Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Special Issue No. 1 (August 2010) pp. 154 – 166

Applications and Applied Mathematics:
An International Journal
(AAM)

Homotopy Perturbation Method and the Stagnation Point Flow

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Received: February 15, 2010; Accepted: January 31, 2011

Abstract

The laminar steady flow of an incompressible, viscous fluid near a stagnation point has been computed using the homotopy perturbation method (HPM). Both the cases, (i) two-dimensional flow and (ii) axisymmetric flow, have been considered. A sequence of successive approximations has been obtained in the solution, and the convergence of the sequence is achieved by using the Padé approximants. It is found that there is a complete agreement between the results obtained by the HPM and the exact numerical solution.

Keywords: Stagnation point flow, homotopy perturbation method, Padé approximants.

AMS Classification: 34L30, 35Q70, 65Q30

Introduction

The stagnation point flow is one of those rare problems in fluid dynamics which finds its place in almost every textbook, such as Schlichting (1968), Shih-i-Pai (1956) and White (1991). Besides being technologically very important in aerodynamics the problem is also ideal pedagogically as it is one of those for which the governing partial differential equations can be reduced to a single ordinary differential equation using similarity variables. The problem has been solved for a two-dimensional flow (Hiemenz, 1911), axi-symmetric flow (Homann, 1936) and the general three-dimensional flow (Howarth, 1951). Since an exact numerical solution of the boundary value problem (BVP) describing the flow is apparently not possible, there have been attempts to solve the problem both numerically and also analytically, though in the latter case the solutions have to be necessarily either approximate or in the form of some series. Using the integral method, Pohlhausen (1921) derived the solution for the Falkner-Skan flow, from which the solutions can be readily derived for two-dimensional and axi-symmetric flows. From the engineering point of view there was a satisfactory agreement between the solution obtained by the integral method by

Pohlhausen and the exact numerical solution. For the Falkner-Skan flow, a detailed numerical solution has been given by Nachtsheim and Swigert (1965).

The analytical solutions of the engineering problems have always been of practical importance to engineers as well as have presented mathematical challenges to the researchers. Whereas a numerical solution gives the values at some discrete values in time and space -- and to obtain the value at some point different than those for which the solution is obtained, some further interpolation is required, in general -- the analytical solutions have the immense advantage in that they give the values at any arbitrarily chosen value of the variables of interest. For the two-dimensional stagnation point flow in hydromagnetics, an approximate solution, making use of the weighted residual method in which the residual of the differential equation is minimized in the least square sense, has been derived by Ariel (1994). The solution has the attractive feature that it is not only accurate for the purely hydrodynamics case, but that its accuracy improves as the magnitude of the Hartmann number is increased. The numerical methods, in general, suffer degradation when the Hartmann number's value is increased as the problem becomes increasingly sensitive numerically.

There is yet another factor that needs to be addressed when a numerical solution is sought of the stagnation point flow. It is related to the infinite domain of the flow. For a numerical solution the infinite domain of integration is usually replaced by a finite value, in which the value is hopefully chosen sufficiently large to minimize the error introduced as a result of replacing infinity by a finite value. There have been attempts to address this issue and one of the techniques suggested in the literature is known as the free boundary value problem (FBVP) formulation. Using FBVP formulation Fazio (1992) obtained the solution of the Blasius flow – the remarkable feature about the solution being that the accuracy of the solution could be improved substantially by choosing sufficiently small value of the skin-friction at the outer boundary. A similar approach was chosen by Ariel (1993) to get the solution of the two-dimensional stagnation point flow.

Of late considerable interest has been generated in obtaining the solution of nonlinear problems in science and technology by invoking the concept of homotopy. The basic idea is to introduce a homotopy parameter p which varies from 0 to 1. When p is zero, the problem simplifies to a somewhat a trivial problem, which is usually linear and whose solution can be found relatively easily. As p is incremented to 1, a family of solutions is generated, which ultimately approaches the desired solution as p takes the value 1. Watson (1979) has been the main exponent of the technique as he and his coworkers (Watson and Wang, 1978; Wang and Watson, 1979a, 1979b; Watson, 1981, 1990) solved a number of difficult problems using the technique numerically.

Recently the idea of using the homotopy technique to derive analytical solutions has attracted a lot of attention of researchers mainly due to efforts of Liao (1992) and He (1999). Liao refined the technique by introducing an auxiliary parameter $\hbar$, besides another optional auxiliary function in his formulation of the problem. The parameter $\hbar$ can be used to control the convergence of the series solution obtained as the power series in p. Liao, Hayat, Abbasbandy and their coworkers have solved a number of important problems (Liao, 1999, 2003, 2004; Wang *et* al, 2003; Hayat *et al* 2004; Hayat and Sajid, 2007; Sajid *et al*, 2007; Tan and Abbasbandy, 2008; Abbasbandy and Hayat, 2009a, 2009b) and succeeded in obtaining purely analytical solutions in lieu of numerical solutions. He, on the other hand, has demonstrated that it is not essential to use the auxiliary parameter $\hbar$. Instead by introducing other parameters and judiciously adjusting their values by requiring some appropriate conditions, He (1998, 2003a, 2003b, 2006a, 2006b, 2008) showed that analytical solutions can be obtained which require very few terms in the power series in p, and yet they are sufficiently accurate.

For the flows caused by moving boundary, the HPM has proved to be very useful and effective. Using only one-term correction Ariel *et al* (2006) derived an analytical solution for the axisymmetric flow past a stretching sheet which was remarkably accurate even when the effects of suction and magnetic field were included separately or jointly. Ariel (2007a) also gave a fully analytical solution for the axisymmetric flow past a stretching sheet when there is a partial slip at the sheet. That the HPM is not limited to the solution of the problems characterized by a single BVP, Ariel (2007b) obtained the solution for the generalized three dimensional flow past a stretching sheet. There has been a criticism that the HPM as developed in (Ariel *et al*, 2006; Ariel 2007a, 2007b) cannot be generalized, i.e., more correction terms cannot be included in the solution (El-Mistikawy, 2009). Ariel (2009a), however, extended the classical version of HPM to derive the extended HPM which can include an arbitrary number of correction terms, and used it to obtain the solution of the problem of axisymmetric flow past a stretching sheet, and also the solution past a rotating disk (Ariel, 2009b).

For the two-dimensional stagnation point flow, He (2004) has used his HPM to obtain an approximate solution. He also used only a one-term correction. His results can be considered satisfactory from an engineering angle, however, if more accurate results are required then it is not quite obvious how the method given by him can be generalized to obtain more correction terms. In any case, as opposed to the flow past a stretching sheet, to date no attempt has been made to derive a solution by HPM of the problem of the stagnation point flow which systematically improves the accuracy of the solution. The present paper is an attempt in that direction. The equations of motion as given in the textbooks are first transformed to a BVP in a finite domain [0, 1] using Crocco's variables, as indicated by Djukić (1974). The resulting equation is reset in the framework of HPM and a power series solution is obtained in terms of p, the homotopy parameter. The convergence of the solution is attained by taking recourse to Pade approximants.

Formulation of the problem

We consider the steady, laminar flow of a viscous, incompressible fluid impinging normally on a plane taken along the *XOY* plane. The equations of motion are the well known Navier-Stokes equations

$$\rho \mathbf{q} \bullet \nabla \mathbf{q} = -\nabla P + \mu \nabla^2 \mathbf{q}, \tag{1}$$

$$\nabla \bullet \mathbf{q} = 0, \tag{2}$$

where ρ and μ are respectively the density and coefficient of viscosity, assumed to be constant, P is the pressure at a point, and $\mathbf{q}$ is the velocity.

The flow far from the plane $z = 0$ is governed by the potential flow given by

$\mathbf{q} = ax\mathbf{e}_x$ for the two-dimensional flow, and

$\mathbf{q} = ar\mathbf{e}_r$ for the axisymmetric flow, (3)

where a is a constant.

We define the similarity variables

$$\eta = \sqrt{\frac{a\rho}{\mu}}z \quad \text{and} \quad f = -\sqrt{\frac{\rho}{a\mu}}\frac{q_z}{s}, \tag{4}$$

where q_z is the velocity of the fluid normal to the plane $z = 0$, and s is a parameter that characterizes the nature of the flow. It is equal to 1 for two-dimensional flow, and equal to 2 for the axisymmetric flow.

In terms of the similarity variables, the equations of motion (1) and (2) reduce to the following BVP

$$f''' + sff'' + 1 - f'^2 = 0, \tag{5}$$

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1. \tag{6}$$

BVP (5)-(6) has been solved numerically and analytically numerous times in the literature. Its counterpart, involving interchange of boundary conditions on f' for the flow past a stretching sheet, has been solved by Ariel *et al* (2006) using the HPM. On the other hand, a fully analytical solution of the BVP (5) and (6) as it stands, with some minor adjustments, has been derived by Liao (2003) using the HAM. As mentioned earlier, He (2004) gave an approximate solution with one correction term for the two-dimensional flow ($s = 1$). But so far in the literature, using the HPM, no solution has been proposed that has been generalized to include an arbitrary number of correction terms.

Following Djukić (1974), we find it convenient to introduce the Crocco's variables

$$\lambda = f',\ \phi = f''^2. \tag{7}$$

As a result, BVP (5)-(6) "reduces" to the following form:

$$\phi\frac{d^2\phi}{d\lambda^2} - \frac{1}{2}\left(\frac{d\phi}{d\lambda}\right)^2 - (1-\lambda^2)\frac{d\phi}{d\lambda} - 2(2-s)\lambda\phi = 0, \tag{8}$$

$$@\ \lambda = 0,\ \frac{d\phi}{d\lambda} = -2, \text{ and } @\lambda = 1,\ \phi = 0. \tag{9}$$

Djukić (1974) solved BVP (8)-(9) by using the integral method due to Pohlhausen (1921) to obtain the solution of the two-dimensional stagnation flow of a power law fluid in hydromagnetics. On the other hand Ariel (2002) solved a similar BVP numerically to obtain the solution of the two-dimensional flow of a power law fluid past a stretching sheet.

There is one more transformation, namely,

$$\xi = 1 - \lambda, \tag{10}$$

that is needed to facilitate the work in the sequel. With this transformation, equations (8) and (9) take the form

$$\phi \frac{d^2\phi}{d\xi^2} - \frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + \xi(2-\xi)\frac{d\phi}{d\xi} - 2(2-s)(1-\xi)\phi = 0, \tag{11}$$

$$@\ \xi = 0,\ \phi = 0,\ \text{and}\ \ @\xi = 1,\ \frac{d\phi}{d\xi} = 2. \tag{12}$$

Homotopy Perturbation Formulation

We set up the homotopy perturbation formulation by rewriting equation (11) as

$$\phi \frac{d^2\phi}{d\xi^2} = p\left[\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 - \xi(2-\xi)\frac{d\phi}{d\xi} + 2(2-s)(1-\xi)\phi\right], \tag{13}$$

where p is the homotopy parameter.

We seek a perturbation solution for φ in the form of a power series in p as under

$$\phi = \phi_0 + \phi_1 p + \phi_2 p^2 + \cdots = \sum_{n=0}^{\infty} \phi_n p^n, \tag{14}$$

Assuming that the series (14) converges for $p = 1$, the final solution for ϕ is given by

$$\phi = \sum_{n=0}^{\infty} \phi_n. \tag{15}$$

The quantity of the greatest physical significant, namely the skin-friction at the plane, and represented by $f''(0)$ is then given by

$$f''(0) = \sqrt{\phi}\Big|_{\xi=1}. \tag{16}$$

The velocity profiles can be computed by

$$\eta = \int_{\xi}^{1} \frac{1}{\sqrt{\phi}} d\xi, \tag{17}$$

which expresses η in terms of ξ or f'. A second integration would give f in terms of η.

HPM Solution

Substituting for φ from equation (14) into equations (13) and (12), and equating like powers of p on both sides, we obtain the following system of BVPs.
Zeroth order system:

$$\phi_0 \frac{d^2\phi_0}{d\xi^2} = 0, \tag{18}$$

$$@\ \xi = 0,\ \phi_0 = 0,\ \text{ and }\ @\xi = 1,\ \frac{d\phi_0}{d\xi} = 2. \tag{19}$$

Higher order systems:

$$\phi_0 \frac{d^2\phi_n}{d\xi^2} = \sum_{m=1}^{n}\left(-\phi_m \frac{d^2\phi_{n-m}}{d\xi^2} + \frac{1}{2}\frac{d\phi_m}{d\xi}\frac{d\phi_{n-m}}{d\xi}\right) - \xi(2-\xi)\frac{d\phi_{n-1}}{d\xi} + 2(2-s)(1-\xi)\phi_{n-1}, \tag{20}$$

$$@\ \xi = 0,\ \phi_n = 0,\ \text{ and }\ @\xi = 1,\ \frac{d\phi_n}{d\xi} = 0. \tag{21}$$

The solution of the BVP (18)-(19) is

$$\phi_0 = 2\xi. \tag{22}$$

The solution of the BVP (20)-(21) is now straightforward. After dividing equation (20) by φ_0, we first integrate equation (20) and use the boundary condition at $\xi = 1$ in equation (21) to obtain $d\phi_n / d\xi$. Next, another integration is carried out and use is made of the boundary condition at $\xi = 0$ in equation (21) to obtain φ_n. This step can be repeated up to any desired value of n. Below we list the first few φ_n for both: s = 1 (two-dimensional flow) and s = 2 (axi-symmetric flow).

Two – dimensional flow:

$$\phi_1 = -\tfrac{1}{6}\xi(3 + \xi^2 - 6\ln\xi), \tag{23}$$

$$\phi_2 = \tfrac{1}{360}\xi[230 - 180\xi + 5\xi^2 + 10\xi^3 - 3\xi^4 + 90\ln\xi + 90(\ln\xi)^2], \tag{24}$$

$$\phi_3 = \tfrac{1}{151200}\xi[82817 - 94500\xi + 15575\xi^2 - 1050\xi^3 - 882\xi^4 + 518\xi^5 - 110\xi^6 + \\ +(67200 - 2100\xi^3 + 630\xi^4)\ln\xi + 28350(\ln\xi)^2 + 6300(\ln\xi)^3] \tag{25}$$

$$\phi_4 = \tfrac{1}{190512000}\xi[110480745 - 136930500\xi + 33093375\xi^2 - 7938000\xi^3 + 2075346\xi^4 - \\ -51450\xi^5 - 217020\xi^6 + 85725\xi^7 - 14350\xi^8 + (94510710 - 7938000\xi^2 - 661500\xi^3 + \\ +1349460\xi^4 - 652680\xi^5 + 138600\xi^6)\ln\xi + (39028500 + 661500\xi^3 - 198450\xi^4)(\ln\xi)^2 + \\ +9922500(\ln\xi)^3 + 992250(\ln\xi)^4] \tag{26}$$

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Axisymmetric Flow

$$\phi_1 = \tfrac{1}{6}\xi(3 - 6\xi + \xi^2 - 6\ln\xi), \tag{27}$$

$$\phi_2 = \tfrac{1}{360}\xi[370 - 450\xi + 115\xi^2 - 25\xi^3 + 3\xi^4 + 270\ln\xi + 90(\ln\xi)^2], \tag{28}$$

$$\phi_3 = \tfrac{1}{151200}\xi[169988 - 217350\xi + 60025\xi^2 - 11025\xi^3 + 357\xi^4 + 77\xi^5 + 10\xi^6 + \\ +(134400 - 12600\xi^2 + 5250\xi^3 - 630\xi^4)\ln\xi + 47250(\ln\xi)^2 + 6300(\ln\xi)^3] \tag{29}$$

$$\phi_4 = \tfrac{1}{381024000}\xi[460984515 - 601303500\xi + 157400250\xi^2 - 9628500\xi^3 - 7491267\xi^4 + \\ +2132970\xi^5 - 413760\xi^6 + 69750\xi^7 - 3850\xi^8 + (383528880 - 66150000\xi^2 + \\ +17860500\xi^3 - 52920\xi^4 - 194040\xi^5 - 25200\xi^6)\ln\xi + (144207000 + 7938000\xi^2 - \\ -3307500\xi^3 + 396900\xi^4)(\ln\xi)^2 + 27783000(\ln\xi)^3 + 1984500(\ln\xi)^4] \tag{30}$$

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There is no doubt that the expansions for the higher order perturbation terms can be listed, if necessary. However, they become increasingly unwieldy, and for the sake of brevity are not given here.

In Table 1, the values of φ at $\xi = 1$ are listed for s = 1 (two-dimensional flow) and s = 2 (axisymmetric flow) for the first thirty orders of the HPM solution.

Table 1. Illustrating the variation of φ at $\xi = 1$ for s = 1 (two-dimensional flow) and s = 2 (axisymmetric flow) for various values of n, the order of the HPM solution

n	s=1	s=2	n	s=1	s=2
0	2	2	16	1.51928797	1.72116497
1	1.33333333	1.66666667	17	1.51929295	1.72117138
2	1.50555556	1.70277778	18	1.51929186	1.72117594
3	1.52121693	1.71654762	19	1.51928810	1.72117894
4	1.52428168	1.72113160	20	1.51928370	1.72118072
5	1.52315311	1.72221721	21	1.51927971	1.72118166
6	1.52135556	1.72214833	22	1.51927655	1.72118203
7	1.52001697	1.72181868	23	1.51927429	1.72118206
8	1.51928242	1.72152230	24	1.51927281	1.72118191
9	1.51899059	1.72132374	25	1.51927193	1.72118168
10	1.51894793	1.72121192	26	1.51927150	1.72118142
11	1.51901297	1.72115924	27	1.51927134	1.72118118
12	1.51910402	1.72114122	28	1.51927137	1.72118098
13	1.51918328	1.72114088	29	1.51927148	1.72118081
14	1.51923897	1.72114795	30	1.51927163	1.72118069
15	1.51927201	1.72115685			

It can be seen from Table 1 that the HPM solution is converging to the desired solution; however the convergence is not as rapid as one would like it to be. The convergence can be accelerated considerably by invoking the Shank's transformation (1955). This has been done advantageously by Ariel (2009a) to compute accurately the axisymmetric flow past a stretching sheet. There is an attractive alternative in seeking the Padé approximants corresponding to the sequence of approximations – the latter has the advantage of zeroing in to the correct limit even if the sequence happens to be divergent (Ariel, 2010). Encouraged by these developments in the present work we have taken recourse to Padé approximants. In Tables 2 and 3, the values of φ at $\xi = 1$ are listed for two-dimensional flow and axisymmetric flow respectively, this time applying the Padé [m, n] rational approximations, where either $m = n$ or $m = n$-1.

Table 2. Illustrating the variation of φ at $\xi = 1$ for the two-dimensional flow for various values of [*m*, *n*], the order of the Padé rational approximations

m	*n*	φ(1)	*m*	*n*	φ(1)
0	0	2	0	1	1.5
1	1	1.4701986755	1	2	1.4868990588
2	2	1.5248466678	2	3	1.5228634480
3	3	1.5046681613	3	4	1.5183082794
4	4	1.5189935995	4	5	1.5191364179
5	5	1.5193342077	5	6	1.5193093097
6	6	1.5192838556	6	7	1.5192755561
7	7	1.5192696459	7	8	1.5192702022
8	8	1.5192723868	8	9	1.5192728152
9	9	1.5192727302	9	10	1.5192731061
10	10	1.5192724002	10	11	1.5192723774
11	11	1.5192723277	11	12	1.5192723242
12	12	1.5192723236	12	13	1.5192723240
13	13	1.5192723330	13	14	1.5192723327
14	14	1.5192723343	14	15	1.5192723320
15	15	1.5192723320	15	16	1.5192723320
16	16	1.5192723321	16	17	1.5192723323
17	17	1.5192723317	17	18	1.5192723317
18	18	1.5192723317	18	19	1.5192723317
19	19	1.5192723317	19	20	1.5192723317
20	20	1.5192723317			

Table 3. Illustrating the variation of φ at $\xi = 1$ for the axisymmetric flow for various values of [m, n], the order of the Padé rational approximations

m	n	$\varphi(1)$	m	n	$\varphi(1)$
0	0	2	0	1	1.714285714
1	1	1.6992481203	1	2	1.7066025532
2	2	1.7234029221	2	3	1.7224651436
3	3	1.7209686239	3	4	1.7210362853
4	4	1.7211936888	4	5	1.7211927799
5	5	1.7211943420	5	6	1.7212437722
6	6	1.7211796692	6	7	1.7211798626
7	7	1.7211804934	7	8	1.7211805402
8	8	1.7211805608	8	9	1.7211805554
9	9	1.7211805104	9	10	1.7211805091
10	10	1.7211805090	10	11	1.7211805091
11	11	1.7211805134	11	12	1.7211805128
12	12	1.7211805126	12	13	1.7211805127
13	13	1.7211805127	13	14	1.7211805127
14	14	1.7211805129	14	15	1.7211805125
15	15	1.7211805126	15	16	1.7211805126
16	16	1.7211805126	16	17	1.7211805126
17	17	1.7211805126	17	18	1.7211805126
18	18	1.7211805126	18	19	1.7211805126
19	19	1.7211805126	19	20	1.7211805126
20	20	1.7211805126			

It is evident from Tables 2 and 3 that a [15, 15] Padé rational approximant accelerates the convergence of the solution sufficiently to ensure the accuracy of ten-digits. The values of $f''(0)$ can be readily obtained by taking the square root of φ at $\xi = 1$. We have

$f''(0) = 1.2325876568$ for the two-dimensional flow,

$f''(0) = 1.3119376939$ for the axisymmetric flow.

The Padé approximation is also in full agreement with the numerical solution obtained by Ariel (1993) for the two-dimensional flow using the FBVP formulation, where the value of $f''(0)$ has been computed to sixteen-digit accuracy. For the axisymmetric flow, it may be added that, we believe that the value of $f''(0)$ calculated here is the most accurate reported in the literature.

Conclusion

In the present work we computed the steady, laminar, flow of an incompressible, viscous fluid near a stagnation point using the HPM. Both the cases (i) the two-dimensional flow, and (ii) axisymmetric flow have been considered. For an efficient implementation of the HPM, the usual BVPs characterizing the flows are first transformed to appropriate BVPs in a finite domain [0, 1] using the Crocco variables. Even though the resulting BVP is such that the zeroth order solution is nonlinear, no difficulty was encountered in developing an HPM solution. The resulting solution in the form of a power series in p turns out to be converging rather slowly at $p=1$. However an application of Padé approximation considerably accelerates the convergence and leads to a solution which is accurate to ten-digit accuracy using only [15, 15] Padé approximant.

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