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Certain Results for the Laguerre-Gould Hopper Polynomials

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Abstract

In this paper, we derive generating functions for the Laguerre-Gould Hopper polynomials in terms of the generalized Lauricella function by using series rearrangement techniques. Further, we derive the summation formulae for that polynomials by using different analytical means on its generating function or by using certain operational techniques. Also, generating functions and summation formulae for the polynomials related to Laguerre-Gould Hopper polynomials are obtained as applications of main results.

Keywords: Laguerre-Gould Hopper polynomials; Lauricella function; Generating function; Summation formulae

MSC 2010 No.: 33B10, 33C45, 33C65

1. Introduction and preliminaries

Very recently (Khan and Al-Gonah, 2012) introduced the Laguerre-Gould Hopper polynomials (LGHP in the following) ${}_LH_n^{(m,s)}(x,y,z)$ and studied their properties. These polynomials are defined as:

$$ {}_LH_n^{(m,s)}(x,y,z) = n! \sum_{k=0}^{[\frac{n}{s}]} \frac{z^k \, {}_mL_{n-sk}(x,y)}{k!(n-sk)!}, \tag{1.1} $$

where the symbol $\left[\frac{n}{s}\right]$ denotes the greatest integer less than or equal $\frac{n}{s}$ and ${}_mL_n(x,y)$ are the 2-variable generalized Laguerre polynomials (2VgLP) defined by the series definition (Dattoli et al., 1999):

$$ {}_mL_n(x,y) = n! \sum_{r=0}^{[\frac{n}{m}]} \frac{x^r y^{n-mr}}{(r!)^2(n-mr)!} \tag{1.2} $$

and by the operational definition

$$ {}_mL_n(x,y) = \exp\left(D_x^{-1} \frac{\partial^m}{\partial y^m}\right) \Big\{y^n\Big\}, \tag{1.3} $$

where D_x^{-1} denotes the inverse of the derivative operator $D_x := \frac{\partial}{\partial x}$ and is defined in such a way that

$$ D_x^{-n}\Big\{f(x)\Big\} = \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} f(\xi) d\xi, \tag{1.4} $$

so that for $f(x) = 1$, we have

$$ D_x^{-n}\Big\{1\Big\} = \frac{x^n}{n!}. \tag{1.5} $$

The polynomials ${}_LH_n^{(m,s)}(x,y,z)$ are defined by the following generating function:

$$ \exp(yt + zt^s) C_0(-xt^m) = \sum_{n=0}^{\infty} {}_LH_n^{(m,s)}(x,y,z) \frac{t^n}{n!}, \tag{1.6} $$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined as (Andrews, 1985):

$$ C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}. \tag{1.7} $$

Also, we recall that the generalized Bessel function or the Bessel-Wright function is defined by (Srivastava and Manocha, 1984)

$$ J_n^{(m)}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!(n+mk)!}, \tag{1.8} $$

which for $x \to -x$ yields (Dattoli et al., 2006, p.33)

$$ W_n(x;m) = \sum_{k=0}^{\infty} \frac{x^k}{k!(n+mk)!} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; m \in \mathbb{N}) \tag{1.9} $$

and for $m = 1$, reduces to the n^{th} order Tricomi functions (1.7). The associated Laguerre polynomials $L_n^{\alpha}(x)$ are defined by the series (Abramowitz and Stegun, 1992)

$$ L_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{(-1)^k \ (n+\alpha)! \ x^k}{(n-k)!(\alpha+k)!k!}, \qquad \alpha = 0, 1, 2, \ldots \quad . \tag{1.10} $$

Also, the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ are defined by the following operational rules:

$$ {}_LH_n^{(m,s)}(x,y,z) = \exp\left(z\frac{\partial^s}{\partial y^s}\right)\Big\{{}_mL_n(x,y)\Big\}, \tag{1.11a} $$

$$ {}_LH_n^{(m,s)}(x,y,z) = \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\Big\{H_n^{(s)}(y,z)\Big\}, \tag{1.11b} $$

where $H_n^{(s)}(x,y)$ are the higher-order Hermite polynomials, some times called the Kampé de Fériet or the Gould-Hopper polynomials (GHP) defined by (Gould and Hopper, 1962, p.58), see also (Dattoli et al., 2000a)

$$ g_n^s(x,y) = H_n^{(s)}(x,y) = n!\sum_{k=0}^{[\frac{n}{s}]}\frac{y^k x^{n-sk}}{k!(n-sk)!}. \tag{1.12} $$

The polynomials $H_n^{(s)}(x,y)$ are specified by the generating function

$$ \exp(xt+yt^s) = \sum_{n=0}^{\infty} H_n^{(s)}(x,y)\frac{t^n}{n!} \tag{1.13} $$

and by the operational definition

$$ H_n^{(s)}(x,y) = \exp\left(y\frac{\partial^s}{\partial x^s}\right)\Big\{x^n\Big\}. \tag{1.14} $$

(Gould and Hopper, 1962) used the notation $g_n^s(x,y)$ for these polynomials, but due to their link with the Hermite polynomials the notation $H_n^{(s)}(x,y)$ was used in most of the works. We also follow the notation $H_n^{(s)}(x,y)$ in this paper. We note the following special cases:

$$ {}_1L_n(-x,y) = L_n(x,y), \tag{1.15} $$

$$ H_n^{(2)}(x,y) = H_n(x,y), \tag{1.16a} $$

$$ H_n^{(s)}(x,0) = H_n(x,0) = x^n, \tag{1.16b} $$

where $L_n(x,y)$ denotes the 2-variable Laguerre polynomials (2VLP) (Dattoli and Torre, 1998) and $H_n(x,y)$ denotes the 2-variable Hermite-Kampé de Fériet polynomials
(2VHKdFP) (Appell and de Fériet, 1926) respectively. Also, we note that

$$ H_n\left(x,-\frac{1}{2}\right) = He_n(x), \tag{1.17} $$

where $He_n(x)$ denotes the classical Hermite polynomials (Andrews, 1985). Further, we have

$$ {}_LH_n^{(1,2)}(-x,y,z) = {}_LH_n(x,y,z), \tag{1.18} $$

$$ {}_LH_n^{(1,2)}\left(-x,y,-\frac{1}{2}\right) = {}_LH_n\left(x,y,-\frac{1}{2}\right) = {}_LH_n^*(x,y) \tag{1.19} $$

and

$$ {}_LH_n^{(1,1)}(-x,y,z) = L_n(x,y+z), \tag{1.20} $$

where ${}_LH_n(x,y,z)$ denotes the 3-variable Laguerre-Hermite polynomials (3VLHP) (Dattoli et al., 2000a) and ${}_LH_n^*(x,y)$ denotes the 2-variable Laguerre-Hermite polynomials (2VLHP) (Dattoli et al., 2000b), respectively. The study of the properties of multi-variable generalized special functions has provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. The relevance of the special functions in physics is well established. Most of the special functions of mathematical physics as well as their generalizations have been suggested by physical problems. We know that the 2VHKdFP $H_n(x,y)$ are solutions of the heat equation

$$\frac{\partial}{\partial y}H_n(x,y)=\frac{\partial^2}{\partial x^2}H_n(x,y), \tag{1.21a}$$

with the initial condition

$$H_n(x,0)=x^n. \tag{1.21b}$$

Also, the 2VLP $L_n(x,y)$ are natural solutions of the equation

$$\frac{\partial}{\partial y}L_n(x,y)=-\left(\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)L_n(x,y), \tag{1.22a}$$

with the initial condition

$$L_n(x,0)=\frac{(-x)^n}{n!}, \tag{1.22b}$$

which is a kind of heat diffusion equation of Fokker-Plank type and is used to study the beam life-time due to quantum fluctuation in storage rings (Wr$\ddot{u}$ullich, 1992). In reference (Dattoli, 2004), it has been shown that the summation formulae of special functions, often encountered in applications ranging from electromagnetic processes to combinatorics, can be written in terms of Hermite polynomials of more than one variable. The work of this paper is motivated by the results on generating functions for the extended generalized Hermite polynomials due to (Greubel, 2006) and the recently derived summation formulae for Hermite and the Gould-Hopper polynomials by (Khan et al., 2008) and (Khan and Al-Saad, 2011). Throughout this work, we use the Pochhammer symbol $(\lambda)_n$, defined by (Srivastava and Manocha, 1984, p.21)

$$(\lambda)_n=\begin{cases} 1, & \text{if} \quad n=0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if} \quad n\in 1,2,3,\cdots, \end{cases} \tag{1.23}$$

also, we note that

$$(\lambda)_{m+n}=(\lambda)_m(\lambda+m)_n, \tag{1.24}$$

$$(n-mk)!=\frac{(-1)^{mk}\ n!}{(-n)_{mk}},\quad 0\le k\le\left[\frac{n}{m}\right] \tag{1.25}$$

and

$$(n-M)!=\frac{(-1)^{M}n!}{(-n)_{M}},\quad 0\le M\le n, \tag{1.26}$$

where M is defined by $M=m_1k_1+m_2k_2+\cdots+m_jk_j,\ m_1,m_2,...,m_j\in\mathbb{N}\ ;k_1,k_2,...,k_j\in\mathbb{N}_0$. We recall that, the Kampé de Fériet function of two variables is defined by (Srivastava and Manocha, 1984, p.63)

$$F^{p:q;k}_{l:m;n}\left[\begin{smallmatrix}(a_p):\ (b_q);(c_k);\\ (\alpha_l):(\beta_m);(\gamma_n);\end{smallmatrix}x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}. \tag{1.27}$$

A further generalization of the Kampé de Fériet function (1.27) is the generalized Lauricella function of several variables, which is defined as (Srivastava and Manocha, 1984, p.64):

$$F^{A:B';B'';...;B^{(n)}}_{C:D';D'';...;D^{(n)}}(z_1,z_2,...,z_n)$$
$$\equiv F^{A:B';B'';...;B^{(n)}}_{C:D';D'';...;D^{(n)}}\left(\begin{matrix}[(a):\theta',\theta'',...,\theta^{(n)}]:[(b'):\phi'];[(b''):\phi''];...;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\psi'',...,\psi^{(n)}]:[(d'):\delta'];[(d''):\delta''];...;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,z_2,...,z_n\right)$$
$$=\sum_{m_1,m_2,...,m_n=0}^{\infty}\Omega(m_1,m_2,...,m_n)\,\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}\cdots\frac{z_n^{m_n}}{m_n!}, \tag{1.28}$$

where

$$\Omega(m_1,m_2,...,m_n)$$
$$:=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+m_2\theta''_j+\cdots+m_n\theta^{(n)}_j}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\prod_{j=1}^{B''}(b''_j)_{m_2\phi''_j}\cdots\prod_{j=1}^{B^{(n)}}(b^{(n)}_j)_{m_n\phi^{(n)}_j}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+m_2\psi''_j+\cdots+m_n\psi^{(n)}_j}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\prod_{j=1}^{D''}(d''_j)_{m_2\delta''_j}\cdots\prod_{j=1}^{D^{(n)}}(d^{(n)}_j)_{m_n\delta^{(n)}_j}}$$

and the coefficients

$$\theta^{(k)}_j, j=1,2,...,A;\quad \phi^{(k)}_j, j=1,2,...,B^{(k)};\quad \psi^{(k)}_j, j=1,2,...,C;\quad \delta^{(k)}_j, j=1,2,...,D^{(k)};$$

for all $k\in\{1,2,...,n\}$ are real and positive, (a) abbreviates the array of A parameters $a_1,a_2,...,a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters $b^{(k)}_j, j=1,2,...,B^{(k)}$; for all $k\in\{1,2,...,n\}$ with similar interpretations for (c) and $(d^{(k)})$, $k=1,2,...,n$; $et\ cetera$. Note that, when the coefficients in equation (1.28) equal to 1, the generalized Lauricella function (1.28) reduces to a direct multivariable extension of the Kampé de Fériet function (1.27). Taking coefficients equal to 1 in definition (1.28) and for $n=3$, we have the Kampé de Fériet function of three variables

$$F^{p:q_1;q_2;q_3}_{l:s_1;s_2;s_3}(z_1,z_2,z_3)\equiv F^{p:q_1;q_2;q_3}_{l:s_1;s_2;s_3}\left(\begin{matrix}(a_p):(b'_{q_1});(b''_{q_2});[(b'''_{q_3});\\ (c_l):(d'_{s_1});(d''_{s_2});(d'''_{s_3});\end{matrix}\; z_1,z_2,z_3\right)$$
$$=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{m_1+m_2+m_3}\prod_{j=1}^{q_1}(b'_j)_{m_1}\prod_{j=1}^{q_2}(b''_j)_{m_2}\prod_{j=1}^{q_3}(b'''_j)_{m_3}}{\prod_{j=1}^{l}(c_j)_{m_1+m_2+m_3}\prod_{j=1}^{s_1}(d'_j)_{m_1}\prod_{j=1}^{s_2}(d''_j)_{m_2}\prod_{j=1}^{s_3}(d'''_j)_{m_3}}$$
$$\times\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}\frac{z_3^{m_3}}{m_3!}. \tag{1.29}$$

As an illustration, we derive the following explicit representation of the LGHP ${}_LH^{(m,s)}_n(x,y,z)$ in terms of the generalized Lauricella function of two variables:

$${}_LH^{(m,s)}_n(x,y,z)=y^nF^{1:0;0}_{0:0;1}\left(\begin{matrix}[-n:s,m]:-;\ -;\\ -:-;[1:1];\end{matrix}\; z\left(\frac{-1}{y}\right)^s,\ x\left(\frac{-1}{y}\right)^m\right). \tag{1.30}$$

In order to derive representation (1.30), we use series definition (1.2) of the 2VgLP ${}_mL_n(x,y)$ in the r.h.s. of definition (1.1), so that we have

$${}_LH^{(m,s)}_n(x,y,z)=n!\sum_{k,r=0}^{sk+mr\le n}\frac{z^kx^ry^{n-sk-mr}}{k!(r!)^2(n-sk-mr)!}. \tag{1.31}$$

Now, using equation (1.26) and the elementary factorial property $n!=(1)_n$ in the r.h.s. of equation (1.31) and in view of definition (1.28) (for $n=2$), we get representation (1.30). In this paper, we derive the generating functions for the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ in terms of the generalized Lauricella function of three variables $F_{C:D';D'';D'''}^{A:B';B'';B'''}[.]$ by using series rearrangement techniques. Further, we derive the summation formulae for the LGHP by using different analytical means on the generating function of the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ or by using certain operational techniques.

2. Generating functions for the Laguerre-Gould Hopper polynomials

First, we prove the following generating function for the LGHP ${}_LH_n^{(m,s)}(x,y,z)$.

Theorem 2.1. *For a suitable bounded sequence* $\{f(n)\}_{n=0}^{\infty}$ *, the following generating function for the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true:*

$$\sum_{n=0}^{\infty} f(n)\ {}_LH_n^{(m,s)}(x,y,z)\ t^n$$
$$= \sum_{n,k,r=0}^{\infty} f(n+sk+mr)\frac{(1)_{n+sk+mr}}{(1)_r}\frac{(yt)^n}{n!}\frac{(zt^s)^k}{k!}\frac{(xt^m)^r}{r!}. \tag{2.1}$$

Proof. Denoting the l.h.s. of equation (2.1) by Δ_1 and using equation (1.31), we find

$$\Delta_1 = n!\sum_{n=0}^{\infty}\sum_{k,r=0}^{sk+mr\le n} f(n)\ \frac{z^k x^r y^{n-sk-mr} t^n}{k!(r!)^2(n-sk-mr)!}.$$

Replacing n by $n+sk+mr$ in the above equation and using the lemma (Srivastava and Manocha, 1984, p.102):

$$\sum_{n=0}^{\infty}\sum_{k_1,k_2,\ldots,k_r=0}^{M\le n}\phi(k_1,k_2,\cdots,k_r;n) = \sum_{n=0}^{\infty}\sum_{k_1,k_2,\ldots,k_r=0}^{\infty}\phi(k_1,k_2,\cdots,k_r;n+M), \tag{2.2}$$

where M is defined by $M = m_1k_1+m_2k_2+\cdots+m_jk_j,\ \ m_1,m_2,...,m_j \in \mathbb{N}\ ; k_1,k_2,...,k_j \in \mathbb{N}_0$, we find

$$\Delta_1 = \sum_{n,k,r=0}^{\infty} f(n+sk+mr)\ \frac{(n+sk+mr)!\ (yt)^n(zt^s)^k(xt^m)^r}{n!k!(r!)^2}.$$

Now, using the elementary factorial property $n! = (1)_n$, we get the r.h.s. of assertion (2.1) of Theorem 2.1.

Remark 2.1. Taking $f(n) = \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n\prod_{j=1}^{l}(c_j)_n}$ in assertion (2.1) of Theorem 2.1 and using definition (1.28) (for $n = 3$), we deduce the following consequence of Theorem 2.1.

Corollary 2.1. *The following generating function for the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true*:

$$\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n\prod_{j=1}^{l}(c_j)_n}{}_LH_n^{(m,s)}(x,y,z)\ t^n$$
$$= F_{q+l:0;0;1}^{p+1:0;0;0}\left(\begin{smallmatrix}[(a)_1^p:1,s,m],[1:1,s,m]:-;-;\ \ -;\\ [(b)_1^q:1,s,m],[(c)_1^l:1,s,m]:-;-;[1:1];\end{smallmatrix}\ yt, zt^s, xt^m\right), \tag{2.3}$$

where the notation $(a)_1^p$ *is used to represent the product* $\prod_{j=1}^{p} a_j$.

Remark 2.2. Taking $f(n) = \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}$ in assertion (2.1) of Theorem 2.1 and using definition (1.28) (for $n = 3$), we deduce the following consequence of Theorem 2.1.

Corollary 2.2. *The following generating function for the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true*:

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} {}_LH_n^{(m,s)}(x,y,z)\ t^n$$

$$= F^{p+1:0;0;0}_{q:0;0;1}\left({}^{[(a)_1^p:1,s,m],[1:1,s,m]:-;-;\quad -;}_{[(b)_1^q:1,s,m]:-;-;[1:1];}\ yt, zt^s, xt^m \right). \tag{2.4}$$

Next, we prove the following bilateral generating function for the LGHP ${}_LH_n^{(m,s)}(x,y,z)$.

Theorem 2.2. *The following bilateral generating function for the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true:*

$$\sum_{n=0}^{\infty} J_n^{(s)}(w)\ {}_LH_n^{(m,s)}(x,y,z)\frac{t^n}{n!} = F^{0:0;0;0}_{1:0;0;1}\left({}^{-:-;-;\quad -;}_{[1:1,s,m]:-;-;[1:1];}\ yt, zt^s - w, xt^m \right). \tag{2.5}$$

Proof. Denoting the l.h.s. of equation (2.5) by Δ_2 and using definitions (1.8) and (1.31), we find

$$\Delta_2 = \sum_{n,p=0}^{\infty} \sum_{k,r=0}^{sk+mr\leq n} \frac{(-1)^p w^p z^k x^r y^{n-sk-mr} t^n}{p!(n+sp)!k!(r!)^2(n-sk-mr)!}\ .$$

Replacing n by $n + sk + mr$ and using equation (2.2) in the resultant equation, we find

$$\Delta_2 = \sum_{n,p,k,r=0}^{\infty} \frac{(-1)^p w^p y^n z^k x^r t^{n+sk+mr}}{(n+sk+sp+mr)!\ p!n!\ k!\ (r!)^2}.$$

Now, replacing k by $k - p$ in the above equation and using equation (1.25) (for $m = 1$) in the resultant equation, we find

$$\Delta_2 = \sum_{n,k,r=0}^{\infty} \frac{(yt)^n(zt^s)^k(xt^m)^r}{(1)_{n+sk+mr}(1)_r n!k!r!} \sum_{p=0}^{k} \frac{(-k)_p}{p!}\left(\frac{w}{zt^s}\right)^p. \tag{2.6}$$

Finally, using the expansion (Srivastava and Manocha, 1984)

$$(1-x)^{-\lambda} = \sum_{n=0}^{\infty} (\lambda)_n\ \frac{x^n}{n!}, \tag{2.7}$$

and definition (1.28) (for $n = 3$) in equation (2.6), we get the r.h.s. of assertion (2.5) of Theorem 2.2.

Remark 2.3. Taking $w = zt^s$ in assertion (2.5) of Theorem 2.2, we deduce the following consequence of Theorem 2.2.

Corollary 2.3. *The following bilateral generating function for the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true:*

$$\sum_{n=0}^{\infty} J_n^{(s)}(zt^s)\ {}_LH_n^{(m,s)}(x,y,z)\frac{t^n}{n!} = F_{1:0;1}^{0:0;0}\left({}_{[1:1,m]:-;[1:1];}^{-:-;\ \ -;}\ \ yt, xt^m\right). \tag{2.8}$$

In the forthcoming section, we establish summation formulae for the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ by using series rearrangement techniques and also by making use of the operational techniques.

3. Summation formulae for the Laguerre-Gould Hopper polynomials

First, we prove the following result involving the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ by using series rearrangement techniques:

Theorem 3.1. *The following summation formula for the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true:*

$${}_LH_{k+l}^{(m,s)}(x,w,v) = \sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}H_{n+r}^{(s)}(w-y,v-z)\ {}_LH_{k+l-n-r}^{(m,s)}(x,y,z). \tag{3.1}$$

Proof. Replacing t by $t+u$ in equation (1.6) and using the formula (Srivastava and Manocha, 1984, p.52):

$$\sum_{n=0}^{\infty} f(n)\frac{(x+y)^n}{n!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.2}$$

in the resultant equation, we find the following generating function for the LGHP ${}_LH_n^{(m,s)}(x,y,z)$:

$$\exp(y(t+u)+z(t+u)^s)C_0(-x(t+u)^m) = \sum_{k,l=0}^{\infty}\frac{t^k}{k!}\frac{u^l}{l!}\ {}_LH_{k+l}^{(m,s)}(x,y,z),$$

which can be written as

$$C_0(-x(t+u)^m) = \exp(-y(t+u)-z(t+u)^s)\sum_{k,l=0}^{\infty}\frac{t^k}{k!}\frac{u^l}{l!}\ {}_LH_{k+l}^{(m,s)}(x,y,z). \tag{3.3}$$

Replacing y by w and z by v in equation (3.3) and equating the resultant equation to itself, we find

$$\sum_{k,l=0}^{\infty}\frac{t^k}{k!}\frac{u^l}{l!}\ {}_LH_{k+l}^{(m,s)}(x,w,v)$$

$$= \exp((w-y)(t+u)+(v-z)(t+u)^s)\sum_{k,l=0}^{\infty}\frac{t^k}{k!}\frac{u^l}{l!}\ {}_LH_{k+l}^{(m,s)}(x,y,z), \tag{3.4}$$

which on using the generating function (1.13) in the exponential on the r.h.s., becomes

$$\sum_{k,l=0}^{\infty}\frac{t^k}{k!}\frac{u^l}{l!}\ {}_LH_{k+l}^{(m,s)}(x,w,v)$$

$$= \sum_{n=0}^{\infty} \frac{(t+u)^n}{n!} H_n^{(s)}(w-y, v-z) \sum_{k,l=0}^{\infty} \frac{t^k}{k!} \frac{u^l}{l!} \ {}_LH_{k+l}^{(m,s)}(x,y,z). \tag{3.5}$$

Again, using formula (3.2) in the first summation on the r.h.s. of equation (3.5), we have

$$\sum_{k,l=0}^{\infty} \frac{t^k}{k!} \frac{u^l}{l!} \ {}_LH_{k+l}^{(m,s)}(x,w,v)$$

$$= \sum_{n,r=0}^{\infty} \frac{t^n u^r}{n! r!} H_{n+r}^{(s)}(w-y, v-z) \sum_{k,l=0}^{\infty} \frac{t^k}{k!} \frac{u^l}{l!} \ {}_LH_{k+l}^{(m,s)}(x,y,z). \tag{3.6}$$

Now, replacing k by $k-n$ and l by $l-r$ in the r.h.s. of equation (3.6) and using the lemma (Srivastava and Manocha, 1984, p.100):

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} A(n, k-n), \tag{3.7}$$

in the resultant equation, we find

$$\sum_{k,l=0}^{\infty} \frac{t^k}{k!} \frac{u^l}{l!} \ {}_LH_{k+l}^{(m,s)}(x,w,v)$$

$$= \sum_{k,l=0}^{\infty} \sum_{n,r=0}^{k,l} \frac{t^k u^l \ H_{n+r}^{(s)}(w-y, v-z)}{n! r! (k-n)! (l-r)!} \ {}_LH_{k+l-n-r}^{(m,s)}(x,y,z). \tag{3.8}$$

Finally, on equating the coefficients of like powers of t and u in equation (3.8), we get assertion (3.1) of Theorem 3.1.

Remark 3.1. Replacing v by z in assertion (3.1) of Theorem 3.1 and using relation (1.16b), we deduce the following consequence of Theorem 3.1.

Corollary 3.1. *The following summation formula for the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true:*

$$ {}_LH_{k+l}^{(m,s)}(x,w,z) = \sum_{n,r=0}^{k,l} \binom{k}{n} \binom{l}{r} (w-y)^{n+r} \ {}_LH_{k+l-n-r}^{(m,s)}(x,y,z). \tag{3.9}$$

Next, we prove the following result involving the product of the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ by using series rearrangement techniques:

Theorem 3.2. *The following summation formula involving product of the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true:*

$$ {}_LH_n^{(m,s)}(x,w,u) \ {}_LH_r^{(m,s)}(X,W,U) = \sum_{k,p=0}^{n,r} \binom{n}{k} \binom{r}{p} H_k^{(s)}(w-y, u-z)$$

$$\times H_p^{(s)}(W-Y, U-Z) \ {}_LH_{n-k}^{(m,s)}(x,y,z) \ {}_LH_{r-p}^{(m,s)}(X,Y,Z). \tag{3.10}$$

Proof. Consider the product of the LGHP generating function (1.6) in the following form:

$$\exp(yt+YT+zt^s+ZT^s)\ C_0(-xt^m)\ C_0(-XT^m)$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}{}_LH_n^{(m,s)}(x,y,z)\ {}_LH_r^{(m,s)}(X,Y,Z)\frac{t^n}{n!}\frac{T^r}{r!}. \tag{3.11}$$

Replacing y by w, z by u, Y by W and Z by U in equation (3.11) and equating the resultant equation to itself, we find

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}{}_LH_n^{(m,s)}(x,w,u)\ {}_LH_r^{(m,s)}(X,W,U)\frac{t^n}{n!}\frac{T^r}{r!}=\exp((w-y)t+(u-z)t^s)$$
$$\times\exp((W-Y)T+(U-Z)T^s)\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}{}_LH_n^{(m,s)}(x,y,z)\ {}_LH_r^{(m,s)}(X,Y,Z)\frac{t^n}{n!}\frac{T^r}{r!}, \tag{3.12}$$

which on using the generating function (1.13) in the exponentials on the r.h.s., becomes

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}{}_LH_n^{(m,s)}(x,w,u)\ {}_LH_r^{(m,s)}(X,W,U)\frac{t^n}{n!}\frac{T^r}{r!}$$
$$=\sum_{n,k=0}^{\infty}\sum_{r,p=0}^{\infty}H_k^{(s)}(w-y,u-z)\ {}_LH_n^{(m,s)}(x,y,z)\ \frac{t^{n+k}}{n!k!}\ H_p^{(s)}(W-Y,U-Z)$$
$$\times\ {}_LH_r^{(m,s)}(X,Y,Z)\ \frac{T^{r+p}}{r!p!}. \tag{3.13}$$

Finally, replacing n by $n-k$, r by $r-p$ and using equation (3.7) in the r.h.s. of the above equation and then equating the coefficients of like powers t and T, we get assertion (3.10) of Theorem 3.2.

Remark 3.2. Replacing u by z and U by Z in assertion (3.10) of Theorem 3.2 and using relation (1.16b), we deduce the following consequence of Theorem 3.2.

Corollary 3.2. *The following summation formula involving product of the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true:*

$${}_LH_n^{(m,s)}(x,w,z)\ {}_LH_r^{(m,s)}(X,W,Z)=\sum_{k,p=0}^{n,r}\binom{n}{k}\binom{r}{p}(w-y)^k(W-Y)^p$$
$$\times\ {}_LH_{n-k}^{(m,s)}(x,y,z)\ {}_LH_{r-p}^{(m,s)}(X,Y,Z). \tag{3.14}$$

Further, we prove the following results connecting the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ with the 2VgLP ${}_mL_n(x,y)$ and the GHP $H_n^{(s)}(x,y)$ by using operational techniques:

Theorem 3.3. *The following summation formula for the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true:*

$${}_LH_{k+l}^{(m,s)}(z,w,y)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}{}_mL_{n+r}(z,w-x)\ H_{k+l-n-r}^{(s)}(x,y). \tag{3.15}$$

Proof. We start by a recently derived summation formula for the GHP $H_n^{(s)}(x,y)$ (Khan and Al-Saad, 2011, p.1538)

$$H_{k+l}^{(s)}(w,y)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}H_{k+l-n-r}^{(s)}(x,y). \tag{3.16}$$

Operating $\exp\left(D_z^{-1}\frac{\partial^m}{\partial w^m}\right)$ on both sides of equation (3.16), we have

$$\exp\left(D_z^{-1}\frac{\partial^m}{\partial w^m}\right)H_{k+l}^{(s)}(w,y)$$
$$=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}H_{k+l-n-r}^{(s)}(x,y)\exp\left(D_z^{-1}\frac{\partial^m}{\partial w^m}\right)(w-x)^{n+r}. \tag{3.17}$$

Using the operational definitions (1.11b) and (1.3) in the l.h.s. and r.h.s., respectively of equation (3.17), we get assertion (3.15) of Theorem 3.3.

Theorem 3.4. *The following summation formula for the LGHP* ${}_LH_n^{(m,s)}(x,y,z)$ *holds true:*

$${}_LH_{k+l}^{(m,s)}(y,w,z)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}H_{n+r}^{(s)}(w-x,z)\ {}_mL_{k+l-n-r}(y,x). \tag{3.18}$$

Proof. Replacing s by m and y by D_y^{-1} in equation (3.16) and then using the following link between the GHP $H_n^{(s)}(x,y)$ and the 2VgLP ${}_mL_n(x,y)$ (Dattoli et al., 1999, p.213):

$$H_n^{(m)}(y,D_x^{-1})={}_mL_n(x,y), \tag{3.19}$$

in the resultant equation, we find the following summation formula for the 2VgLP ${}_mL_n(x,y)$:

$${}_mL_{k+l}(y,w)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}{}_mL_{k+l-n-r}(y,x). \tag{3.20}$$

Now, operating $\exp\left(z\frac{\partial^s}{\partial w^s}\right)$ on equation (3.20) and using operational definitions (1.11a) and (1.14) in the l.h.s. and r.h.s., respectively of the resultant equation, we get assertion (3.18) of Theorem 3.4.

4. Applications

I. Taking $p=q=l=1$ in equation (2.3), we get

$$\sum_{n=0}^{\infty}\frac{(a_1)_n}{(b_1)_n(c_1)_n}{}_LH_n^{(m,s)}(x,y,z)\ t^n=F_{2:0;0;1}^{2:0;0;0}\left(\begin{matrix}[a_1:1,s,m],[1:1,s,m]:-;-; & -; \\ [b_1:1,s,m],[c_1:1,s,m]:-;-;[1:1]; \end{matrix}\ yt,zt^s,xt^m\right). \tag{4.1}$$

Further, taking $b_1=c_1=1$ and replacing a_1 by $a+1$ in equation (4.1), we get

$$\sum_{n=0}^{\infty}\binom{a+n}{n}{}_LH_n^{(m,s)}(x,y,z)\ \frac{t^n}{n!}=F_{1:0;0;1}^{1:0;0;0}\left(\begin{matrix}[a+1:1,s,m]:-;-; & -; \\ [1:1,s,m]:-;-;[1:1]; \end{matrix}\ yt,zt^s,xt^m\right). \tag{4.2}$$

II. Taking $p = q = 1$ in equation (2.4), we get

$$\sum_{n=0}^{\infty} \frac{(a_1)_n}{(b_1)_n} {}_LH_n^{(m,s)}(x,y,z)\ t^n = F^{2:0;0;0}_{1:0;0;1}\left({}^{[a_1:1,s,m],[1:1,s,m]:-;-;\ \ -;}_{[b_1:1,s,m]:-;-;[1:1];}\ yt, zt^s, xt^m \right). \tag{4.3}$$

Next, taking $b_1 = 1$ and replacing a_1 by a in equation (4.3) and using equations (1.24) and (2.7) in the r.h.s. of the resultant equation, we get

$$\sum_{n=0}^{\infty} (a)_n\ {}_LH_n^{(m,s)}(x,y,z)\ \frac{t^n}{n!}$$
$$= (1-yt)^{-a}\ F^{1:0;0}_{0:0;1}\left({}^{[a:s,m]:-;\ \ -;}_{-:-;[1:1];}\ z\left(\frac{t}{1-yt}\right)^s, x\left(\frac{t}{1-yt}\right)^m \right), \tag{4.4}$$

which for $a = 1$, becomes

$$\sum_{n=0}^{\infty} {}_LH_n^{(m,s)}(x,y,z)\ t^n$$
$$= (1-yt)^{-1} F^{1:0;0}_{0:0;1}\left({}^{[1:s,m]:-;\ -;}_{-:-;[1:1];}\ z\left(\frac{t}{1-yt}\right)^s, x\left(\frac{t}{1-yt}\right)^m \right). \tag{4.5}$$

Again, taking $s = m$ in equation (4.4), we get the following generating function for the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ in terms of the Kampé de Fériet function of two variables $F^{p:q;k}_{l:m;n}[.]$:

$$\sum_{n=0}^{\infty} (a)_n\ {}_LH_n^{(m,m)}(x,y,z)\ \frac{t^n}{n!}$$
$$= (1-yt)^{-a} F^{m:0;0}_{0:0;1}\left({}^{\Delta(m,a):-;-;}_{-:-;\ 1;}\ z\left(\frac{m}{1-yt}\right)^m, x\left(\frac{m}{1-yt}\right)^m \right), \tag{4.6}$$

where $\Delta(m;a)$ abbreviates the array of m parameters $\frac{a}{m}, \frac{a-1}{m}, \ldots, \frac{a-m+1}{m}$ $\ m \geqslant 1$.

III. Replacing w by $-w$ in equation (2.5) and using definition (1.9), we get

$$\sum_{n=0}^{\infty} W_n(w;s)\ {}_LH_n^{(m,s)}(x,y,z)\frac{t^n}{n!} = F^{0:0;0;0}_{1:0;0;1}\left({}^{-:-;-;\ \ -;}_{[1:1,s,m]:-;-;[1:1];}\ yt, zt^s + w, xt^m \right). \tag{4.7}$$

Again, taking $s = m = 1$ and replacing x by $-x$ in equation (2.5) and using equations (1.7) and (1.20) in the l.h.s. of the resultant equation, we get the following bilateral generating function for the 2VLP $L_n(x,y)$ in terms of the Kampé de Fériet function of three variables:

$$\sum_{n=0}^{\infty} C_n(w)\ L_n(x, y+z)\frac{t^n}{n!} = F^{0:0;0;0}_{1:0;0;1}\left({}^{-:-;-;-;}_{1:-;-;\ 1;}\ yt, zt - w, -xt \right), \tag{4.8}$$

IV. Taking $l = 0$ in equation (3.1) and replacing w by $w + y$, v by $v + z$ in the resultant equation, we get

$${}_LH_k^{(m,s)}(x, w+y, v+z) = \sum_{n=0}^{k} \binom{k}{n} H_n^{(s)}(w,v)\ {}_LH_{k-n}^{(m,s)}(x,y,z), \tag{4.9}$$

which on taking $v = 0$ and using relation (1.16b), yields

$$ {}_LH_k^{(m,s)}(x, w+y, z) = \sum_{n=0}^{k} \binom{k}{n} w^n \ {}_LH_{k-n}^{(m,s)}(x, y, z). \tag{4.10} $$

Next, taking $m = 1$, $s = 2$ and replacing x by $-x$ in equations (3.1), (3.9), (4.9) and (4,10) and using equation (1.18), we get the following summation formulae for the 3VLHP ${}_LH_n(x, y, z)$:

$$ {}_LH_{k+l}(x, w, v) = \sum_{n,r=0}^{k,l} \binom{k}{n}\binom{l}{r} H_{n+r}(w-y, v-z) \ {}_LH_{k+l-n-r}(x, y, z), \tag{4.11} $$

$$ {}_LH_{k+l}(x, w, z) = \sum_{n,k=0}^{k,l} \binom{k}{n}\binom{l}{r} (w-y)^{n+r} \ {}_LH_{k+l-n-r}(x, y, z), \tag{4.12} $$

$$ {}_LH_k(x, w+y, v+z) = \sum_{n=0}^{k} \binom{k}{n} H_n(w, v) \ {}_LH_{k-n}(x, y, z), \tag{4.13} $$

$$ {}_LH_k(x, w+y, z) = \sum_{n=0}^{k} \binom{k}{n} w^n \ {}_LH_{k-n}(x, y, z). \tag{4.14} $$

Again, taking $z = -\frac{1}{2}$ in equations (4.12) and (4.14) and using relation (1.19), we get the following summation formulae for the 2VLHP ${}_LH_n^*(x, y)$:

$$ {}_LH_{k+l}^*(x, w) = \sum_{n,r=0}^{k,l} \binom{k}{n}\binom{l}{r} (w-y)^{n+r} \ {}_LH_{k+l-n-r}^*(x, y), \tag{4.15} $$

$$ {}_LH_k^*(x, w+y) = \sum_{n=0}^{k} \binom{k}{n} w^{k-n} \ {}_LH_n^*(x, y). \tag{4.16} $$

Further, taking $v = z = -\frac{1}{2}$ in equation (4.13) and using relations (1.17) and (1.19), we get

$$ {}_LH_k(x, w+y, -1) = \sum_{n=0}^{k} \binom{k}{n} He_{k-n}(w) \ {}_LH_n^*(x, y). \tag{4.17} $$

V. Taking $m = 1$, $s = 2$ and replacing x and X by $-x$ and $-X$, respectively in equations (3.10) and (3.14) and using relations (1.18) and (1.16a), we get the following summation formulae involving product of the 3VLHP ${}_LH_n(x, y, z)$:

$$ {}_LH_n(x, w, u) \ {}_LH_r(X, W, U) = \sum_{k,p=0}^{n,r} \binom{n}{k}\binom{r}{p} H_k(w-y, u-z) $$
$$ \times H_p(W-Y, U-Z) \ {}_LH_{n-k}(x, y, z) \ {}_LH_{r-p}(X, Y, Z), \tag{4.18} $$

$$ {}_LH_n(x, w, z) \ {}_LH_r(X, W, Z) = \sum_{k,p=0}^{n,r} \binom{n}{k}\binom{r}{p} (w-y)^k (W-Y)^p $$
$$ \times \ {}_LH_{n-k}(x, y, z) \ {}_LH_{r-p}(X, Y, Z). \tag{4.19} $$

Further, taking $z = Z = -\frac{1}{2}$ in equation (4.19) and using relation (1.19), we get the following summation formula involving product of the 2VLHP ${}_LH_n^*(x,y)$:

$$ {}_LH_n^*(x,w)\ {}_LH_r^*(X,W) = \sum_{k,p=0}^{n,r} \binom{n}{k}\binom{r}{p}(w-y)^k(W-Y)^p \\ \times\ {}_LH_{n-k}^*(x,y)\ {}_LH_{r-p}^*(X,Y). \tag{4.20} $$

VI. Taking $l = 0$ in equations (3.15) and (3.18), we get

$$ {}_LH_k^{(m,s)}(z,w,y) = \sum_{n=0}^{k} \binom{k}{n} {}_mL_n(z,w-x)H_{k-n}^{(s)}(x,y) \tag{4.21} $$

and

$$ {}_LH_k^{(m,s)}(y,w,z) = \sum_{n=0}^{k} \binom{k}{n} H_n^{(s)}(w-x,z)\ {}_mL_{k-n}(y,x). \tag{4.22} $$

Next, taking $m = 1$, $s = 2$ and replacing z by $-z$ in equations (3.15) and (4.21) and then using equations (1.18), (1.15) and (1.16a), we get the following summation formulae for the 3VLHP ${}_LH_n(x,y,z)$:

$$ {}_LH_{k+l}(z,w,y) = \sum_{n,r=0}^{k,l} \binom{k}{n}\binom{l}{r} L_{n+r}(z,w-x)H_{k+l-n-r}(x,y), \tag{4.23} $$

$$ {}_LH_k(z,w,y) = \sum_{n=0}^{k} \binom{k}{n} L_n(z,w-x)H_{k-n}(x,y). \tag{4.24} $$

Again, taking $m = 1$, $s = 2$ and replacing y by $-y$ in equations (3.18) and (4.22) and then using equations (1.18), (1.15) and (1.16a), we get the following summation formulae for the 3VLHP ${}_LH_n(x,y,z)$:

$$ {}_LH_{k+l}(y,w,z) = \sum_{n,r=0}^{k,l} \binom{k}{n}\binom{l}{r} H_{n+r}(w-x,z)\ L_{k+l-n-r}(y,x), \tag{4.25} $$

$$ {}_LH_k(y,w,z) = \sum_{n=0}^{k} \binom{k}{n} H_n(w-x,z)\ L_{k-n}(y,x). \tag{4.26} $$

Further, taking $y = -\frac{1}{2}$ in equations (4.23), (4.24) and $z = -\frac{1}{2}$ in equations (4.25), (4.26), respectively and using relations (1.19) and (1.17), we get the following summation formulae for the 2VLHP ${}_LH_n^*(x,y)$:

$$ {}_LH_{k+l}^*(z,w) = \sum_{n,r=0}^{k,l} \binom{k}{n}\binom{l}{r} L_{n+r}(z,w-x)He_{k+l-n-r}(x), \tag{4.27} $$

$$ {}_LH_k^*(z,w) = \sum_{n=0}^{k} \binom{k}{n} L_n(w-x,z)He_{k-n}(x), \tag{4.28} $$

and

$$ {}_LH^*_{k+l}(y,w) = \sum_{n,r=0}^{k,l} \binom{k}{n}\binom{l}{r} He_{n+r}(w-x)\ L_{k+l-n-r}(y,x), \tag{4.29}$$

$$ {}_LH^*_{k}(y,w) = \sum_{n=0}^{k} \binom{k}{n} He_{n}(w-x)\ L_{k-n}(y,x), \tag{4.30}$$

respectively.

5. Concluding remarks

In Section 3, we have established the summation formula (3.10), containing the product of the LGHP ${}_LH_n^{(m,s)}(x,y,z)$. The formula was proved by using series rearrangement techniques on the product of the generating functions of the LGHP ${}_LH_n^{(m,s)}(x,y,z)$. To explore another possibility, we consider the product of the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ generating function (1.6) in the following form:

$$\begin{aligned}&\exp((Y+y)t+(Z+z)t^s)\ C_0(-Xt^m)\ C_0(-xt^m)\\ &\qquad = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} {}_LH_r^{(m,s)}(X,Y,Z)\ {}_LH_n^{(m,s)}(x,y,z)\ \frac{t^{n+r}}{n!r!}.\end{aligned} \tag{5.1}$$

Replacing n by $n-r$ and using equation (3.7) in the r.h.s. of the above equation, we find

$$\begin{aligned}&\exp((Y+y)t+(Z+z)t^s)\ C_0(-Xt^m)\ C_0(-xt^m)\\ &\qquad = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \binom{n}{r} {}_LH_r^{(m,s)}(X,Y,Z)\ {}_LH_{n-r}^{(m,s)}(x,y,z)\frac{t^{n}}{n!},\end{aligned} \tag{5.2}$$

which on shifting the exponential to the r.h.s. and using the generating function (1.12), becomes

$$\begin{aligned}C_0(-Xt^m)\ C_0(-xt^m) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n} \binom{n}{r} H_k^{(s)}(-(Y+y),-(Z+z))\\ &\qquad \times {}_LH_r^{(m,s)}(X,Y,Z)\ {}_LH_{n-r}^{(m,s)}(x,y,z)\frac{t^{n+k}}{n!k!}.\end{aligned} \tag{5.3}$$

Again, replacing n by $n-k$ and using equation (3.7) in the r.h.s. of the above equation, we get

$$\begin{aligned}C_0(-Xt^m)\ C_0(-xt^m) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k} \binom{n}{k}\binom{n-k}{r}\\ &\times H_k^{(s)}(-(Y+y),-(Z+z))\ {}_LH_r^{(m,s)}(X,Y,Z)\ {}_LH_{n-k-r}^{(m,s)}(x,y,z)\frac{t^{n}}{n!}.\end{aligned} \tag{5.4}$$

Taking $m=1$, $s=2$ and replacing X and x by $-X$ and $-x$, respectively, in equation (5.4) and using relations (1.16a) and (1.18), we find

$$C_0(Xt)\ C_0(xt) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k} \binom{n}{k}\binom{n-k}{r}$$

$$\times H_k(-(Y+y),-(Z+z))\ {}_LH_r(X,Y,Z)\ {}_LH_{n-k-r}(x,y,z)\frac{t^n}{n!}. \tag{5.5}$$

Now, using the following generating function for the 2-variable Legendre polynomials $R_n(x,y)$ (Dattoli et al., 2001):

$$C_0(-yt)\ C_0(xt)=\sum_{n=0}^{\infty}R_n(x,y)\ \frac{t^n}{(n!)^2} \tag{5.6}$$

in the l.h.s. of equation (5.5) and then equating the coefficients of like powers of t in the resultant equation, we get the following summation formula for $R_n(x,y)$:

$$R_n(X,-x)=n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}\ H_k(-(Y+y),-(Z+z))$$

$$\times\ {}_LH_r(X,Y,Z)\ {}_LH_{n-k-r}(x,y,z), \tag{5.7}$$

which on using the following link between the 2-variable Legendre polynomials $R_n(x,y)$ and the classical Legendre polynomials $P_n(x)$ (Andrews, 1985):

$$R_n\left(\frac{1-x}{2},\frac{1+x}{2}\right)=P_n(x), \tag{5.8}$$

yields the following summation formula for $P_n(x)$:

$$P_n(x)=n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}H_k(-(Y+y),-(Z+z))$$

$$\times{}_LH_r\left(\frac{1-x}{2},Y,Z\right)\ {}_LH_{n-k-r}\left(\frac{-1-x}{2},y,z\right). \tag{5.9}$$

6. Conclusion

In this paper, we have established some generating functions for the Laguerre-Gould Hopper polynomials by using series rearrangement techniques. Also, some summation formulae for that polynomials are derived by using certain operational techniques and by using different analytical means on its generating function. Further, many generating functions and summation formulae for the polynomials related to Laguerre-Gould Hopper polynomials are obtained as applications of main results. The approach presented in this paper is general and can be extended to establish other properties of special polynomials.

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