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Variational Iteration Method for Solving Initial and Boundary Value Problems of Bratu-type

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Abstract

In this paper, we present a reliable framework to solve the initial and boundary value problems of Bratu-type which are widely applicable in fuel ignition of the combustion theory and heat transfer. The algorithm rests mainly on a relatively new technique, the variational iteration method. Several examples are given to confirm the efficiency and the accuracy of the proposed algorithm.

Keywords: Variational iteration method, Bratu's problem, boundary value problems, initial value problems

AMS 2000 Subject Classification Numbers: 65 N10, 34 Bxx

1. Introduction

This paper is concerned with boundary value problems and an initial value problem of the Bratu-type. It is well known that Bratu's boundary value problem in one-dimensional planar coordinates is of the form, (see Ascher (1995), Boyd (2003), (1986), Buckmire (2003), Jacobson (2002), Wazwaz (2005), Noor and Mohyud-Din (2007a))

$$u'' + \lambda e^{u} = 0, \qquad 0 < x < 1, \tag{1}$$

with boundary conditions

$$u(0) = 0, \qquad u(1) = 0.$$

The standard Bratu type problem (1) was used to model a combustion problem in a numerical slab. The Bratu models appear in a number of applications such as the fuel ignition of the thermal combustion theory and in the Chandrasekhar model of the expansion of the universe, see Buckmire (2003). It stimulates a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction; see Argebesola (2005), Wazwaz (2005), Noor and Mohyud-Din (2007). A substantial amount of research work has been directed for the study of the Bratu problem, see Ascher (1995), Boyd (2003, 1986), Buckmire (2003), Jacobson (2002), Wazwaz (2005), Noor and Mohyud-Din (2007a) and the references therein.

Several numerical techniques, such as the finite difference method, finite element approximation, weighted residual method and the shooting method have been implemented independently to handle the Bratu model numerically, see Ascher (1995), Argebesola (2005), (1986), Buckmire (2003), Jacobson (2002). In addition, Boyd (2003, 1985) employed Chebyshev polynomial expansions and the Gegenbauer polynomials as base functions. Recently, Wazwaz (2005) used decomposition method for solving such models. More recently, Noor and Mohyud-Din (2007b) employed homotopy perturbation method and the variational iteration decomposition method for finding the solution of these problems. Inspired and motivated by the ongoing research in this area, we applied a relatively new technique, the variational iteration method for solving initial and boundary value problems of Bratu type models.

The exact solution to (1) is given as: (see Ascher (1995), Boyd (2003), (1986), Buckmire (2003), Jacobson (2002), Wazwaz (2005), Noor and Mohyud-Din (2007a))

$$u(x) = 2\ln\left[\frac{\cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right],$$

where θ satisfies

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right).$$

The Bratu problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_{c,}$ and $\lambda < \lambda_c$, respectively, where the critical value λ_c satisfies the equation

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\ \sinh\left(\frac{\theta_c}{4}\right).$$

It was evaluated in Ascher (1995), Boyd (2003, 1986), Buckmire (2003), Jacobson (2002), and Wazwaz (2005) that the critical value λ_c is given by

$$\lambda_c = 3.513830719 .$$

The basic motivation of the present work is to introduce a reliable treatment of two boundary value problems of Bratu-type model, given by

$$u'' - \pi^2 e^u = 0, \quad 0 < x < 1,$$

with boundary conditions

$$u(0) = u(1) = 0,$$

and

$$u'' + \pi^2 e^{-u} = 0, \qquad 0 < x < 1,$$

with boundary conditions

$$u(0) = u(1) = 0,$$

in addition, an initial value problem of the Bratu-type

$$u'' - 2e^u = 0, \quad 0 < x < 1,$$

with initial conditions

$$u(0) = u(1) = 0.$$

In this paper, our work stems mainly from the variational iteration method (see He (1999, 2000, 2006), He and Wu (2006), Inokuti et al. (1978), Mohyud-Din (2007), Noor and Mohyud-Din (2007a,b)). The basic motivation of this paper is to propose mathematical technique without imposing perturbation, restrictive assumptions or linearization. The variational iteration method which accurately computes the series solution is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with easily computable components. The main advantage of the method is that it can be applied directly for all types of nonlinear differential and integral equations, homogeneous or inhomogeneous, with constant or variable coefficients. Moreover, the proposed method is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution. Several examples are given to gauge the effectiveness and the usefulness of the suggested variational iteration method. The fact that variational iteration technique solves nonlinear problems without using Adomian polynomials can be considered as a clear advantage of this method over the decomposition method.

He (1999, 2000, 2006) developed the variational iteration method for solving linear, nonlinear, initial and boundary value problems. It is worth mentioning that the method was first considered by Inokuti, Sekine and Mura (1978) but the true potential of the variational iteration method was explored by He. In this method the solution is given in an infinite series usually converging to an accurate solution, see He (1999, 2000, 2006), He and Wu (2006), Inokuti et al. (1978), Noor and Mohyud-Din (2007a) and the references therein. We apply the variational iteration method to solve the initial and boundary value problems of the Bratu type. It is shown that the proposed technique provides the solution in a rapid convergent series with easily computable components. It is observed that the suggested method solve effectively, easily and accurately a large class of linear, nonlinear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions.

2. Variational Iteration Method

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{2}$$

where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to variational iteration method He (1999, 2000, 2006), He and Wu (2006), Inokuti et al. (1978), we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (L u_n(s) + N \tilde{u}_n(s) - g(s))\, ds, \tag{3}$$

where λ is a Lagrange multiplier, see He (1999, 2000, 2006), He and Wu (2006), Inokuti et al (1978), which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta \tilde{u}_n = 0$; (3) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in He (1999, 2000, 2006), He and Wu (2006), Inokuti et al. (1978), Noor and Mohyud-Din (2007a). In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by

$$u = \lim_{n\to\infty} u_n.$$

3. Numerical Applications

In this section, we apply the variational iteration method reviewed in section 2 for solving initial and boundary value problems of the Bratu-type. Numerical results indicate that the proposed variational iteration method is very reliable and efficient. The fact that the suggested technique solves nonlinear problems without using Adomian polynomials can be considered as a clear advantage of this method over the decomposition method. Moreover, we have also considered an example of higher dimensional initial boundary value problem (Noor and Mohyud-Din (2007a)). For the sake of comparison we take the same examples as considered by Wazwaz (2005), Noor and Mohyud-Din (2007b).

Example 3.1 Consider the following Bratu-type model

$$u'' - \pi^2 e^u = 0, \quad 0 < x < 1,$$

with initial conditions

$$u(0) = 0, \qquad u(1) = 0.$$

The correct functional is given as

$$u_{n+1}(x) = \alpha x + \int_0^x \lambda(\xi)\left(\frac{d^2 u_n(x)}{dx^2} - \pi^2 e^{\tilde{u}_n}\right) d\xi,$$

where $\alpha = u'(0) \neq 0$ is not given but will be determined by using the other boundary conditions; $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \xi - x,$ yields the following iteration formula

$$u_{n+1}(x) = \alpha x + \int_0^x (\xi - x)\left(\frac{d^2 u_n(x)}{dx^2} - \pi^2 e^{\tilde{u}_n}\right) d\xi.$$

Consequently, the following approximants are obtained

$$u_0(x) = \alpha x,$$

$$u_1(x) = \alpha x - \frac{\pi^2}{a^2}\left(e^{\alpha x} + \alpha x + 1\right)$$

$$u_2(x) = \alpha x - \frac{\pi^2}{\alpha^2}\left(e^{\alpha x} + \alpha x + 1\right) - \frac{\pi^4}{4\alpha^4}\left(e^{2\alpha x} + 4\alpha x e^{Ax} - 4e^{\alpha x} + 2\alpha x + 5\right)$$

$$u_3(x) = \alpha x - \frac{\pi^2}{a^2}\left(e^{\alpha x} + \alpha x + 1\right) - \frac{\pi^4}{4\alpha^4}\left(e^{2\alpha x} + 4\alpha x e^{\alpha x} - 4e^{\alpha x} + 2\alpha x + 5\right)$$
$$+ \frac{\pi^6}{12\alpha^6}\left(e^{3\alpha x} + 6e^{2\alpha x}\left(-\alpha x\right) + 3e^{\alpha x}\left(2\alpha^2 x^2 - 6\alpha x + 5\right) - 6\alpha x - 22\right),$$
$$\vdots$$

The series solution is given by

$$u(x) = \alpha x - \frac{\pi^2}{\alpha^2}\left(e^{\alpha x} + \alpha x + 1\right) - \frac{\pi^4}{4\alpha^4}\left(e^{2\alpha x} + 4\alpha x e^{\alpha x} - 4e^{\alpha x} + 2\alpha x + 5\right)$$
$$+ \frac{\pi^6}{12\alpha^6}\left(e^{3\alpha x} + 6e^{2\alpha x}\left(-\alpha x\right) + 3e^{\alpha x}\left(2\alpha^2 x^2 - 6\alpha x + 5\right) - 6\alpha x - 22\right) + \cdots,$$

or equivalently

$$u(x) = \alpha x + \frac{\pi^2}{2!}x^2 + \frac{\pi^2\alpha}{3!}x^3 + \left(\frac{\pi^2\alpha^2 + \pi^4}{4!}\right)x^4 + \left(\frac{\pi^2\alpha^3 + 4\pi^4\alpha}{5!}\right)x^5$$
$$+ \left(\frac{11\pi^4\alpha^2 + \pi^2\alpha^4 + 4\pi^6}{6!}\right)x^6 + \left(\frac{26\pi^4\alpha^3 + \pi^2\alpha^5 + 34\pi^6\alpha}{6!}\right)x^7 + \cdots.$$

Imposing the boundary condition $u\,(1) = 0$ leads to obtain $\alpha = \pi,$ and consequently, the closed form solution is given as

$$u(x) = -\ln\left(1 + \cos\left(\left(\frac{1}{2} + x\right)\pi\right)\right).$$

Example 3.2. Consider the following Bratu-type model

$$u'' + \pi^2 e^{-u} = 0, \qquad 0 < x < 1,$$

with initial conditions

$$u(0) = 0, \qquad u(1) = 0.$$

The correct functional is given as

$$u_{n+1}(x) = \alpha x + \int_0^x \lambda(\xi)\left(\frac{d^2 u_n(x)}{d x^2} - \pi^2 e^{-\tilde{u}_n}\right) d\xi,$$

where $\alpha = u'(0) \neq 0$ is not given but will be determined by using the other boundary conditions; $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \xi - x$, yields the following iteration formula

$$u_{n+1}(x) = \alpha x + \int_0^x (\xi - x)\left(\frac{d^2 u_n(x)}{d x^2} - \pi^2 e^{-\tilde{u}_n}\right) d\xi.$$

Consequently, the following approximants are obtained

$$\begin{aligned}
u_0(x) &= \alpha x, \\
u_1(x) &= \alpha x - \frac{\pi^2}{A^2}\left(e^{-\alpha x} + \alpha x - 1\right), \\
u_2(x) &= \alpha x - \frac{\pi^2}{\alpha^2}\left(e^{-\alpha x} + \alpha x - 1\right) - \frac{\pi^4}{4\alpha^4}\left(e^{-2\alpha x} + 4\alpha x e^{-Ax} + 4e^{-\alpha x} + 2\alpha x - 5\right), \\
u_3(x) &= \alpha x - \frac{\pi^2}{A^2}\left(e^{-\alpha x} + \alpha x - 1\right) - \frac{\pi^4}{4\alpha^4}\left(e^{-2\alpha x} + 4\alpha x e^{-\alpha x} + 4e^{-\alpha x} + 2\alpha x - 5\right) \\
&\quad - \frac{\pi^6}{12\alpha^6}\left(e^{-3\alpha x} + 6e^{-2\alpha x}\left(1 + Ax\right) + 3e^{-\alpha x}\left(2\alpha^2 x^2 + 6\alpha x + 5\right) + 6\alpha x - 22\right), \\
&\ \vdots
\end{aligned}$$

The series solution is given as

$$\begin{aligned}
u(x) &= \alpha x - \frac{\pi^2}{\alpha^2}\left(e^{-x} + \alpha x - 1\right) - \frac{\pi^4}{4\alpha^4}\left(e^{-2\alpha x} + 4\alpha x e^{-Ax} + 2\alpha x - 5\right) \\
&\quad - \frac{\pi^6}{12\alpha^6}\left(e^{-3\alpha x} + 6e^{-2\alpha x}\left(1 + \alpha x\right) + 3e^{-\alpha x}\left(2\alpha^2 x^2 + 6\alpha x + 5\right) + 6\alpha x - 22\right) + \cdots,
\end{aligned}$$

or equivalently

$$\begin{aligned}
u(x) &= \alpha x - \frac{\pi^2}{2!}x^2 + \frac{\pi^2\alpha}{3!}x^3 - \left(\frac{\pi^2\alpha^2 + \pi^4}{4!}\right)x^4 + \left(\frac{\pi^2\alpha^3 + 4\pi^4\alpha}{5!}\right)x^5 \\
&\quad - \left(\frac{11\pi^4\alpha^2 + \pi^2\alpha^4 + 4\pi^6}{6!}\right)x^6 + \left(\frac{26\pi^4\alpha^3 + \pi^2\alpha^5 + 34\pi^6\alpha}{6!}\right)x^7 + \cdots.
\end{aligned}$$

Imposing the boundary conditions at $u(1) = 0$ leads to obtain $\alpha = \pi$, and consequently, the closed form solution is given as

$$u(x) = \ln\left(1 + \sin(1 + \pi x)\right).$$

Example 3.3. Consider the following initial value problem of the Bratu-type

$$u'' - 2e^{u} = 0, \quad 0 < x < 1,$$

with initial conditions

$$u(0) = 0, \qquad u(1) = 0.$$

The correct functional is given by

$$u_{n+1}(x) = \int_0^x \lambda(\xi)\left(\frac{d^2 u_n(x)}{dx^2} - \pi^2 e^{\tilde{u}_n}\right) d\xi,$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \xi - x$, yields the following iteration formula

$$u_{n+1}(x) = \int_0^x (\xi - x)\left(\frac{d^2 u(x)}{dx^2} - \pi^2 e^{\tilde{u}_n}\right) d\xi.$$

Consequently, the following approximants are obtained

$$\begin{aligned}
u_0(x) &= 0, \\
u_1(x) &= x^2, \\
u_2(x) &= x^2 + \frac{1}{6}x^4, \\
u_3(x) &= x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6, \\
u_4(x) &= x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6 + \frac{17}{1260}x^8, \\
u_5(x) &= x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6 + \frac{17}{1260}x^8 + \frac{62}{14175}x^{10}, \\
u_6(x) &= x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6 + \frac{17}{1260}x^8 + \frac{62}{14175}x^{10} + \frac{691}{467775}x^{12}, \\
&\vdots
\end{aligned}$$

The series solution is given as

$$u(x) = x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6 + \frac{17}{1260}x^8 + \frac{31}{14175}x^{10} + \frac{691}{467775}x^{12} + \cdots,$$

or equivalently

$$u(x) = -2\left(-\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 - \frac{17}{2520}x^8 - \frac{31}{14175}x^{10} - \frac{691}{935550}x^{12} + \cdots\right).$$

The exact solution is given by

$$u(x) = -2\ln(\cos(x)).$$

Example 3.4. Consider the three dimensional initial boundary value problem

$$u_{tt} = \frac{1}{45}x^2u_{xx} + \frac{1}{45}y^2u_{yy} + \frac{1}{45}z^2u_{zz} - u, \qquad 0 < x, y < 1, t < 0,$$

subject to the Neumann boundary conditions

$$u_x(0, y, z, t) = 0, \qquad u_x(1, y, z, t) = 6y^6z^6\sinh t, \qquad u_y(x, 0, z, t) = 0,$$
$$u_y(x, 1, z, t) = 6x^6z^6\sinh t, \qquad u_z(x, y, 0, t) = 0, \qquad u_z(x, y, 1, t) = 6x^6y^6\sinh t,$$

and
the initial conditions

$$u(x, y, z, 0) = 0, \qquad u_t(x, y, z, 0) = x^6y^6z^6.$$

The correct functional is given as

$$u_{n+1}(x, y, z, t) = x^6y^6z^6\ t + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n}{\partial t^2} - \frac{1}{45}\left(x^2(\tilde{u}_n)_{xx} + y^2(\tilde{u}_n)_{yy} + z^2(\tilde{u}_n)_{zz}\right) + \tilde{u}_n\right)d\xi,$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \xi - t$, yields the following iteration formula

$$u_{n+1}(x, y, z, t) = x^6y^6z^6\ t + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n}{\partial t^2} - \frac{1}{45}\left(x^2(\tilde{u}_n)_{xx} + y^2(\tilde{u}_n)_{yy} + z^2(\tilde{u}_n)_{zz}\right) + \tilde{u}_n\right)d\xi.$$

Consequently, the following approximants are obtained

$$u_0(x, y, z, t) = x^6y^6z^6\,t,$$
$$u_1(x, y, z, t) = x^6y^6z^6\left(t + \frac{t^3}{3!}\right),$$

$$u_2(x,y,z,t)=x^6y^6z^6\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}\right),$$
$$u_3(x,y,z,t)=x^6y^6z^6\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}\right),$$
$$u_4(x,y,z,t)=x^6y^6z^6\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+\frac{t^9}{9!}\right),$$
$$\vdots$$

The series solution is given by

$$u(x,y,z,t)=x^6y^6z^6\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+\frac{t^9}{9!}+\cdots\right),$$

and in a closed form by

$$u(x,y,z,t)=x^6y^6z^6\sinh t.$$

4. Conclusion

In this paper, we applied the variational iteration method for solving the initial and boundary value problems of Bratu-type. The results clearly indicate the reliability and accuracy of the proposed technique. We also applied the proposed technique on a three dimensional initial boundary value problem. The suggested method is used directly without using perturbation, linearization or restrictive assumptions. Moreover, the variational iteration method is more reliable and efficient than the decomposition method. The fact that the method solves nonlinear problems without using Adomian polynomials can be considered as a clear advantage of this method over the decomposition method.

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