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Exact solitary-wave Special Solutions for the Nonlinear Dispersive *K(m,n)* Equations by Means of the Homotopy Analysis Method

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Abstract

In this paper, we study the nonlinear dispersive *K(m,n)* equations which exhibit solutions with solitary patterns. New exact solitary solutions are found. The two special cases, *K(2, 2)* and *K(3, 3)*, are chosen to illustrate the concrete features of the homotopy analysis method in *K(m,n)* equations. The nonlinear equations *K(m,n)* are studied for two different cases, namely when $m = n$ being odd and even integers. General formulas for the solutions of *K(m,n)* equations are established.

Keywords: Homotopy analysis method, nonlinear dispersive *K(m,n)* equations

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1. Introduction

Searching for solitary solutions for nonlinear equation in mathematical physics is attractive in the solitary theory. For example, Wadati (1972, 1973) developed the solitons for *KdV* and MKdV equations. In 1993, Rosenau and Hyman (1993) presented a family of fully nonlinear *KdV* equations *K(m,n)*

$$u_t + \; u^m \;{}_x + \; u^n \;{}_{xxx} = 0, \quad m > 0, \quad 1 < n \leq 3 \tag{1}$$

and introduced a class of solitary waves with compact support that are solutions of a two parameter family of fully nonlinear dispersive partial different equations such as *K*(2, 2) equation

$$u_t + \; u^2 \;{}_x + \; u^2 \;{}_{xxx} = 0, \tag{2}$$

Recently, Wazwaz (2002a) gave exact special solutions with solitary patterns for the nonlinear dispersive *K(m,n)* equations

$$u_t - \; u^m \;{}_x + \; u^n \;{}_{xxx} = 0, \quad m,n > 1 \tag{3}$$

The new solitary-wave special solutions with compact support for the nonlinear dispersive $K(m,n)$ equations

$$u_t + \; u^m \;{}_x + \; u^n \;{}_{xxx} = 0, \quad m,n > 1 \tag{4}$$

are presented by Wazwaz (2002b). Of course, other solitary-wave solutions of *K*(*m*,*n*) equations were also found by many authors [Rosenau (1994, 1997)].

In recent, Wazwaz (2001, 2002a, 2002b, 2004) have successfully used the Adomian decomposition method to construct solitary solutions for many nonlinear equations.

The main goal of this paper is to investigate the *K(m,n)* equations of the form

$$u_t + \; u^m \;{}_x - \; u^n \;{}_{xxx} = 0, \quad m,n > 1 \tag{5}$$

and we would like to extend the homotopy analysis method (HAM) to seek exact special solutions with solitary patterns for (5).

The HAM is developed in 1992 by Liao (1995, 1999, 2004). This method has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors [Abbasbandy (2007), Ayub et al. (2003), Hayat et al. (2004)]. By the present method, numerical results can be obtained with using a few iterations. The HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series for large values of t. Other numerical methods are given low degree of accuracy for large values of t. Therefore, the HAM handles linear and nonlinear problems without any assumption and restriction.

2. The homotopy analysis method (HAM)

We apply the HAM to the nonlinear dispersive *K(m,n)* equations (5) . We consider the following differential equation

$$N\left[u(x,t)\right]=0, \tag{6}$$

where N is a nonlinear operator for this problem, x and t denote independent variables, $u(x,t)$ is an unknown function.

In the frame of HAM, we can construct the following zeroth-order deformation:

$$(1-q)L\left[U(x,t;q)-u_0(x,t)\right]=q\hbar H(x,t)N\left[U(x,t;q)\right], \tag{7}$$

where $q\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is an auxiliary parameter, $H(x,t)\neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $U(x,t;q)$ is an unknown function on the independent variables x, t and q.

Obviously, when $q=0$ and $q=1$, it holds

$$U(x,t;0)=u_0(x,t),\quad U(x,t;1)=u(x,t). \tag{8}$$

Using the parameter q, we expand $U(x,t;q)$ in Taylor series as follows:

$$U(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{9}$$

where

$$u_m=\frac{1}{m!}\frac{\partial^m U(x,t;q)}{\partial^m q}\Bigg|_{q=0}. \tag{10}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function $H(x,t)$ are selected such that the series (9) is convergent at $q=1$, then due to (8) we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{11}$$

Let us define the vector

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}. \tag{12}$$

Differentiating (7) m times with respect to the embedding parameter q, then setting $q = 0$ and finally dividing them by $m!$, we have the so-called m^{th}-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}) \tag{13}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[U(x,t;q)]}{\partial^{m-1} q} \Bigg|_{q=0}, \tag{14}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases} \tag{15}$$

Finally, for the purpose of computation, we will approximate the HAM solution (11) by the following truncated series:

$$\phi_m(x,t) = \sum_{k=0}^{m-1} u_k(x,t). \tag{16}$$

3. Applications

In this section, we would like to choose two special equations, namely $K(2, 2)$ and $K(3, 3)$ with specific initial conditions, to illustrate the above-mentioned scheme.

Example 1. Consider the $K(2, 2)$ equation with the initial condition

$$u_t + (u^2)_x - (u^2)_{xxx} = 0, \tag{17a}$$

$$u(x,0) = \frac{4}{3} v \sinh^2\left(\frac{x}{4}\right), \tag{17b}$$

where v is an arbitrary constant.

According to (7), the zeroth-order deformation can be given by

$$(1-q)\,L\left[U(x,t;q)-u_0(x,t)\right]=q\hbar H(x,t)\left[U_t+\left(U^2\right)_x-\left(U^2\right)_{xxx}\right]. \tag{18}$$

We can start with an initial approximation $u_0(x,t)=\frac{4}{3}v\sinh^2\left(\frac{x}{4}\right)$ and choose the auxiliary linear operator

$$L\left[U(x,t;q)\right]=\frac{\partial U(x,t;q)}{\partial t},$$

with the property

$$L(C)=0,$$

where C is an integral constant. We also choose the auxiliary function to be

$$H(x,t)=1.$$

Hence, the m^{th}-order deformation can be given by

$$L\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=\hbar H(x,t)R_m\left(\vec{u}_{m-1}\right),$$

where

$$R_m\left(\vec{u}_{m-1}\right)=\frac{\partial\left(u_{m-1}\right)}{\partial t}+\frac{\partial}{\partial x}\left(\sum_{i=0}^{m-1}u_iu_{m-1-i}\right)-\frac{\partial^3}{\partial x^3}\left(\sum_{i=0}^{m-1}u_iu_{m-1-i}\right). \tag{19}$$

Now the solution of the m^{th}-order deformation equations (19) for $m\geq 1$ becomes

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar L^{-1}\left[R_m\left(\vec{u}_{m-1}\right)\right]. \tag{20}$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,t)=\frac{4}{3}v\sinh^2\left(\frac{x}{4}\right),$$

$$u_1(x,t)=-\frac{\hbar}{3}v^2t\sinh\left(\frac{x}{2}\right),$$

$$u_2(x,t)=-\frac{\hbar}{3}v^2t\sinh\left(\frac{x}{2}\right)-\frac{\hbar^2}{3}v^2t\sinh\left(\frac{x}{2}\right)+\frac{\hbar^2}{12}v^3t^2\cosh\left(\frac{x}{2}\right),$$

.

and so on. Hence, the HAM series solution (for $\hbar=-1$) is

$$\begin{aligned} u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+.....\\ &=\frac{4}{3}v\sinh^2\left(\frac{x}{4}\right)+\frac{1}{3}v^2t\sinh\left(\frac{x}{2}\right)+\frac{1}{12}v^3t^2\cosh\left(\frac{x}{2}\right)+\frac{1}{72}v^4t^3\,\text{sinhh}\left(\frac{x}{2}\right)+...... \end{aligned} \tag{21}$$

Using Taylor series into (21), we find the closed form solution

$$u(x,t)=\frac{4}{3}v\sinh^2\left(\frac{x+vt}{4}\right). \tag{22}$$

In addition, we can develop another exact solution for the $K(2, 2)$ equation. Now we consider another initial value problem of $K(2, 2)$ equation

$$u_t+\left(u^2\right)_x-\left(u^2\right)_{xxx}=0, \tag{23a}$$

$$u(x,0)=-\frac{4}{3}v\cosh^2\left(\frac{x}{4}\right), \tag{23b}$$

Using the manner as discussed above, we obtain another exact solution given by

$$u(x,t)=-\frac{4}{3}v\cosh^2\left(\frac{x+vt}{4}\right). \tag{24}$$

Example 2. Consider the initial value problem $K(3, 3)$

$$u_t+\left(u^3\right)_x-\left(u^3\right)_{xxx}=0, \tag{25a}$$

$$u(x,0)=\frac{\sqrt{6v}}{2}\sinh\left(\frac{x}{3}\right), \tag{25b}$$

where v is an arbitrary constant.

According to (7), the zeroth-order deformation can be given by

$$(1-q)L\left[U(x,t;q)-u_0(x,t)\right]=q\hbar H(x,t)\left[U_t+\left(U^3\right)_x-\left(U^3\right)_{xxx}\right] \tag{26}$$

We can start with an initial approximation $u_0(x,t)=\frac{\sqrt{6v}}{2}\sinh\left(\frac{x}{3}\right)$ and we choose the auxiliary linear operator

$$L\left[U(x,t;q)\right]=\frac{\partial U(x,t;q)}{\partial t},$$

with the property

$$L(C)=0,$$

where C is an integral constant. We also choose the auxiliary function to be

$$H(x,t)=1.$$

Hence, the mth-order deformation can be given by

$$L\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=\hbar H(x,t)R_m\left(\vec{u}_{m-1}\right),$$

where

$$R_m\left(\vec{u}_{m-1}\right)=\frac{\partial\left(u_{m-1}\right)}{\partial t}+\frac{\partial}{\partial x}\left(\sum_{i=0}^{m-1}u_i\left(\sum_{k=0}^{m-1-i}u_k u_{m-1-i-k}\right)\right)-\frac{\partial^3}{\partial x^3}\left(\sum_{i=0}^{m-1}u_i\left(\sum_{k=0}^{m-1-i}u_k u_{m-1-i-k}\right)\right) \quad (27)$$

Now the solution of the mth-order deformation equations (27) for $m\geq 1$ become

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar L^{-1}\left[R_m\left(\vec{u}_{m-1}\right)\right]. \quad (28)$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,t)=\frac{\sqrt{6v}}{2}\sinh\left(\frac{x}{3}\right),$$
$$u_1(x,t)=-\frac{\sqrt{6}}{6}\hbar v^{3/2}t\cosh\left(\frac{x}{3}\right),$$
$$u_2(x,t)=-\frac{\sqrt{6}}{6}\hbar v^{3/2}t\cosh\left(\frac{x}{3}\right)-\frac{\sqrt{6}}{6}\hbar^2 v^{3/2}t\cosh\left(\frac{x}{3}\right)+\frac{\sqrt{6}}{36}\hbar^2 v^{5/2}t^2\sinh\left(\frac{x}{3}\right),$$

.

and so on. Hence, the HAM series solution (for $\hbar=-1$) is

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\ldots..$$
$$=\frac{\sqrt{6v}}{2}\sinh\left(\frac{x}{3}\right)+\frac{\sqrt{6}}{6}v^{3/2}t\cosh\left(\frac{x}{3}\right)+\frac{\sqrt{6}}{36}v^{5/2}t^2\sinh\left(\frac{x}{3}\right)+\frac{\sqrt{6}}{324}v^{7/2}t^3\cosh\left(\frac{x}{3}\right)+\ldots.. \quad (29)$$

Using Taylor series into (29), we find the closed form solution

$$u(x,t) = \frac{\sqrt{6v}}{2}\sinh\left(\frac{x+vt}{3}\right) \tag{30}$$

To obtain another exact solution for *K(3, 3)*, we consider the initial value problem of *K(3, 3)* equation

$$u_t + (u^3)_x - (u^3)_{xxx} = 0, \tag{31a}$$

$$u(x,0) = -\frac{\sqrt{6v}}{2}\sinh\left(\frac{x}{3}\right), \tag{31b}$$

According to the similar steps as discussed above, we have another exact solution given by

$$u(x,t) = -\frac{\sqrt{6v}}{2}\sinh\left(\frac{x+vt}{3}\right)$$

4. More exact solutions

4.1. *The K(2,2) type*

In Section 3, two exact solitary patterns solutions were developed in the form

$$u(x,t) = \frac{4}{3}v\sinh^2\left(\frac{x+vt}{4}\right) \tag{32}$$

and

$$u(x,t) = -\frac{4}{3}v\cosh^2\left(\frac{x+vt}{4}\right) \tag{33}$$

By combining the two results, we will find that

$$u(x,t) = \frac{4}{3}Mv\sinh^2\left(\frac{x+vt}{4}\right) - \frac{4}{3}Nv\cosh^2\left(\frac{x+vt}{4}\right), \tag{34}$$

satisfies the *K(2, 2)* equation, where M and N are constants if

$$M=N \text{ or } M = 1-N \tag{35}$$

(a) When $M = N$, we can obtain the trivial solution

$$u(x,t) = -\frac{4}{3}Nv \tag{36}$$

(b) When *M = 1-N,* we can obtain the new exact solution

$$u(x,t) = \frac{4}{3}(1-N)v\sinh^2\left(\frac{x+vt}{4}\right) - \frac{4}{3}Nv\cosh^2\left(\frac{x+vt}{4}\right), \tag{37}$$

Moreover, adding a constant to the arguments in (32) and (33) will exhibit more exact solutions. In other words, we introduce the exact solutions

$$u(x,t) = \frac{4}{3}v\sinh^2\left(\frac{x+vt}{4}+c\right) \tag{38}$$

and

$$u(x,t) = -\frac{4}{3}v\cosh^2\left(\frac{x+vt}{4}+c\right) \tag{39}$$

where c is a constant.

4.2. *The K(3,3) type*

As discussed before, the exact solutions for the *K(3, 3)* equation are given in the form

$$u(x,t) = \frac{\sqrt{6v}}{2}\sinh\left(\frac{x+vt}{3}\right) \tag{40}$$

and

$$u(x,t) = -\frac{\sqrt{6v}}{2}\sinh\left(\frac{x+vt}{3}\right) \tag{41}$$

We can obtain a new exact solution by combining the two results (40) and (41) and we find that

$$u(x,t) = \frac{\sqrt{6v}}{2}M\sinh\left(\frac{x+vt}{3}\right) - \frac{\sqrt{6v}}{2}N\sinh\left(\frac{x+vt}{3}\right) \tag{42}$$

satisfies the *K(3, 3)* equation if

$$M = N,\ M = 1+N,\ M = -1+N, \tag{43}$$

(a) When *M = N,* we can obtain the trivial solution

$$u(x,t) = 0 \tag{44}$$

(b) When $M = 1+N$, we can obtain the new exact solution

$$u(x,t) = \frac{\sqrt{6v}}{2}(1+N)\sinh\left(\frac{x+vt}{3}\right) - \frac{\sqrt{6v}}{2}N\sinh\left(\frac{x+vt}{3}\right) \tag{45}$$

(c) When $M = -1+N$, we can obtain *the* new exact solution

$$u(x,t) = \frac{\sqrt{6v}}{2}(-1+N)\sinh\left(\frac{x+vt}{3}\right) - \frac{\sqrt{6v}}{2}N\sinh\left(\frac{x+vt}{3}\right) \tag{46}$$

Moreover, adding a constant to the arguments in (40) and (41) will exhibit more exact solutions

$$u(x,t) = \frac{\sqrt{6v}}{2}\sinh\left(\frac{x-vt}{3}+c\right) \tag{47}$$

and

$$u(x,t) = -\frac{\sqrt{6v}}{2}\sinh\left(\frac{x-vt}{3}+c\right) \tag{48}$$

where c is a constant.

5. General solutions for *K(m,n)*

5.1. The *K*(*m,n*), *m*= *n* Being Even Integer

When $m = n$ being even integer, we can test several initial value problems by using homotopy analysis method. By properly observing several results, we introduce the general formulas

$$u(x,t) = \left(\sqrt{\frac{2vp}{p+1}}\sinh\left[\frac{n-1}{2n}(x+vt)+c\right]\right)^{2/(n-1)} \tag{49}$$

and

$$u(x,t) = -\left(\sqrt{\frac{2vp}{p+1}}\cosh\left[\frac{n-1}{2n}(x+vt)+c\right]\right)^{2/(n-1)}, \tag{50}$$

where c is a constant.

5.2. The *K*(*m*,*n*), *m*= *n* Being Odd Integer

For $m = n$ being odd integer and $n > 1$, several problems were tested by using homotopy analysis method and by carefully observing several results, we introduce the general formulas

$$u(x,t) = \left(\sqrt{\frac{2vp}{p+1}} \sinh\left[\frac{n-1}{2n}(x+vt) + c \right] \right)^{2/(n-1)} \tag{51}$$

and

$$u(x,t) = -\left(\sqrt{\frac{2vp}{p+1}} \sinh\left[\frac{n-1}{2n}(x+vt) + c \right] \right)^{2/(n-1)}, \tag{52}$$

where c is a constant. And by combining the two results (51) and (52), we will find that

$$u(x,t) = M\left(\sqrt{\frac{2vp}{p+1}} \sinh\left[\frac{n-1}{2n}(x+vt) + c \right] \right)^{2/(n-1)} - N\left(\sqrt{\frac{2vp}{p+1}} \sinh\left[\frac{n-1}{2n}(x+vt) + c \right] \right)^{2/(n-1)}, \tag{53}$$

satisfies the *K*(*n*,*n*) equation with n being odd integer >1, where M and N are constants if

$$M = N,\ M = 1+N,\ M = -1+N \tag{54}$$

Thus this completes our goal by establishing general formulas for solutions of the nonlinear dispersive *K*(*n*,*n*) that work for all values of $n > 1$.

6. Conclusion

In summary, we have presented the application of homotopy analysis method to the nonlinear dispersive *K(m,n)* equations. We chose two special cases, *K*(2, 2) and *K*(3, 3) equations to illustrate the scheme such that new exact solutions with solitary patterns are of important significance. We developed the new exact solutions which are generated by combining two distinct solutions of the *K*(2, 2) and *K(3, 3)* equations. At last, we establish the general formulas for exact solutions of equations *K*(*m*,*n*) when $m = n$ being even and odd integers for $n > 1$. It is worth noting that other solitary solutions of *K*(*m,n)* equation may be also constructed by using the homotopy analysis method.

References

Abbasbandy, S. (2007). Homotopy analysis method for heat radiation equations. *Int. Comm. Heat. Mass. Transfer Vol.* **34**, pp. 380–387.

Ayub, M., Rasheed, A. and Hayat, T. (2003). Exact flow of a third grade fluid past a porous plate using homotopy analysis method. *Int. J. Eng. Sci.* Vol. **41**, pp. 2091–2103.

Hayat, T., Khan, M. and Asghar, S. (2004). Magnetohydrodynamic flow of an Oldroyd 6–constant fluid. *Appl. Math. Comput.Vol.* **155**, pp. 417–425.

Liao, S.J.(1995). An approximate solution technique which does not depend upon small parameters: a special example. *Int. J. Nonlinear. Mech Vol.* **30**, pp. 371–380.

Liao, S.J.(1999). An explicit, totally analytic approximation of Blasius viscous flow problems. *Int. J. Nonlinear. Mech. Vol.* **34**, pp. 759–778.

Liao, S.J.(2004). On the homotopy analysis method for nonlinear problems. *Appl. Math. Comput.* Vol. **147**, pp. 499–513.

Rosenau, P. and Hyman, J.M.(1993). Compactons: solitons with finite wavelengths. *Phys. Rev. Lett.* Vol. **70**, pp. 564–567.

Rosenau, P.(1994). Nonlinear dispersion and compact structures. *Phys. Rev. Lett. Vol.* **73**, pp. 1737–1741.

Rosenau, P. (1997). On nonanalytic solitary waves formed by a nonlinear dispersion. Phys. Lett. A.Vol. **230**, pp. 305–318.

Wadati, M. (1972). The exact solution of the modified Kortweg–de Vries equation. *J. Phys. Soc. Jpn. Vol.* **32**, pp. 1681-1687.

Wadati, M. (1973). The modified Kortweg–de Vries equation. *J. Phys. Soc. Jpn. Vol.* **34**, pp. 1289–1296.

Wazwaz, A.M.(2001) Construction of soliton solutions and periodic solutions of the Boussinesq equation by the modified decomposition method. *Chaos, Solitons & Fractals*. Vol. **12**, pp. 1549–1556.

Wazwaz, A.M. (2002a). Exact special solutions with solitary patterns for the nonlinear dispersive K(m,n) equations. *Chaos, Solitons &Fractals Vol.* **13**, pp. 161–170.

Wazwaz, A.M. (2002b). New solitary-wave special solutions with compact support for the nonlinear dispersive K(m,n) equations. *Chaos, Solitons & Fractals.Vol.* **13**, pp. 321–330.

Yonggui, Zhu. (2004) Exact special solutions with solitary patterns for Boussinesq-like B(m,n) equations with fully nonlinear dispersion. *Chaos, Solitons & Fractals.* Vol. **22**, pp. 213–220.