf{z}) = z_1 \overline{\tilde{P}(-\mathbf{z})}. \quad (38)$$

The graphs of both ϕ and $\tilde{\psi}$ are illustrated in Fig. 1, where ϕ was plotted and approximated by ϕ_8 , the eighth iteration of the following cascade algorithm:

$$\begin{aligned} \phi_n(\mathbf{x}) &= \\ & \phi_{n-1}(A\mathbf{x}) + \frac{1}{4} \left[\phi_{n-1} \left(A\mathbf{x} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + \phi_{n-1} \left(A\mathbf{x} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right. \\ & \quad \left. + \phi_{n-1} \left(A\mathbf{x} - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) + \phi_{n-1} \left(A\mathbf{x} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \right], \\ \phi_0(\mathbf{x}) &= \frac{1}{4} \chi_{[-1,1] \times [-1,1]}(\mathbf{x}), \end{aligned}$$

for $n = 1, 2, \dots$, while $\tilde{\psi}$ was plotted by the fifth iteration of the cascade algorithm with respect to $\tilde{\phi}$. Regarding their Fourier transforms $\hat{\phi}$ and $e^{j(\omega_1 + \omega_2)/2} \hat{\tilde{\psi}}$, it follows from (7), (8), and (17)

(a) $\tilde{\phi}$ (b) ψ

Fig. 1. An illustration of the 2-d hat quincunx-scaling function ϕ determined by the 5-tap bi-linear phase filter in (34), and a wavelet $\tilde{\psi}$ biorthogonal to ϕ .

(a) $\tilde{\phi}$ (b) ψ

Fig. 2. An illustration of $\tilde{\phi}$ and ψ .

that

$$\begin{aligned} \hat{\phi}(\omega) &= P(e^{-jA^{-T}\omega}) \left[\prod_{k=1}^{\infty} P(e^{-j\omega/2^k}) P(e^{-jA^{-T}\omega/2^{k+1}}) \right], \\ \hat{\psi}(\omega) &= \tilde{Q}(e^{-jA^{-T}\omega}) \left[\prod_{k=1}^{\infty} \tilde{P}(e^{-j\omega/2^k}) \tilde{P}(e^{-jA^{-T}\omega/2^{k+1}}) \right], \end{aligned}$$

where P is in (35) and \tilde{Q} is in (38). We plot both $\hat{\phi}$ and $e^{j(\omega_1+\omega_2)/2}\hat{\psi}$ in Fig. 3, where the infinite products were truncated up to the seventh factor.

Meanwhile, by applying the *Matlab* routines in Jiang [19], we have $\tilde{\phi} \in \mathbb{W}^{0.3141}$. The graphs of $\tilde{\phi}$ and ψ are illustrated in Fig. 2, where both $\tilde{\phi}$ and ψ were plotted by using the fifth iteration of its cascade algorithm with respect to ϕ . Again, it follows from (7), (8), and (17) that

$$\tilde{\phi}(\omega) = \tilde{P}(e^{-jA^{-T}\omega}) \left[\prod_{k=1}^{\infty} \tilde{P}(e^{-j\omega/2^k}) \tilde{P}(e^{-jA^{-T}\omega/2^{k+1}}) \right],$$

$$(a) \widehat{\phi}(\omega) \qquad (b) e^{j(\omega_1+\omega_2)/2}\widehat{\psi}(\omega)$$

Fig. 3. An illustration of $\widehat{\phi}(\omega)$ and $e^{j(\omega_1+\omega_2)/2}\widehat{\psi}(\omega)$.

$$(a) \widehat{\widetilde{\phi}}(\omega) \qquad (b) e^{j(\omega_1+\omega_2)/2}\widehat{\widetilde{\psi}}(\omega)$$

Fig. 4. An illustration of $\widehat{\widetilde{\phi}}(\omega)$ and $e^{j(\omega_1+\omega_2)/2}\widehat{\widetilde{\psi}}(\omega)$.

$$\widehat{\widetilde{\psi}}(\omega) = Q(e^{-jA^{-\top}\omega}) \left[\prod_{k=1}^{\infty} P(e^{-j\omega/2^k}) P(e^{-jA^{-\top}\omega/2^{k+1}}) \right],$$

where \widetilde{P} is determined from (37) and Q is given in (38). Both $\widehat{\widetilde{\phi}}(\omega)$ and $e^{j(\omega_1+\omega_2)/2}\widehat{\widetilde{\psi}}(\omega)$ are plotted in Fig. 4, where the infinite products were truncated up to the seventh factor.

It is worthwhile to point out the following. First, the bivariate bi-linear phase filter $\{p_{\mathbf{k}}\}$ in (34) can be generalized to the $(2s+1)$ -tap s -dimensional s -linear phase filter, namely, in terms of its two-scale symbol denoted by $P_{s,2}$,

$$P_{s,2}(\mathbf{z}) = \frac{1}{2} \left(1 + \frac{1}{2s} \left(z_1 + z_1^{-1} + \cdots + z_s + z_s^{-1} \right) \right), \quad (39)$$

which satisfies $P_{s,2} \in \mathbb{SR}_2$ and determines a scaling function, to be called the s - d hat function. In other words, the s - d hat function, denoted by $\phi_{s,2}$, is a A -refinable scaling function with s -dimensional dilation matrix A satisfying $|\det(A)| = 2$ and $A^s = 2I_s$. Secondly, if we denote by

$P_2(z) = ((1+z)/2)^2/z$ the two-scale symbol of the second order cardinal B -spline N_2 , (39) is the direct consequence of the McClellan transformation [25], an efficient method of generating 2-d filters from 1-d prototype filters. (See also algorithms in Charoenlarnopparut & Bose [5] by using Gröbner bases.) More precisely,

$$P_{s,2}(\mathbf{z}) = f\left(\frac{z_1 + z_1^{-1} + \cdots + z_s + z_s^{-1}}{2s}\right),$$

where

$$f\left(\frac{z + z^{-1}}{2}\right) = \frac{1}{2}\left(1 + \frac{z + z^{-1}}{2}\right) = P_2(z).$$

However, our dual filter in (37) is not obtained from an McClellan transformation [25] of any 1-d filter dual to $\{1/2, 1, 1/2\}$. Thirdly, it was shown in Kovačević & Vetterli [20] that the following bi-linear phase filter

$$\tilde{p}_{\mathbf{k}} = \frac{1}{16} \begin{bmatrix} & & -1 & & \\ & -2 & 4 & -2 & \\ -1 & 4 & 28 & 4 & -1 \\ & -2 & 4 & -2 & \\ & & -1 & & \end{bmatrix}$$

is also dual to $\{p_{\mathbf{k}}\}$ in (34). But, by applying the *Matlab* routines in Jiang [19], the corresponding $\tilde{\phi} \in \mathbb{W}^{-0.3370}$. That is, $\tilde{\phi}$ is not even a function. It was also shown in Cohen & Daubechies [11] that the size of such a diamond-shaped dual filter was as large as 57 in order for the corresponding scaling function to be *regular*. For more details of the design of multidimensional multirate filters and filter banks derived from 1-d prototype filters, the reader is referred to Dudgeon & Mersereau [12], Chen & Vaidyanathan [6], Vetterli & Kovačević [30], Charoenlarnopparut & Bose [5] and the references therein.

7. Conclusion

Algorithms for constructing multidimensional PR FIR QMF filter banks induced from systems of different kinds of compactly supported multivariate scaling functions and wavelets were established. In particular, when dilation is an expansive integer matrix whose determinant is two in absolute value, bidimensional PR FIR QMF filter banks corresponding to bivariate scaling functions and wavelets were studied. To demonstrate the algorithm, new quincunxial PR FIR QMF filter banks are constructed. Meanwhile, we plan to continue working on: (1) multidimensional PR FIR QMF filter banks with respect to multivariate scaling functions and wavelets with *various* dilation matrices; (2) developing more robust methods for the design of multidimensional multirate filters and filter banks derived from 1-d and 2-d prototype filters; and (3) various PR FIR QMF filter banks with respect to *all* four orthorhombic lattices: simple, base-centered, body-centered, and face-centered.

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References

- [1] D. Bakić, I. Krishtal, and E. N. Wilson, "Parseval frame wavelets with $E_n^{(2)}$ -dilations," *Appl. Comp. Harmonic Analysis* **19** (2005), 386–431.
- [2] E. Belogay and Y. Wang, "Arbitrarily smooth orthogonal nonseparable wavelets in \mathbb{R}^2 ," *SIAM Math. Analysis* **30** (1999), 678–697.
- [3] C. de Boor, R. DeVore, and A. Ron, "Approximation orders of FSI spaces in $L^2(\mathbb{R}^d)$," *Constr. Approx.* **14** (1998), 411–427.
- [4] C. de Boor, K. Höllig, and S. Riemenschneider, "BOX SPLINES," Springer-Verlag, New York, NY, 1993.
- [5] C. Charoenlarnpopparut and N. K. Bose, "Multidimensional FIR filter bank design using Gröbner bases," *IEEE Trans. Circuits Sys.-II: Express Briefs* **46** (1999), 1475–1486.
- [6] T. Chen and P. P. Vaidyanathan, "Recent development in multidimensional multirate systems," *IEEE Trans. Circuits Sys. Video Tech.* **3** (1993), 116–137.
- [7] C. K. Chui, "MULTIVARIATE SPLINES," CBMS-NSF Series in Applied Mathematics #54, SIAM, Philadelphia, 1988.
- [8] C. K. Chui, K. Jetter, and J. Stöckler, "Wavelets and frames on the four-directional mesh," in: WAVELETS: THEORY, ALGORITHMS AND APPLICATIONS, C. K. Chui, L. Montefusco and L. Puccio (eds.), Academic Press, NY, 1994, 213–230.
- [9] C. K. Chui and Q. T. Jiang, "Multivariate balanced vector-valued refinable functions," in: MODERN DEVELOPMENT IN MULTIVARIATE APPROXIMATION, ISNM 145, V. W. Haussmann, K. Jetter, M. Reimer, and J. Stöckler (eds.), Birkhäuser Verlag, Basel, 2003, 71–102.
- [10] C. K. Chui, J. Stöckler, and J. D. Ward, "Compactly supported box spline wavelets," *Appl. Theory Appl.* **8** (1992), 77–100.
- [11] A. Cohen and I. Daubechies, "Nonseparable bidimensional wavelet bases," *Rev. Mat. Iberoamericana* **9** (1993), 51–137.
- [12] D. E. Dudgeon and R. M. Mersereau, "MULTIDIMENSIONAL DIGITAL SIGNAL PROCESSING," Prentice Hall, NJ, 1984.
- [13] G. Fix and G. Strang, "Fourier analysis of the finite element method in Ritz-Galerkin theory," *Studies in Appl. Math.* **48** (1969), 265–273.
- [14] K. Grochenig and W. R. Madych, "Multiresolution analysis, Haar bases, and self-similar tilings of \mathbb{R}^n ," *IEEE Trans. Inform. Theory* **38** (1992), 556–568.
- [15] B. Han and R. Q. Jia, "Quincunx fundamental refinable functions and quincunx biorthogonal wavelets," *Math. Comp.* **71** (2002), 165–196.
- [16] W. J. He and M. J. Lai, "Construction of bivariate compactly supported biorthogonal box spline wavelets with arbitrarily high regularities," *Appl. Comp. Harmonic Analysis*: **6** (1999), 53–74.
- [17] R. Q. Jia, "Approximation properties of multivariate wavelets," *Math. Comp.* **67** (1998), 647–665.
- [18] Q. T. Jiang, "Multivariate matrix refinable functions with arbitrary matrix dilation," *Trans. of Amer. Math. Soc.* **351** (1999), 2407–2438.
- [19] Q. T. Jiang (2001, April). "Matlab routines for Sobolev and Hölder smoothness computations of refinable functions." [Online]. Available: <http://www.cs.umsl.edu/~jiang/Jsoftware.htm>.
- [20] J. Kovačević and M. Vetterli, "Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for \mathbb{R}^n ," *IEEE Trans. Info. Theory* **38** (1992), 533–555.
- [21] J.-A. Lian, "On the order of polynomial reproduction for multi-scaling functions," *Appl. Comp. Harmonic Analysis* **3** (1996), 358–365.
- [22] J.-A. Lian, "On multivariate multiwavelets," in: GEOMETRIC MODELING AND COMPUTING, M. L. Lucian and M. Neamtu (eds.), Nashboro Press, Brentwood, TN, 2005, 377–386.
- [23] J.-A. Lian, Multidimensional PR filter banks with FIR filters, in: PROCEEDINGS OF THE IEEE 2006 INTERNATIONAL CONFERENCE ON COMMUNICATIONS, CIRCUITS, AND SYSTEMS, June 25–28, 2006, Guilin, China, 2006, 162–166.
- [24] J.-A. Lian, On bivariate scaling functions and wavelets, in: WAVELETS AND SPLINES: ATHENS 2005, Guanrong Chen and Ming-Jun Lai (eds.), Nashboro Press, Brentwood, TN, 2006, 328–345.
- [25] J. McClellan, "The design of two-dimensional filters by transformations," in: PROC. 7TH ANN. PRINCETON CONF. ISS, Princeton, NJ, 1973, 274–251.
- [26] S. D. Riemenschneider and Z. W. Shen, "Box splines, cardinal series, and wavelets," in: APPROXIMATION THEORY AND FUNCTIONAL ANALYSIS, C. K. Chui (eds.), Academic Press, New York, NY, 1991, 133–149.
- [27] S. D. Riemenschneider and Z. W. Shen, "Wavelets and pre-wavelets in low dimensions," *J. Approx. Theory* **71** (1992), 18–38.
- [28] P. P. Vaidyanathan, "MULTIRATE SYSTEMS AND FILTER BANKS," Prentice Hall, NJ, 1993.
- [29] L. Velho and D. Zorin, "4-8 subdivision," *Computer-Aided Geometric Design* **18** (2001), 397–427.
- [30] M. Vetterli and J. Kovačević, "WAVELETS AND SUBBAND CODING," Prentice Hall PTR, Englewood Cliffs, NJ, 1995.