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Pattern Avoiding Partitions, Sequence A054391 and the Kernel Method

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Abstract

Sequence A054391 in OEIS, which we will denote by a_n, counts a certain two-pattern avoidance class of the permutations of size n. In this paper, we provide additional combinatorial interpretations for these numbers in terms of finite set partitions. In particular, we identify six classes of the partitions of size n, all of which have cardinality a_n and each avoiding two classical patterns. We use both algebraic and combinatorial methods to establish our results. In one apparently more difficult case, to show the result, we make use of the kernel method in solving a system of three functional equations which arises after a certain parameter is introduced. We also define an algorithmic bijection between the avoidance class in this case and another which systematically replaces the occurrences of a given pattern with those of another having the same length.

Keywords: Pattern avoidance, set partition, kernel method

MSC 2010 No.: 05A15, 05A18

1. Introduction

If $j \geq 1$, then let s_j denote the sequence (see Barcucci et al. (2000)) which counts the permutations of size n avoiding the patterns 321 and $(j+2)\bar{1}(j+3)2\cdots(j+1)$. Letting j vary produces different sequences, the $j=1$ and $j=2$ cases, for example, corresponding to the Motzkin numbers and to enumerators of Motzkin left factors (which was shown in Barcucci et al. (2000)). Letting j go to infinity produces the Catalan sequence, and so there is a ``discrete continuity" between the Motzkin and Catalan sequences, as noted in Barcucci et al. (2000). In the present correspondence, we are concerned with the case $j=3$, the terms of which we will denote by a_n. In particular, we will show that a_n also counts certain avoidance classes of *set partitions*. The sequence a_n occurs as A054391 in Sloane and has generating function given by

$$\sum_{n\geq 0} a_n x^n = 1 - \frac{2x^2}{2x^2 - 3x + 1 - \sqrt{1-2x-3x^2}}. \tag{1}$$

The first few a_n values, starting with $n=0$, are given by 1, 1, 2, 5, 14, 41, 123, 374, ...

If $n \geq 1$, then a *partition* of $[n]=\{1,2,\ldots,n\}$ is any collection of non-empty, pairwise disjoint subsets, called *blocks*, whose union is $[n]$. (If $n=0$, then there is a single empty partition which has no blocks.) Let P_n denote the set of all partitions of $[n]$. A partition Π is said to be in *standard form* if it is written as $\Pi = B_1/B_2/\cdots$, where the blocks are arranged in ascending order according to the size of the smallest elements. One may also represent $\Pi = B_1/B_2/\cdots \in P_n$, expressed in the standard form, equivalently as $\pi = \pi_1\pi_2\cdots\pi_n$, wherein $j \in B_{\pi_j}, 1 \leq j \leq n$, called the *canonical sequential form*; and, in such case, we will write $\Pi = \pi$. For example, the partition $\Pi = 1,5/2,3,7/4/6,8$ has the canonical sequential form $\pi = 12231424$. Note that $\pi = \pi_1\pi_2\cdots\pi_n$ possesses the *restricted growth property* (see, e.g., Stanton and White (1986) or Wagner (1996) for details), meaning that it satisfies the following three conditions: (i) $\pi_1 = 1$, (ii) π is onto $[k]$ for some $k \geq 1$, and (iii) $\pi_{i+1} \leq \max\{\pi_1, \pi_2, \ldots, \pi_i\} + 1$ for all i, $1 \leq i \leq n-1$. In what follows, we will represent set partitions as words using their canonical sequential forms and consider some particular cases of the general problem of counting the members of a partition class having various restrictions imposed on the order of the letters.

A *classical pattern* τ is a member of $[\ell]^m$ which contains all of the letters in $[\ell]$. We say that a word $\sigma \in [k]^n$ *contains* the classical pattern τ if σ contains a subsequence isomorphic to τ. Otherwise, we say that σ *avoids* τ. For example, a word $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ avoids the pattern 231 if it has no subsequence $\sigma_i\sigma_j\sigma_k$ with $i<j<k$ and $\sigma_k < \sigma_i < \sigma_j$ and avoids the pattern 1221 if it has no subsequence $\sigma_i\sigma_j\sigma_k\sigma_\ell$ with $\sigma_i = \sigma_\ell < \sigma_j = \sigma_k$. The pattern avoidance question has been the topic of much research in enumerative combinatorics, starting with Knuth (1974) and Simion and Schmidt (1985) on permutations and considered, more recently, on further structures such as words and compositions. The avoidance problem can be extended to set partitions upon considering the question on the associated canonical sequential forms. We refer the reader to the papers by Klazar (1996), Sagan (2010), and Jelínek and Mansour (2008) and to the references therein. We will use the following notation. If $\{w_1, w_2, \ldots\}$ is a set of classical patterns, then let $P_n(w_1, w_2, \ldots)$ be the subset of P_n which avoids all of the patterns, whose cardinality we will denote by $p_n(w_1, w_2, \ldots)$.

In this paper, we identify six classes of the partitions of $[n]$ each avoiding a classical pattern of length four and another of length five and each enumerated by the number a_n. In addition to providing new combinatorial interpretations for a sequence, this addresses specific cases of a general question raised by Goyt (2008), for example, regarding the enumeration of avoidance classes of partitions corresponding to two or more patterns. Analogous results concerning the avoidance of two patterns by a permutation have been given, for example, by West (1995). Our main result is the following theorem which we prove in the next section as a series of propositions.

Theorem 1.1. If $n \geq 0$, then $p_n(u,v) = a_n$ for the following sets (u,v):

$$(1)\,(1212,12221) \qquad (2)\,(1212,11222) \qquad (3)\,(1212,11122)$$
$$(4)\,(1221,12311) \qquad (5)\,(1221,12112) \qquad (6)\,(1221,12122).$$

To show this, we give algebraic proofs for cases (1), (3), (4), and (6) and find one-to-one correspondences between cases (1) and (2) and (5) and (6). To establish (4) and (6), we make use of the kernel method (see Banderier et al. (2002) and Hou and Mansour (2011)), in the latter case, to solve a system of functional equations which arises once a certain parameter has been introduced. Our bijection between cases (5) and (6) is of an algorithmic nature and systematically replaces occurrences of 12122 with ones of 12112 without introducing 1221.

2. Proof of the Main Result

2.1. The cases $\{1212,12221\}$, $\{1212,11222\}$, and $\{1212,11122\}$

In this section, we consider the cases of avoiding $\{1212,12221\}$, $\{1212,11222\}$, and $\{1212,11122\}$.

Proposition 2.1. The generating function for $p_n(1212,12221)$ and $p_n(1212,11122)$, where $n \geq 0$, is given by

$$1 - \frac{2x^2}{2x^2 - 3x + 1 - \sqrt{1-2x-3x^2}}.$$

Proof:

Note first that each non-empty partition $\pi \in P_n(1212,12221)$ may be decomposed as either $\pi = 1\pi'$, where π' contains no 1's, or as $\pi = 1\alpha_1 1\alpha_2 \cdots 1\alpha_r$, for some $r \geq 2$, where the α_i contain no 1's. The α_i must avoid $\{1212,111\}$ if $1 \leq i \leq r-1$, with α_r avoiding $\{1212,12221\}$.

Furthermore, note that each letter in α_i is greater than each letter in α_j if $i > j$ in order to avoid an occurrence of 1212 . Let $f(x) = \sum_{n\geq 0} p_n(1212,12221)x^n$ and $g(x) = \sum_{n\geq 0} p_n(1212,111)x^n$. From the foregoing observations, we obtain the relation

$$f(x) = 1 + xf(x) + \frac{x^2 f(x)g(x)}{1 - xg(x)},$$

or

$$f(x) = \frac{1 - xg(x)}{1 - x - xg(x)}. \tag{2}$$

To compute $g(x)$, observe that $\pi = P_n(1212,111)$ must be of the form $\pi = 1\pi'$ or $\pi = 1\pi'1\pi''$, where π' and π'' contain no 1's and avoid $\{1212,111\}$. Furthermore, all of the letters of π'' are greater than all of the letters of π' in the second case. This implies

$$g(x) = 1 + xg(x) + x^2 g^2(x),$$

or

$$g(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}. \tag{3}$$

Substituting (3) into (2) and simplifying yields the first case above.

To compute the generating function $h(x)$ for $p_n(1212,11122)$, we use the same cases as we did for finding $p_n(1212,11222)$. In the second case, however, consider further whether or not α_2 contains a repeated letter. Note that if it does, then α_1 must avoid $\{1212,111\}$. Furthermore, no letters greater than one occurring after the third 1 can be repeated. This yields the relation

$$h(x) = 1 + xh(x) + \frac{x^2}{1-x} h(x)\left(1 + \frac{x}{1-2x}\right) + x^2 g(x)\left(h(x) - \frac{1}{1-x}\right)\left(1 + \frac{x}{1-2x}\right),$$

or

$$h(x) = \frac{1 - 2x - x^2 g(x)}{1 - 3x + x^2 - x^2(1-x)g(x)},$$

from which the desired result follows from (3).

There is a direct bijection showing the equivalence of the pairs $\{1212,12221\}$ and $\{1212,11222\}$.

Proposition 2.2. If $n \geq 0$, then $p_n(1212,12221) = p_n(1212,11222)$.

Proof:

Let $A_n = P_n(1212,12221)$ and $B_n = P_n(1212,11222)$. We will define a bijection f between the sets A_n and B_n in an inductive manner, the cases $n = 1,2,3$ clear. Suppose $n \geq 4$ and $\pi \in A_n$. If $\pi = 12\cdots n$, then $\pi \in B_n$ and let $f(\pi) = \pi$. If $\pi \neq 12\cdots n$, then let $m \geq 1$ be the smallest letter of π which occurs at least twice. Thus, we may write $\pi = 12\cdots(m-1)m\alpha_1 m\alpha_2 \cdots m\alpha_r$, where $r \geq 2$ and the α_i contain only letters in $\{m+1, m+2, \ldots\}$ but are otherwise just as in the proof of Proposition 2.1 above.

Given a finite word w on the alphabet of positive integers having m distinct letters, let $stan(w)$ denote the equivalent word on $[m]$ having all of the same relative comparisons with regard to its positions (often called the *standardization of* w). Let $\beta_r = f(stan(\alpha_r))$ and β_r^o be obtained from β_r by adding m to each letter. If $1 \leq i \leq r-1$, then let α'_i be obtained from α_i by adding ℓ to each letter, where ℓ is the number of distinct letters of α_r. Let $f(\pi)$ be given by

$$f(\pi) = 12\cdots(m-1)m\beta_r^o m\alpha'_1 m\alpha'_2 \cdots m\alpha'_{r-1}.$$

Then $f(\pi)$ belongs to B_n since β_r^o avoids $\{1212,11222\}$ and each α'_i avoids $\{1212,111\}$. One may verify that the mapping f is a bijection, which completes the proof.

Remark: The mapping f is seen to preserve the number of blocks; thus the members of $P_n(1212,12221)$ and $P_n(1212,11222)$ having the same prescribed number of blocks are equinumerous.

2.2. The Case $\{1221,12311\}$

Let $\pi = \pi_1\pi_2\cdots\pi_n$ denote a partition of $[n]$, represented canonically. Recall that empty sums take the value zero, by convention. To establish this case, we divide up the set of partitions in question according to a certain statistic, namely, the one which records the length of the maximal increasing initial run. To do so, given $k \geq 1$, let $f_k(x)$ denote the generating function for the number of partitions π of $[n]$ having at least k letters and avoiding the patterns 1221 and 12311 such that $\pi_1\pi_2\cdots\pi_k = 12\cdots k$ with $\pi_{k+1} \leq k$ (if there is a $(k+1)$-st letter). We have the following relation involving the generating functions $f_k(x)$.

Lemma 2.3. If $k \geq 2$, then

$$f_k(x) = x^k + x^k \overline{f}_1(x) + x^{k-1}\overline{f}_2(x) + \sum_{j=2}^{k-1} x^{k+1-j}\overline{f}_j(x), \tag{4}$$

with initial conditions $f_0(x) = 1$ and $f_1(x) = x + x\overline{f}_1(x)$, where $\overline{f}_k(x) = \sum_{i \geq k} f_i(x)$.

Proof:

Note that $f_1(x) = x + x\overline{f}_1(x)$, since a partition in this case may just have one letter or start 11. If $k = 2$, then a partition π enumerated by $f_2(x)$ must either be 12 or start with 121 or 122, which implies

$$f_2(x) = x^2 + x\overline{f}_2(x) + x^2\overline{f}_1(x).$$

Note that in the second case, the second letter 1 is extraneous (concerning possible occurrences of 12311 or 1221) and therefore can be removed without affecting the enumeration, while in the third case, the letters 1 and 2 at the beginning are extraneous.

If $k \geq 3$, then we consider the following cases concerning the partitions enumerated by $f_k(x)$:

(i) $12\cdots k,$
(ii) $12\cdots kj\pi',$ where $1 \leq j \leq k-2,$
(iii) $12\cdots k(k-1)\pi',$
(iv) $12\cdots kk\pi'.$

The first case contributes x^k. Note that in the second case, the word π' contains no letters in $[j]$, for otherwise there would be an occurrence of 1221 if it contained a letter in $[j-1]$ or an occurrence of 12311 if it contained the letter j. Thus, the letters $(j+1)(j+2)\cdots k\pi'$, taken together, comprise a partition of the form enumerated by $\overline{f}_{k-j}(x)$, which implies the contribution in this case is $x^{j+1}\overline{f}_{k-j}(x)$. Similar reasoning in the third case yields a contribution of $x^{k-1}\overline{f}_2(x)$, since π' can contain no letters in $[k-2]$ and thus $(k-1)k\pi'$ is a partition of the form enumerated by $\overline{f}_2(x)$ (the factor of x^{k-1} accounts for the letters $12\cdots(k-2)$ as well as the second occurrence of $k-1$). Finally, in the fourth case, no member of $[k-1]$ can occur in π', with the second k extraneous, which implies a contribution of $x^k\overline{f}_1(x)$. Combining all of the cases yields (4), which is also seen to hold in the case $k = 2$ as well.

Proposition 2.4. The generating function for $p_n(1221,12311)$, $n \geq 0$, is given by

$$1-\frac{2x^2}{2x^2-3x+1-\sqrt{1-2x-3x^2}}.$$

Proof:

Define the generating function

$$F(x,y)=\sum\nolimits_{k\geq 0} f_k(x)y^k.$$

First note that

$$\overline{f}_1(x)=f_1(x)+f_2(x)+\cdots=F(x,1)-1,\ f_1(x)=x+x\overline{f}_1(x)=xF(x,1), \text{ and}$$
$$\overline{f}_2(x)=F(x,1)-f_1(x)-1=(1-x)F(x,1)-1.$$

Multiplying (4) by y^k and summing over $k\geq 2$ yields

$$\begin{aligned}
F(x,y) &= 1+f_1(x)y+\sum_{k\geq 2}\left(x^k+x^k\overline{f}_1(x)+x^{k-1}\overline{f}_2(x)+\sum_{j=2}^{k-1}x^{k+1-j}\overline{f}_j(x)\right)y^k\\
&= 1+xyF(x,1)+\frac{x^2y^2}{1-xy}+\frac{x^2y^2}{1-xy}[F(x,1)-1]+\frac{xy^2}{1-xy}[(1-x)F(x,1)-1]\\
&\quad +\sum_{k\geq 2}\left(\sum_{j=2}^{k-1}x^{k+1-j}\overline{f}_j(x)\right)y^k\\
&= 1+xyF(x,1)+\frac{xy^2}{1-xy}[F(x,1)-1]+\frac{x^2y}{1-xy}\sum_{j\geq 2}\overline{f}_j(x)y^j\\
&= 1+xyF(x,1)+\frac{xy^2}{1-xy}[F(x,1)-1]+\frac{x^2y}{1-xy}\sum_{i\geq 2}f_i(x)\sum_{j=2}^{i}y^j\\
&= 1+xyF(x,1)+\frac{xy^2}{1-xy}[F(x,1)-1]\\
&\quad +\frac{x^2y}{1-xy}\left[\frac{y^2}{1-y}(F(x,1)-f_1(x)-1)-\frac{y}{1-y}(F(x,y)-f_1(x)y-1)\right],
\end{aligned}$$

which implies

$$\left(1+\frac{x^2y^2}{(1-xy)(1-y)}\right)F(x,y)=\frac{1-xy-xy^2+x^2y^2}{1-xy}+\frac{xy(1-xy-y^2+2xy^2)}{(1-xy)(1-y)}F(x,1). \tag{5}$$

This type of functional equation can be solved systematically using the *kernel method* (see Banderier et al. (2002)). In this case, if we assume that $y = y_0$ in (5), where y_0 satisfies

$$1+\frac{x^2y_0^2}{(1-xy_0)(1-y_0)}=0\text{, i.e.,}$$

$$y_0=\frac{1+x-\sqrt{1-2x-3x^2}}{2x(1+x)},$$

then

$$\sum_{n\geq 0}p_n(12311,1221)x^n = F(x,1) = -\frac{(1-y_0)(1-xy_0-xy_0^2+x^2y_0^2)}{xy_0(1-xy_0-y_0^2+2xy_0^2)}$$
$$=1-\frac{2x^2}{2x^2-3x+1-\sqrt{1-2x-3x^2}},$$

as required. (Note that $F(0,1)=1$ dictates our choice of root for y_0.)

Remark*:* Substituting the expression above for $F(x,1)$ into (5) recovers the expression for $F(x,y)$, from which one can compute an explicit formula for the coefficient of $x^n y^k$.

2.3. The Cases $\{1221,12112\}$ and $\{1221,12122\}$

For these cases, we show first that $p_n(1221,12122) = p_n(1221,12112)$ through a direct bijection and then show that the generating function for $p_n(1221,12122)$ is given by (1) above. Before defining the bijection, we will need the following two lemmas.

Lemma 2.5. Suppose $\pi \in P_n(1221,12122)$ has at least one occurrence of the pattern 12112. Let $b \geq 2$ be the smallest letter such that there exists $a \in [b-1]$ for which there is a subsequence in π given by $abaab$. Then (i) the element b occurs exactly twice in π and (ii) the element a is uniquely determined.

Proof:

For (i), suppose, to the contrary, that b occurs at least three times in π. Let α denote an occurrence of the subsequence $abaab$ in π; we may assume that a and b in α correspond to initial occurrences of letters of their kind. If an additional b occurs to the right of the third a in α, then there would be an occurrence of 12122 in π, which is not allowed. If an additional b occurs to the left of the third a in α, then there would be an occurrence of 1221, which is also not allowed. Thus, there are exactly two b's in π and they correspond to an occurrence of 12112. Note that the minimality of b is not needed for this part.

To show (ii), suppose, to the contrary, that there exists a subsequence β in π involving the two b's given by $\beta = cbccb$ for some $c \in [b-1], c \neq a$. Suppose first $c < a$. If the third c of β occurs to the right of the second a in α above, then there would be an occurrence of 1221, and if it occurs to the left of the second a in α, then there would be an occurrence of 12122. Thus, there are no occurrences of *cbccb* with $c < b$ (in fact, this shows that *any* letter $c < a$ in π can only occur prior to the left-most b in π).

Now suppose $a < c < b$ and let β again denote a possible occurrence of *cbccb*. If the third c of β occurs after the third a in α, then there would be an occurrence of 12112 of the form *acaac*, with $c < b$, contradicting the assumed minimality of b. On the other hand, if the third c of β occurs before the third a of α, then there would be an occurrence of 1221 of the form *acca*. Thus, no element of $[b-1]$ can form an occurrence of 12112 with b, which completes the proof of (ii).

Lemma 2.6. Suppose $\pi = \pi_1\pi_2\cdots\pi_n \in P_n(1221,12122)$ contains at least one occurrence of the pattern 12112 and let a and b be as defined in Lemma 2.5 above. Write $\pi = W_1bW_2bW_3$, where W_3 is possibly empty. Then we have the following: (i) only letters greater than b can occur in W_2, with the exception of a, and no letter other than a can occur more than once in W_2; and (ii) any letter occurring in W_2 can occur at most once in W_3, all of whose letters are greater than b.

Proof:

To prove (i), write $W_2 = \alpha_1 a\alpha_2 a\cdots\alpha_{r-1}a\alpha_r$ for some $r \geq 3$, where the α_i are possibly empty and contain no a's. Suppose, to the contrary, that $c < b$ occurs in W_2, where $c \neq a$. If $c < a$ is in α_i for some $i \geq 2$, then there would be an occurrence of 1221, whereas if it is in α_1, then there would be an occurrence of 12122. If $a < c < b$ and c belongs to α_i for some $1 \leq i < r$, then there would be an occurrence of 1221 with *acca*, whereas if c belongs to α_r, then there would be an occurrence of 12112 of the form *acaac* with $c < b$, contradicting the minimality of b. Thus, only letters greater than b can occur in W_2, with the exception of a, and each letter greater than b can occur in W_2 at most once so as to avoid 1221. Statement (ii) is also a consequence of π belonging to $P_n(1221,12122)$.

We now establish the equivalence of avoiding $\{1221,12122\}$ and $\{1221,12112\}$.

Proposition 2.7. If $n \geq 0$, then $p_n(1221,12122) = p_n(1221,12112)$.

Proof:

Let $A = P_n(1221,12122)$ and $B = P_n(1221,12112)$. We will describe a bijection f between the sets A and B, algorithmically, as follows. Suppose $\pi \in A$. If $\pi \in B$, then let $f(\pi) = \pi$. Otherwise, $\pi_0 = \pi$ contains at least one occurrence of the pattern 12112. Let b_0 denote the smallest letter b in π_0 for which there exists a subsequence *abaab* for some $a < b$ and let a_0 denote the corresponding letter a, which is uniquely determined, by Lemma 2.5. Suppose we write π_0 as $\pi_0 = W_1 b_0 W_2 b_0 W_3$ as in Lemma 2.6 above, where $W_2 = \alpha_1 a_0 \alpha_2 a_0 \cdots \alpha_{r-1} a_0 \alpha_r$. Let $\pi_1 = W_1 b_0 W'_2 b_0 W_3$, where $W'_2 = \alpha_1 a_0 \alpha_2 b_0 \cdots \alpha_{r-1} b_0 \alpha_r$ in which we have changed all but the first a_0 occurring in W_2 to b_0. Note that this replaces all of the occurrences of 12112 involving a_0 and b_0 with ones of 12122. Using Lemma 2.6, one can verify that no occurrences of 1221 are introduced.

If π_1 has no occurrences of 12112, then let $f(\pi) = \pi_1$. Otherwise, let b_1 denote the smallest letter b in π_1 for which there exists a subsequence *abaab* for some $a < b$. One can verify $b_1 > b_0$. By reasoning similar to that used in the proof of Lemma 2.5 above, one can also show that a letter $a < b_1$ for which the subsequence ab_1aab_1 occurs in π_1 is uniquely determined, which we'll denote by a_1. Let π_2 denote the partition obtained by changing all of the letters a_1, except the first, coming after the leftmost b_1 to b_1. No 1221 subsequences are introduced, which follows from the minimality of b_1. Now repeat the above process, considering π_2.

Since $b_0 < b_1 < b_2 < \cdots$, the procedure described must end in a finite number, say t, of steps, with the resulting partition π_t belonging to $P_n(1221,12112)$. Let $f(\pi) = \pi_t$. Note that the largest b for which there exists $a < b$ such that *ababb* occurs in π_t is $b = b_{t-1}$ whenever $t \geq 1$. This follows from the fact one can verify that no occurrences of 12122 in which the 2 corresponds to a letter greater than b_i are introduced in the transition from π_i to π_{i+1} for all i. If $t \geq 1$, then one can also verify that the largest $a < b_{t-1}$ for which there is a subsequence of the form $ab_{t-1}ab_{t-1}b_{t-1}$ in π_t is $a = a_{t-1}$. So to reverse the algorithm, we first consider the largest letter b (if it exists) for which *ababb* occurs in π_t for some $a < b$ and then consider the largest such a corresponding to this b. One can then change the letters accordingly to reverse the final step of the algorithm describing f and the other steps can be similarly reversed, going from last to first.

Note that the above bijection preserves the number of blocks. Below we provide an example when $n = 15$:

$$\pi = \pi_0 = 123224253647567 \to \pi_1 = 123234353647567 \to \pi_2 = 123234354647567 \to$$
$$\pi_3 = 123234354657567 \to \pi_4 = 123234354657667 \to \pi_5 = 123234354657677 = f(\pi).$$

We now find an explicit formula for the generating function for the number of partitions in $P_n(1221,12122)$. In order to achieve this, we will consider the following three types of generating functions:

- For all $k \geq 1$, let $F_k(x)$ be the generating function for the number of partitions $\pi = \pi_1\pi_2\cdots\pi_n \in P_n(1221,12122)$ such that $\pi_1\pi_2\cdots\pi_k = 12\cdots k$ and $\pi_{k+1} \leq k$. We define $F_0(x) = 1$.

- For all $k \geq 2$, let $H_k(x)$ be the generating function for the number of partitions $\pi = \pi_1\pi_2\cdots\pi_n \in P_n(1221,12122)$ such that $\pi_1\pi_2\cdots\pi_k = 12\cdots k$ and $\pi_{k+1} = 1$.

- For all $k \geq 2$, let $G_k(x)$ be the generating function for the number of partitions $\pi = \pi_1\pi_2\cdots\pi_n \in P_n(1221,12122)$ such that $\pi_1\pi_2\cdots\pi_k = 12\cdots k$, $\pi_{k+1} = 1$ and $\pi_j \neq 1$ for all $j = k+2,\ldots,n$.

We define the further generating functions $F(x,y) = \sum_{k\geq 0} F_k(x)y^k$, $H(x,y) = \sum_{k\geq 2} H_k(x)y^k$, and $G(x,y) = \sum_{k\geq 2} G_k(x)y^k$. Our goal will be to find $F(x,1)$, which is the generating function for the sequence $p_n(1221,12122)$, $n \geq 0$. The next three lemmas provide relations which we will need between these generating functions.

Lemma 2.8. We have

$$F(x,y) = 1 + \frac{xy}{1-xy}F(x,1) + \frac{1}{1-xy}H(x,y).$$

Proof:

Let $a_\tau(x)$ be the generating function for the number of partitions $\pi = \pi_1\pi_2\cdots\pi_n \in P_n(1221,12122)$ such that $\pi_1\pi_2\cdots\pi_k = \tau$, where τ is some word. By the definitions, we have

$$\begin{aligned}
F_0(x) &= 1,\\
F_1(x) &= x + a_{11}(x) = x + x\sum_{j\geq 1} F_j(x) = x + x(F(x,1)-1),\\
F_k(x) &= x^k + \sum_{j=1}^{k} a_{12\cdots kj}(x) = x^k + \sum_{j=1}^{k-1} x^{j-1} a_{12\cdots(k+1-j)1}(x) + x^k a_1(x)\\
&= x^k + \sum_{j=1}^{k-1} x^{j-1} H_{k+1-j}(x) + x^k(F(x,1)-1), \qquad k \geq 2.
\end{aligned}$$

Hence,

$$F(x,y)=\frac{1}{1-xy}+\sum_{j\geq 2}\frac{y^j}{1-xy}H_j(x)+\frac{xy}{1-xy}(F(x,1)-1),$$

which is equivalent to

$$F(x,y)=1+\frac{xy}{1-xy}F(x,1)+\frac{1}{1-xy}H(x,y),$$

as required.

Lemma 2.9. We have

$$H(x,y)=G(x,y)+\frac{xy}{1-y}(yH(x,1)-H(x,y)).$$

Proof:

Let us write an equation for the generating function $H_k(x)$. Suppose $\pi=\pi_1\pi_2\cdots\pi_n$ is any member of $P_n(1221,12122)$ such that $\pi_1\pi_2\cdots\pi_{k+1}=12\cdots k1$. Consider the following two cases: (1) $\pi_j\neq 1$ for all $j=k+2,k+3,\ldots,n$ or (2) there exists at least one index $j>k+1$ such that $\pi_j=1$. Clearly, the first case contributes $G_k(x)$. For the second case, we write $\pi=12\cdots k1\pi'1\pi''$, observing that since π avoids 1221, there exists some ℓ such that $\pi'=(k+1)\cdots\ell$. Since the members of $P_n(1221,12122)$ of the form $12\cdots k1(k+1)\cdots\ell 1\pi''$ are in one-to-one correspondence with the members of $P_{n-1}(1221,12122)$ of the form $12\cdots\ell 1\pi''$, we see that the second case contributes $x\sum_{\ell\geq k}H_\ell(x)$. Thus,

$$H_k(x)\quad=G_k(x)+x\sum_{\ell\geq k}H_\ell(x),\qquad k\geq 2.$$

Multiplying this relation by y^k, and summing over all $k\geq 2$, yields

$$H(x,y)=G(x,y)+x\sum_{j\geq 2}\frac{y^2-y^{j+1}}{1-y}H_j(x),$$

which is equivalent to

$$H(x,y)=G(x,y)+\frac{xy}{1-y}(yH(x,1)-H(x,y)),$$

as required.

Lemma 2.10. We have

$$\left(1+\frac{x^2y^2}{(1-y)(1-xy)}\right)G(x,y)$$
$$=\frac{x^3y^2}{1-xy}+\frac{x^2y^2}{(1-y)(1-xy)}G(x,1)+\frac{x^4y^2(2-x)}{(1-x)(1-xy)}F(x,1)+\frac{x^3y^2}{(1-x)(1-xy)}H(x,1).$$

Proof:

Let $a_\tau(x)$ (respectively, $b_\tau(x)$) be the generating function for the number of partitions $\pi=\pi_1\pi_2\cdots\pi_n\in P_n(1221,12122)$ such that $\pi_1\pi_2\cdots\pi_k=\tau$ (respectively, $\pi_1\pi_2\cdots\pi_k=\tau$ and $\pi_j\neq 1$ for all $j\geq k+1$). From the definitions, we have

$$\begin{aligned}G_k(x) &= x^{k+1}+\sum_{j=2}^{k-1}b_{12\cdots k1j}(x)+b_{12\cdots k1k}(x)+b_{12\cdots k1(k+1)}(x)\\ &= x^{k+1}+\sum_{j=2}^{k-1}x^jG_{k+1-j}(x)+x^{k+2}F(x,1)+b_{12\cdots k1(k+1)}(x),\end{aligned}$$

and for all $\ell\geq k+1$,

$$\begin{aligned}&b_{12\cdots k1(k+1)\cdots\ell}(x)\\ &=x^{\ell+1}+\sum_{j=2}^{k}b_{12\cdots k1(k+1)\cdots\ell j}(x)+\sum_{j=k+1}^{\ell}b_{12\cdots k1(k+1)\cdots\ell j}(x)+b_{12\cdots k1(k+1)\cdots(\ell+1)}(x)\\ &=x^{\ell+1}+\sum_{j=2}^{k}x^jG_{\ell+1-j}(x)+\sum_{j=k+1}^{\ell-1}x^jH_{\ell+1-j}(x)+x^{\ell+1}(F(x,1)-1)+b_{12\cdots k1(k+1)\cdots(\ell+1)}(x).\end{aligned}$$

Hence, by summing over all $\ell\geq k+1$, we obtain

$$\begin{aligned}G_k(x)=&\frac{x^{k+1}}{1-x}+x^{k+2}F(x,1)+\frac{x^{k+2}}{1-x}(F(x,1)-1)+\frac{x^{k+1}}{1-x}H(x,1)\\ &+\sum_{j=2}^{k-1}x^jG_{k+1-j}(x)+\sum_{j=3}^{k}\frac{x^j-x^{k+1}}{1-x}G_{k+2-j}(x)+\frac{x^2-x^{k+1}}{1-x}\sum_{\ell\geq k}G_\ell(x).\end{aligned}$$

Multiplying the above recurrence by y^k, and summing over all $k\geq 2$, yields

$$
\begin{aligned}
G(x,y) &= \frac{x^3y^2}{(1-x)(1-xy)} + \frac{x^4y^2}{1-xy}F(x,1) + \frac{x^4y^2}{(1-x)(1-xy)}(F(x,1)-1) \\
&\quad + \frac{x^3y^2}{(1-x)(1-xy)}H(x,1) + \frac{x^2y}{1-xy}G(x,y) + \frac{x^3y}{1-xy}G(x,y) \\
&\quad + \frac{x^2y}{(1-x)(1-y)}(yG(x,1)-G(x,y)) - \frac{x^3y^2}{(1-x)(1-xy)}G(x,1),
\end{aligned}
$$

which leads to the required result.

Now we are ready to find an explicit formula for the generating function $F(x,1)$ for the number of partitions in $P_n(1221,12122)$. Lemmas 2.8, 2.9, and 2.10 give rise to a system of functional equations which we will solve using the kernel method (see Hou and Mansour (2011)). Lemma 2.8 implies

$$
F(x,1) = \frac{1-x}{1-2x} + \frac{1}{1-2x}H(x,1), \tag{6}
$$

and replacing y by $\frac{1}{1-x}$ in Lemma 2.9 gives

$$
H(x,1) = (1-x)G\left(x, \frac{1}{1-x}\right). \tag{7}
$$

Replacing y first by $y_1 = \frac{1+x-\sqrt{1-2x-3x^2}}{2x(1+x)}$ and then by $y_2 = \frac{1}{1-x}$ in Lemma 2.10 gives

$$
G(x,1) = \frac{x^3y_1^2}{1-xy_1} + \frac{x^4y_1^2(2-x)}{(1-x)(1-xy_1)}F(x,1) + \frac{x^3y_1^2}{(1-x)(1-xy_1)}H(x,1), \tag{8}
$$

$$
G\left(x, \frac{1}{1-x}\right) = \frac{x^3y_2}{1-3x} + \frac{x^4y_2^2(2-x)}{1-3x}F(x,1) + \frac{x^3y_2^2}{1-3x}H(x,1) - \frac{x}{1-3x}G(x,1). \tag{9}
$$

Solving the system of equations (6)-(9) yields the following result.

Proposition 2.11. The generating function for $p_n(1221,12122)$, $n \geq 0$, *is given by*

$$
F(x,1) = 1 - \frac{2x^2}{2x^2-3x+1-\sqrt{1-2x-3x^2}}.
$$

3. Conclusion

In this paper, we have identified six subsets of the partitions of size n, each avoiding a classical pattern of length four and another of length five and each enumerated by the sequence a_n. We have thereby obtained new classes of combinatorial structures enumerated by a_n, apparently the first such examples related to set partitions. Furthermore, numerical evidence shows that there are no other members of the (4, 5) Wilf-equivalence class for set partitions corresponding to the sequence a_n. We have used both algebraic and combinatorial methods to establish our results. In a couple of the seemingly more difficult cases, we make use of the kernel method to solve the functional equations that are satisfied by the related generating functions. It would be interesting to see if any of the other sequences in the "discrete continuity" mentioned in the introduction enumerate restricted subsets of partitions when $j > 3$ (such as those avoiding three or more patterns). Finally, it seems that the technique of introducing an auxiliary parameter and solving the functional equations which arise as a result using the kernel method would have wider applicability to other questions of avoidance not only for set partitions but also for other finite discrete structures, such as k-array words or permutations.

REFERENCES

Banderier, C., Bousquet-Mélou, M., Denise, A., Flajolet, P., Gardy, D. and Gouyou-Beauchamps, D. (2002). Generating functions for generating trees, (*Formal Power Series and Algebraic Combinatorics*, Barcelona, 1999), *Discrete Math.* **246:1-3**, 29-55.

Barcucci, E., Lungo, A.D., Pergola, E. and Pinzani, R. (2000). From Motzkin to Catalan permutations, *Discrete Math.* **217**, 33-49.

Goyt, A. (2008). Avoidance of partitions of a three-element set, *Adv. in Appl. Math.* **41,** 95-114.

Hou, Q.H. and Mansour, T. (2011). Kernel method and systems of functional equations with several conditions, *J. Comput. Appl. Math.* **235:5**, 1205-1212.

Jelínek, V. and Mansour, T. (2008). On pattern-avoiding partitions, *Electron. J. Combin.* **15**, # R39.

Klazar, M. (1996). On $abab$-free and $abba$-free set partitions, *European J. Combin.* **17**, 53-68.

Knuth, D.E. (1974). *The Art of Computer Programming, Vol's. 1 and 3*, Addison-Wesley, Reading, Massachusetts.

Sagan, B.E. (2010). Pattern avoidance in set partitions, *Ars Combin.* **94**, 79-96.

Simion, R. and Schmidt, F.W. (1985). Restricted permutations, *European J. Combin.* **6**, 383-406.

Sloane, N.J. The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

Stanton, D. and White, D. (1986). *Constructive Combinatorics*, Springer, New York.

Wagner, C. (1996). Generalized Stirling and Lah numbers, *Discrete Math.* **160**, 199-218.

West, J. (1995). Generating trees and the Catalan and Schröder numbers, *Discrete Math.* **146**, 247-262.