Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 3, Issue 1 (June 2008), pp. 128 – 136
(Previously, Vol. 3, No. 1)

Applications and Applied Mathematics:
An International Journal
(AAM)

Certain Expansion Formulae Involving a Basic Analogue of Fox's H-Function

S. D. Purohit
Department of Basic-Sciences (Mathematics)
College of Technology and Engineering
M.P. University of Agriculture and Technology, Udaipur-313001, India
sunil_a_ purohit@yahoo.com

R. K. Yadav
Department of Mathematics and Statistics
J. N. Vyas University, Jodhpur-342005, India
rkmdyadav@yahoo.co.in

S.L. Kalla
Department of Mathematics and Computer Science
P.O. Box-5969, Safat 13060, Kuwait
shyamkalla@yahoo.com

Received December 3, 2007; accepted May 7, 2008

Abstract

Certain expansion formulae for a basic analogue of the Fox's H-function have been derived by the applications of the q-Leibniz rule for the Weyl type q-derivatives of a product of two functions. Expansion formulae involving a basic analogue of Meijer's G-function and MacRobert's E-function have been derived as special cases of the main results.

Keywords: q-Leibniz rule, Weyl fractional, q-integral operator, Fox's H-function

AMS 2000 Subject Classification Numbers: 33D60 and 26A33

1. Introduction

Recently, Purohit (2007) introduced a new q-extension of the Leibniz rule for the derivatives of a product of two basic functions in terms of a finite q-series involving Weyl type q-derivatives of the functions in the following manner:

$$ {}_zD^{\alpha}_{\infty,q}\{U(z)\,V(z)\} = \sum_{r=0}^{\alpha} \frac{(-1)^r q^{r(r+1)/2}(q^{-\alpha};q)_r}{(q;q)_r}\; {}_zD^{\alpha-r}_{\infty,q}\{U(z)\}\; {}_zD^{\alpha}_{\infty,q}\left\{V(zq^{\alpha-r})\right\}, \quad (1) $$

where $U(z)$ and $V(z)$ are two regular functions and the fractional q-differential operator ${}_zD^{\alpha}_{\infty,q}(.)$ of Weyl type is given by

$$ {}_zD^{\alpha}_{\infty,q}\{f(z)\} = \frac{q^{-\alpha(1+\alpha)/2}}{\Gamma_q(-\alpha)} \int_z^{\infty} (t-z)_{-\alpha-1} f(tq^{1+\alpha})\, d(t;q)\,, \quad (2) $$

where $\mathrm{Re}(\alpha) < 0$ and

$$ (x-y)_{\nu} = x^{\nu} \prod_{n=0}^{\infty} \left[\frac{1-(y/x)q^{n}}{1-(y/x)q^{\nu+n}} \right], \quad (3) $$

the basic integration cf. Gasper and Rahman (1990), is defined as:

$$ \int_z^{\infty} f(t)\, d(t;q) = z(1-q) \sum_{k=1}^{\infty} q^{-k} f(zq^{-k})\,. \quad (4) $$

In view of the relation (4), operator (2) can be expressed as:

$$ {}_zD^{\alpha}_{\infty,q}\{f(z)\} = \frac{q^{\alpha(1-\alpha)/2} z^{-\alpha}(1-q)}{\Gamma_q(-\alpha)} \sum_{k=0}^{\infty} q^{\alpha k} (1-q^{k+1})_{-\alpha-1} f(zq^{\alpha-k})\,, \quad (5) $$

where $\mathrm{Re}(\alpha) < 0$.

In particular, for $f(z) = z^{-p}$, the equation (5) yields to

$$ {}_zD^{\alpha}_{\infty,q}\left\{z^{-p}\right\} = \frac{\Gamma_q(p+\alpha)}{\Gamma_q(p)} q^{-\alpha p+\alpha(1-\alpha)/2} z^{-p-\alpha}\,, \quad (6) $$

where $\mathrm{Re}(\alpha) < 0$.

We shall make use of the following notations and definitions in the sequel:

For real or complex a and $|q|<1$, the q-shifted factorial is defined as:

$$(a;q)_n = \begin{cases} 1, & if \;\; n=0 \\ (1-a)(1-aq)...(1-aq^{n-1}), & if \;\; n \in N. \end{cases} \tag{7}$$

In terms of the q-gamma function, (7) can be expressed as

$$(a;q)_n = \frac{\Gamma_q(a+n)\,(1-q)^n}{\Gamma_q(a)} \;, n>0, \tag{8}$$

where the q-gamma function cf. Gasper and Rahman (1990), is given by

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty (1-q)^{a-1}} \;, \tag{9}$$

where $a \neq 0,-1,-2,...$.

Saxena, et al. (1983), introduced a basic analogue of the H-function in terms of the Mellin-Barnes type basic contour integral in the following manner:

$$H_{A,B}^{m,n}\left[z;q\middle|\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s})\prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\;\pi\, z^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+\beta_j s})\;\prod_{j=n+1}^{A} G(q^{a_j-\alpha_j s}) G(q^{1-s}) \sin \pi s}\, ds, \tag{10}$$

where

$$G(q^\alpha) = \left\{\prod_{n=0}^{\infty}(1-q^{\alpha+n})\right\}^{-1} = \frac{1}{(q^\alpha;q)_\infty}, \tag{11}$$

and $0 \le m \le B,\; 0 \le n \le A$; α_j and β_i are all positive integers. The contour C is a line parallel to $\mathrm{Re}(ws)=0$, with indentations, if necessary, in such a manner that all the poles of $G(q^{b_j-\beta_j s})$, $1 \le j \le m$, are to the right, and those of $G(q^{1-a_j+\alpha_j s})$, $1 \le j \le n$ to the left of C. The integral converges if $\mathrm{Re}[s\log(z)-\log\sin \pi s] < 0$ for large values of $|s|$ on the contour C, that is, if $\left|\left\{\arg(z) - w_2 w_1^{-1}\log|z|\right\}\right| < \pi$, where $|q|<1$, $\log q = -w = -(w_1+iw_2)$, w, w_1, w_2 are definite quantities. w_1 and w_2 being real.

For $\alpha_j = \beta_i = 1,\ j = 1,\cdots,A;\ i = 1,\cdots,B$ the definition (10) reduces to the q-analogue of the Meijer's G-function due to Saxena, et al. (1983), namely

$$H_{A,B}^{m,n}\left[z;q \,\middle|\, \begin{matrix}(a,1)\\(b,1)\end{matrix}\right] \equiv G_{A,B}^{m,n}\left[z;q \,\middle|\, \begin{matrix}a_1,\cdots,a_A\\b_1,\cdots,b_B\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m} G(q^{b_j - s}) \prod_{j=1}^{n} G(q^{1-a_j+s})\ \pi z^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+s}) \prod_{j=n+1}^{A} G(q^{a_j - s}) G(q^{1-s}) \sin \pi s} ds, \tag{12}$$

where $0 \le m \le B,\ 0 \le n \le A$ and $\mathrm{Re}[s \log(z) - \log \sin \pi s] < 0.$

Further, if we set $n = 0$ and $m = B$ in the equation (12), we get the basic analogue of MacRobert's E-function due to Agarwal (1960), namely

$$G_{A,B}^{B,0}\left[z;q \,\middle|\, \begin{matrix}a_1,\cdots,a_A\\b_1,\cdots,b_B\end{matrix}\right] \equiv \mathrm{E_q[B;b_j:A;a_j:}z]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{B} G(q^{b_j - s})\ \pi z^s}{\prod_{j=1}^{A} G(q^{a_j - s}) G(q^{1-s}) \sin \pi s} ds, \tag{13}$$

where $\mathrm{Re}[s \log(z) - \log \sin \pi s] < 0.$

The Fox's H-function and Meijer's G-function have been studied in detail by several mathematicians for their theoretical and applications point of view. These functions have found wide ranging applications in mathematical, physical, biological and statistical sciences. It would be interesting to observe that almost all the classical special functions are the particular cases of the Fox's H-function. A detailed account of various classical special functions expressible in terms of Meijer's G-function or Fox's H-function along with their applications to the aforementioned fields can be found in the research monographs by Mathai et al. (1973, 1978).

A new generalization was considered by Saxena et al. (1983) in the form of the *q*-extensions of the Fox's H- function and Meijer's G-function by means of the Mellin- Barne's type of basic integral. The advantage of these new extensions of the Fox's H and Meijer's G-functions lies in the fact that a number of q-special functions including the basic hypergeometric functions, happens to be the particular cases of the $H_q(.)$ and $G_q(.)$ functions, thus widening the scope for further applications. In a paper, Saxena, et al. (1990), besides proving some interesting relations, have established an important limit formula for the $H_q(.)$ function when q tends to 1. Various basic functions expressible in terms of the basic analogue of Fox's H-function or basic Meijer's G-function with their applications can be found in the research papers due to Saxena, et al. (2005) and Yadav et al. (2006).

In the present paper, we shall explore the possibility for derivation of some expansion formulae involving the basic analogue of the Fox's H-function by the applications of the q-Leibniz rule for the Weyl type q-derivatives of a product of two functions. We also investigate the expansion formulae involving the basic analogues of Meijer's G-function and MacRobert's E-function.

2. Main Results

In this section, we shall establish certain results associated with the basic analogue of Fox's H-function by assigning suitable values to the functions $U(z)$, $V(z)$, and α in the q-Leibniz rule (1). The main results to be established are as under:

$$H^{m+1,n}_{A+1,B+1}\left[\rho(zq^{\mu})^{k};q\left|\begin{matrix}(a,\alpha),(\lambda,k)\\(\mu+\lambda,k),(b,\beta)\end{matrix}\right.\right]=\sum_{r=0}^{\mu}\frac{(-1)^{r}q^{r(r+1)/2+\lambda r}(q^{-\mu};q)_{r}(q^{\lambda};q)_{\mu-r}}{(q;q)_{r}}$$

$$H^{m+1,n}_{A+1,B+1}\left[\rho(zq^{\mu})^{k};q\left|\begin{matrix}(a,\alpha),(0,k)\\(r,k),(b,\beta)\end{matrix}\right.\right], \tag{14}$$

where $0\le m\le B,\ 0\le n\le A,\ \mathrm{Re}[s\log(z)-\log\sin\pi s]<0,\ k\ge 0$ and ρ being any complex quantity.

$$H^{m,n+1}_{A+1,B+1}\left[\rho(zq^{\mu})^{k};q\left|\begin{matrix}(1-\mu-\lambda,-k),\ (a,\alpha)\\(b,\beta),\ (1-\lambda,-k)\end{matrix}\right.\right]=\sum_{r=0}^{\mu}\frac{(-1)^{r}q^{r(r+1)/2+\lambda r}(q^{-\mu};q)_{r}}{(q;q)_{r}}$$

$$(q^{\lambda};q)_{\mu-r}\,H^{m,n+1}_{A+1,B+1}\left[\rho(zq^{\mu})^{k};q\left|\begin{matrix}(1-r,-k),\ (a,\alpha)\\(b,\beta),\ (1,-k)\end{matrix}\right.\right], \tag{15}$$

where $0\le m\le B,\ 0\le n\le A,\ \mathrm{Re}[s\log(z)-\log\sin\pi s]<0,\ k<0$ and ρ being any complex quantity.

Proof of the main results:

To prove the results (14) and (15), we begin with $U(z)=z^{-\lambda}$ and

$$V(z)=H^{m,n}_{A,B}\left[\rho z^{k};q\left|\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right.\right],$$

in the equation (1) to obtain

$$ {}_zD^{\mu}_{\infty,q}\left\{z^{-\lambda}\, H^{m,n}_{A,B}\left[\rho z^k ;q\left|\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right.\right]\right\} = \sum_{r=0}^{\mu}\frac{(-1)^r q^{r(r+1)/2}(q^{-\mu};q)_r}{(q;q)_r}\, {}_zD^{\mu-r}_{\infty,q}\left\{z^{-\lambda}\right\} $$

$$ {}_zD^{\alpha}_{\infty,q}\left\{H^{m,n}_{A,B}\left[\rho (zq^{\mu-r})^k ;q\left|\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right.\right]\right\}. \tag{16} $$

n view of the definition (10), the left-hand side of equation (16) becomes

$$ {}_zD^{\mu}_{\infty,q}\left\{z^{-\lambda}\, H^{m,n}_{A,B}\left[\rho z^k ;q\left|\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right.\right]\right\} $$

$$ = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s})\prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\ \pi\rho^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+\beta_j s})\prod_{j=n+1}^{A} G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}\, {}_zD^{\mu}_{\infty,q}\left\{z^{-(\lambda-ks)}\right\} ds\,. \tag{17} $$

On making use of fractional q-derivative formula (6) in the above equation (17), we obtain following interesting transformation for the $H_q(.)$ function after certain simplifications:

$$ {}_zD^{\mu}_{\infty,q}\left\{z^{-\lambda}\, H^{m,n}_{A,B}\left[\rho z^k ;q\left|\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right.\right]\right\} = \frac{z^{-\lambda-\mu} q^{-\mu\lambda+\mu(1-\mu)/2}}{(1-q)^{\mu}} $$

$$ H^{m+1,n}_{A+1,B+1}\left[\rho (zq^{\mu})^k ;q\left|\begin{matrix}(a,\alpha),(\lambda,k)\\(\mu+\lambda,k),(b,\beta)\end{matrix}\right.\right], \tag{18} $$

where $k \geq 0$.

Again, if we take $k<0$, we obtain the following fractional q-derivative formula for the $H_q(.)$ function, namely

$$ {}_zD^{\mu}_{\infty,q}\left\{z^{-\lambda}\, H^{m,n}_{A,B}\left[\rho z^k ;q\left|\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right.\right]\right\} = \frac{z^{-\lambda-\mu} q^{-\mu\lambda+\mu(1-\mu)/2}}{(1-q)^{\mu}} $$

$$ H^{m,n+1}_{A+1,B+1}\left[\rho (zq^{\mu})^k ;q\left|\begin{matrix}(1-\mu-\lambda,-k),\,(a,\alpha)\\(b,\beta),\,(1-\lambda,-k)\end{matrix}\right.\right]. \tag{19} $$

We now substitute $\lambda=0$ and replace μ by r and then z by $zq^{\mu-r}$ respectively, in equation (18) to obtain the following transformation for the $H_q(.)$ function:

$$
{}_zD^{r}_{\infty,q}\left\{H^{m,n}_{A,B}\left[\rho(zq^{\mu-r})^{k};q\left|\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right.\right]\right\}=\frac{z^{-r}q^{r(r+1)/2-\mu r}}{(1-q)^{r}}
$$

$$
H^{m+1,n}_{A+1,B+1}\left[\rho(zq^{\mu})^{k};q\left|\begin{matrix}(a,\alpha),(0,k)\\(r,k),(b,\beta)\end{matrix}\right.\right]. \tag{20}
$$

Further, in view of the result (6), one can easily obtain the following relation

$$
{}_zD^{\mu-r}_{\infty,q}\left\{z^{-\lambda}\right\}=\frac{\Gamma_q(\lambda+\mu-r)}{\Gamma_q(\lambda)}q^{(\mu-r)(1-\mu+r-2\lambda)/2}\ z^{-\lambda-\mu+r}. \tag{21}
$$

On substituting the values of the various expressions involved in the equation (16), from equations (18), (20) and (21), we arrive at the main result (14).

The proof of the result (15) follows similarly when $k<0$ and by the usages of the transformation formula (19) and the relation (21).

3. Special Cases

In this section, we shall consider some special cases of the main results and deduce certain expansion formulae involving the basic analogue of Meijer's G-function and basic analogue of MacRobert's E-function.

If we set $\alpha_j=\beta_i=1,\ j=1,\cdots,A;\ i=1,\cdots,B$ and $k=1$, in the main result (14), we obtain the following interesting expansion formula involving Meijer's $G_q(.)$ function, namely

$$
G^{m+1,n}_{A+1,B+1}\left[\rho zq^{\mu};q\left|\begin{matrix}a_1,\cdots,a_A,\lambda\\ \mu+\lambda,b_1,\cdots,b_B\end{matrix}\right.\right]=\sum_{r=0}^{\mu}\frac{(-1)^r q^{r(r+1)/2+\lambda r}(q^{-\mu};q)_r(q^{\lambda};q)_{\mu-r}}{(q;q)_r}
$$

$$
G^{m+1,n}_{A+1,B+1}\left[\rho zq^{\mu};q\left|\begin{matrix}a_1,\cdots,a_A,0\\ r,b_1,\cdots,b_B\end{matrix}\right.\right], \tag{22}
$$

where $0\le m\le B,\ 0\le n\le A$, $\mathrm{Re}[s\log(z)-\log\sin\pi s]<0$ and ρ being any complex quantity.

Similarly, for $\alpha_j=\beta_i=1,\ j=1,\cdots,A;\ i=1,\cdots,B$ and $k=-1$, the main result (15) reduces to yet another expansion formula associated with the basic analogue of Meijer's G-function, namely

$$
G^{m,n+1}_{A+1,B+1}\left[\rho/zq^{\mu};q\left|\begin{matrix}1-\mu-\lambda,a_1,\cdots,a_A\\ b_1,\cdots,b_B,1-\lambda\end{matrix}\right.\right]=\sum_{r=0}^{\mu}\frac{(-1)^r q^{r(r+1)/2+\lambda r}(q^{-\mu};q)_r}{(q;q)_r}
$$

$$(q^{\lambda};q)_{\mu-r}\, G_{A+1,B+1}^{m,n+1}\left[\rho/zq^{\mu};q\left|\begin{matrix}1-r,a_1,\cdots,a_A\\ b_1,\cdots,b_B,1\end{matrix}\right.\right], \tag{23}$$

where $0\le m\le B,\ 0\le n\le A$, $\mathrm{Re}[s\log(z)-\log\sin\pi s]<0$ and ρ being any complex quantity.

Finally, if we set $n=0$ and $m=B$, the result (22), yields to an expansion formula involving MacRobert's $E_q(.)$ function, namely

$$E_q[B+1;b_j,\mu+\lambda:A+1;\,a_j,\lambda:\rho zq^{\mu}]=\sum_{r=0}^{\mu}\frac{(-1)^r q^{r(r+1)/2+\lambda r}(q^{-\mu};q)_r(q^{\lambda};q)_{\mu-r}}{(q;q)_r}$$

$$E_q[B+1;b_j,r:A+1;\,a_j,0:\rho zq^{\mu}], \tag{24}$$

where $\mathrm{Re}[s\log(z)-\log\sin\pi s]<0$ and ρ being any complex quantity.

Acknowledgement:
The authors sincerely acknowledge the valuable suggestions made by one of the referees.

REFERENCES

Agarwal, N., (1960). A q-analogue of MacRobert's generalized E-function, Ganita 11, 49-63.

Agarwal, R.P., (1976). Fractional q-derivatives and q-integrals and certain hypergeometric transformations, Ganita 27, 25-32.

Al-Salam, W.A., (1966). Some fractional q-integrals and q-derivatives, Proc. Edin. Math. Soc. 15, 135-140.

Gasper, G., M. Rahman, (1990). Basic Hypergeometric Series, Cambridge University Press, Cambridge.

Mathai, A. M., R. K. Saxena, (1973). Generalized Hypergeometric Functions With Applications in Statistics and Physical Sciences, Springer-Verlag, Berlin.

Mathai, A. M., R. K. Saxena, (1978). The H-function With Application in Statistics and Other Disciplines, John Wiley and Sons. Inc. New York.

Purohit, S.D. (2007). On a q-extension of the Leibniz rule via Weyl type of q-derivative operator, (Communicated).

Saxena, R.K., G.C. Modi., S.L. Kalla, (1983). A basic analogue of Fox's H-function, Rev. Tec. Ing. Univ. Zulia 6, 139-143.

Saxena, R.K., R. Kumar, (1990). Recurrence relations for the basic analogue of the H-function, J. Nat. Acad. Math. 8, 48-54.

Saxena, R.K., R.K. Yadav, S.D. Purohit, S.L. Kalla, (2005). Kober fractional q-integral operator of the basic analogue of the H-function, Rev. Tec. Ing. Univ. Zulia 28(2), 154-158.

Slater, L.J., (1966). Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, London and New York.

Yadav, R.K., S.D. Purohit, (2006). On fractional q-derivatives and transformations of the generalized basic hypergeometric functions, J. Indian Acad. Math. 28(2), 321-326.

Yadav, R.K., and S.D. Purohit, (2006). On applications of Weyl fractional q-integral operator to generalized basic hypergeometric functions, Kyungpook Math. J. 46, 235-245.