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Multistage Homotopy Analysis Method for Solving Nonlinear Integral Equations

H. Jafari and M. A. Firoozjaee

Department of Mathematics
University of Mazandaran
Babolsar, IRAN
jafari@umz.ac.ir, m64arab@math.com

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Abstract

In this paper, we present an efficient modification of the homotopy analysis method (HAM) that will facilitate the calculations. We then conduct a comparative study between the new modification and the homotopy analysis method. This modification of the homotopy analysis method is applied to nonlinear integral equations and mixed Volterra-Fredholm integral equations, which yields a series solution with accelerated convergence. Numerical illustrations are investigated to show the features of the technique. The modified method accelerates the rapid convergence of the series solution and reduces the size of work.

Keywords: Homotopy analysis method; Multistage Homotopy analysis method; Variational iteration method; Volterra-Fredholm integral equations.

MSC (2000) No: 34K28; 45G99

1. Introduction

The mixed Volterra-Fredholm integral equations arise in the theory of parabolic boundary value problems, the mathematical modeling of the spatio-temporal development of an epidemic, and various physical and biological problems. A discussion of the formulation of these models is given in Wazwaz (2002) and the references therein.

The nonlinear mixed Volterra-Fredholm integral equation is given in Wazwaz (2002) as

$$u(x,y) = f(x,y) + \int_0^y \int_\Omega G(x,y,s,t,u(s,t))dsdt, \qquad (x,y) \in [0,1]\times\Omega, \tag{1}$$

where $u(x,y)$ is an unknown function, the functions $f(x,y)$ and $G(x,y,s,t,u)$ are analytic on $D=\Omega\times[0,T]$ and where Ω is a closed subset of (R, n = 1,2,3). The existence and uniqueness results for equation (1) may be found in Diekmann (1978), Han et al. (1994), and Pachpatta (1986). However, few numerical methods for equation (1) are known in the literature Wazwaz (2002). For the linear case, the time collocation method was introduced in Pachpatta (1986) and the projection method was presented in Hacia (1996, 2002). In Brunner (1990) the results of Pachpatta (1986) have been extended to nonlinear Volterra-Hammerstein integral equations. In Maleknejad et al. (1999) and Wazwaz (2002), a technique based on the Adomian decomposition method was used for the solution of (1). A new method for solving (1) by means of the Legendre wavelets method is introduced in Yousefi et al. (2005). Another type of mixed Volterra-Fredholm integro-differential equation and Fredholm integro-differential equation that we are going to solve approximately by the multistage homotopy analysis method is given as

$$\begin{aligned} y(x) &= f(x) + \lambda_1 \int_0^x K_1(x,t)F(y(t))dt + \lambda_2 \int_0^x K_1(x,t)G(y(t))dt, \qquad 0 \le x \le 1, \\ y^{(n)}(x) &= f(x) + \lambda_1 \int_{-1}^1 K_1(x,t)G(y(t))dt, \end{aligned} \tag{2}$$

where $f(x)$ and the kernels $K_1(x,t)$ and $K2(x,t)$ are assumed to be in $L^2(R)$ on the interval $0 \le x,t \le 1$ and $y^{(n)}(x)$ is n^{th} derivative of $y(x)$.

The integro-differential equations (IDEs) arise from the mathematical modeling of many scientific phenomena. Nonlinear phenomena that appear in many applications in scientific fields can be modeled by nonlinear integro-differential equations.

In 1992, Liao Rashidi et al. (2009) employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely Homotopy Analysis Method (HAM), Liao (2003), Nadeem et al. (2010), and Rashidi et al. (2009). This method has been successfully applied to solve many types of nonlinear problems Jafari et al. (2008), Liao (1997), and Rashidi et al. (2009). In the present paper we use the Multistage Homotopy analysis method to obtain solutions of nonlinear mixed Volterra-Fredholm integral equations and nonlinear integral equations.

2. Basic Idea of HAM

Consider the following differential equation

$$N[u(r)] = 0, \tag{3}$$

where $\boldsymbol{N}$ is a nonlinear operator, r denotes independent variable, and $u(r)$ is an unknown function. For simplicity, we ignore all boundaries or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao (2003) constructs the so called zero order deformation equation

$$(1-p)L[\phi(r,p)-u_0(r)] = p\hbar H(r)N[\phi(r,p)], \tag{4}$$

where $\boldsymbol{p} \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $H(r) \neq 0$ is an auxiliary function, $\boldsymbol{L}$ is an auxiliary linear operator, $u_0(r)$ is an initial guess of $u(r)$, $u(r,p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\varphi(r;0) = u_0(r), \qquad \varphi(r;1) = u(r). \tag{5}$$

Thus, as p increases from 0 to 1, the solution $u(r;p)$ varies from the initial guesses $u_0(r)$ to the solution $u(r)$. Expanding $u(r,p)$ in Taylor series with respect to p, we have

$$\phi(r;p) = u_0(r) + \sum_{m=1}^{+\infty} u_m(r)\, p^m, \tag{6}$$

where

$$u_m(r) = \frac{1}{m!} \frac{\partial^m \phi(r;p)}{\partial p^m} \Big|_{p=0} . \tag{7}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (4) converges at $p = 1$, then we have

$$u(r) = u_0(r) + \sum_{m=1}^{+\infty} u_m(r), \tag{8}$$

which must be one of solutions of original nonlinear equation, as proved by Liao (2003). As $\hbar = -1$ and $H(r) = 1$, equation (4) becomes

$$(1-p)L[\phi(r,p)-u_0(r)] + pN[\phi(r,p)] = 0, \tag{9}$$

which is used mostly in the homotopy perturbation method He (2006), where as the solution obtained directly, without using Taylor series He (2006). According to the Definition (7), the governing equation can be deduced from the zero-order deformation equation (6). Define the vector

$$\vec{u}_n = \{u_0(r), u_1(r), \ldots, u_n(r)\}.$$

Differentiating equation (4) m times with respect to the embedding parameter p and then setting $p = 0$ and finally, dividing them by $m!$, we obtain the m^{th}-order deformation equation

$$L[u_m(r) - \chi_m u_{m-1}(r)] = \hbar H(r) \Re_m(\vec{u}_{m-1}), \tag{10}$$

where

$$\Re_m(\overrightarrow{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(r,p)]}{\partial\, p^{m-1}}\Big|_{p=0} \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & \boldsymbol{m} \le 1, \\ 1, & \boldsymbol{m} > 1, \end{cases} \tag{12}$$

It should be emphasized that $u_m(r)$ for $m \ge 1$ is governed by the linear equation (10) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Mathematica. For the convergence of the above method we refer the reader to Liao's work Liao (2003). If equation (3) admits unique solution, then this method will produce the unique solution. If equation (3) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3. HAM for Mixed Volterra-Fredholm Integral Equations

To verify the validity and the potential of HAM in solving mixed Volterra-Fredholm integral equations, we apply it to equation (1).

For equation (1), first we chose

$$L[\phi(x,y;p)] = \phi(x,y;p)$$

and

$$H(x,y) = 1.$$

We define a nonlinear operator as follows:

$$N[\phi(x,y;p)] = \phi(x,y;p) - f(x,y) - \int_0^y\int_\Omega G(x,y,s,t,\phi(s,t;p))dsdt \quad . \tag{13}$$

The zero order deformation equation is:

$$(1-p)L[\phi(x,y;p) - u_0(x,y)] = p\hbar N[\phi(x,y;p)]. \tag{14}$$

Differentiating equation (14), m times with respect to the embedding parameter p and then setting p = 0 and finally dividing them by m!, we have the so-called m^{th}-order deformation equation

for $m \ge 1$:

$$u_m(x,y) = \chi_m\, u_{m-1}(x,y) + h\, R_m(\overrightarrow{u}_{m-1}), \tag{15}$$

where

$$R_m(\overrightarrow{u_{m-1}}) = u_{m-1}(x,y) - \int_0^y \int_\Omega \frac{\partial^{m-1} G(x,y,s,t,\phi(s,t;p))}{\partial p^{m-1}} |_{p=0} dsdt - (1-\chi_m) f(x,y) .$$

Therefore, we can obtain HAM terms by equation (15). Now, we have:

$$u(x,y) = u_0(x,y) + u_1(x,y) + u_2(x,y) + \dots .$$

Example 1. Consider the nonlinear mixed integral equation Yousefi et al. (2005):

$$u(x,y) = xy - e^y + y + 1 + \int_0^y \int_0^1 t e^{u(s,t)} dsdt, \quad 0 \le y \le 1, \tag{16}$$

with the exact solution $u(x,y) = xy$.

For start HAM we choose $u_0(x) = 0$, and, then, of two repeat homotopy analysis method we will obtained terms HAM:

$$u_1(x,y) = -\frac{hy^2}{2} - hy - hxy + e^y h - h$$

$$u_2(x,y) = \frac{h^2 y^4}{8} + \frac{h^2 y^3}{2} - \frac{hy^2}{2} - e^y h^2 y - h^2 y - hy - h^2 xy - hxy + 2e^y h^2 - 2h^2 + e^y h - h$$

$$\vdots$$

$$\begin{aligned} u(x,y) &= u_0(x,y) + u_1(x,y) + u_2(x,y) + \dots \\ &= -hy^2 - 2hy - 2hxy + 2e^y h - 2h + h(\frac{hy^4}{8} + \frac{hy^3}{2} \\ &\quad - e^y hy - hy - hxy + 2e^y h - 2h) \end{aligned}$$

The *h*-curves for this example is presented in Figure 1 which were obtained based on the three order HAM approximations solutions. By *h*-curves, it is easy to discover the valid region of *h*, which corresponds to the line segments nearly parallel to the horizontal axis.

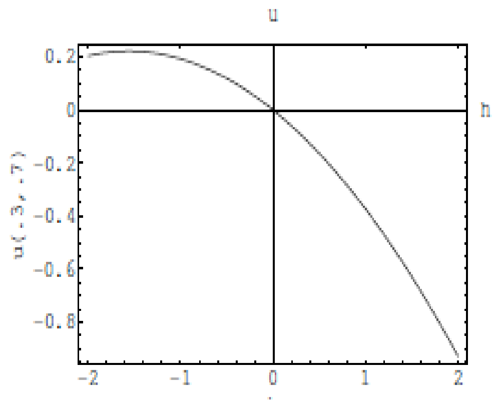

Figure 1. *h*-curve diagram

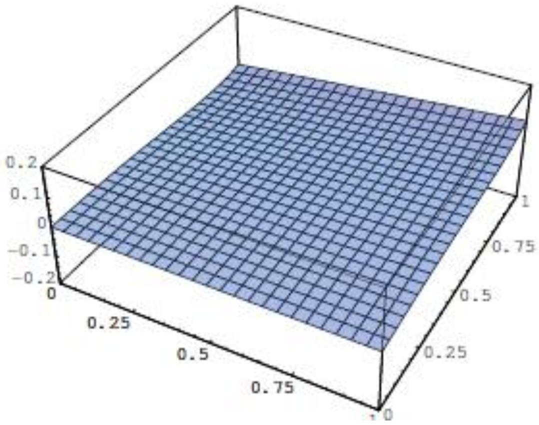

Figure 2. Relative error (computed exact)/exact solution

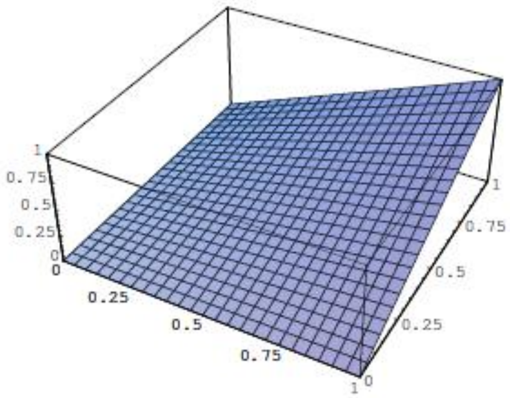

Figure 3. Exact solution

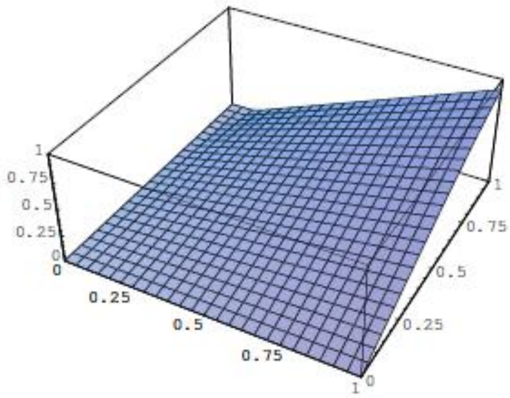

Figure 4. Approximate solutions by HAM (h =-1.3)

The three-dimensional plot of the error HAM is shown in Figure 2.Using the first three terms of the HAM, Figure 4 will be generated. This example is solved by nine terms He's variational iteration method incite Nadeem et al. (2010) and Legendre wavelets method Youseif et al. (2005).

4. The Multistage HAM for Nonlinear Integral Equations

The solution given in HAM is local in nature. To extend this solution over the interval $I=[x_0,T]$ we divide the interval I into sub-intervals $I_j=[x_{j-1},x_j), \quad j=1,2,3,\cdots,p,$, where $0\le x_0<x_1<...\le x_p=T$. We solve the equation (2) in each subinterval I_j. Let $y_1(x)$ be solution of equation (2) in the subinterval I_1. For $2\le i\le p,\ y_i(x)$ is solution of equation (2) in the subinterval I_i with initial conditions by obtaining the initial conditions from the interval I_{i-1}.

$$y_i(x_{i-1})=y_{i-1}(x_{i-1}), \qquad for \quad i=2,...,\ p.$$

The solution of equation (2) for $x \in [x_0, T]$ is given by

$$y(x) = \sum_{i=1}^{p} \chi I_i y_i(x), \tag{17}$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{18}$$

For equation (2), first we choose

$$L[\phi_i(x;p)] = \frac{\partial^n \phi_i(x;p)}{\partial\ x^n}, \qquad n = 0,1,2,...$$

and

$$H_i(x) = 1$$

$$y_{i0}(x\ ) = \sum_{j=0}^{m}\ y\ (\ i-1)_j(x), \qquad for \qquad i = 2,.... .$$

We define a nonlinear operator for mixed Volterra-Fredholm integral equations as follows:

$$\begin{aligned} N[\varphi_i(x;p)] = \varphi_i(x;p) - f(x) - \lambda_1 \int_0^x K_1(x,t) F(\varphi_i(t;p))dt \\ - \lambda_2 \int_0^1 K_2(x,t) G(\varphi_i(t;p))dt \end{aligned} \tag{19}$$

Also for Fredholm integro-differential equation:

$$N[\phi_i(x;p)] = \frac{\partial^n \phi_i(x;p)}{\partial x^n} - f(x) - \lambda_1 \int_{-1}^{1} K_1(x,t) F(\phi_i(t;p))dt \quad . \tag{20}$$

The zero order deformation equation is:

$$(1-p)L[\phi_i(x;p) - u_{i0}(x)] = p\hbar_i N[\phi_i(x;p)] .$$

Differentiating equation (19) and (20), *m* times with respect to the embedding parameter *p* and then setting *p* = 0 and finally dividing them by *m*! and we have the so-called m^{th}-order deformation equation for $m \ge 1$:

$$L[y_{im}(x) - \chi_m y_{im-1}(x)] = h_i R_m(\vec{y}_{im-1}) .$$

Therefore, we can obtain MHAM terms by equation (22) in the interval I_i :

$$y_i(x) = y_{i0}(x) + y_{i1}(x) + y_{i2}(x) + ...$$

Then, of equation (17) we obtained solution of equation (2).

5. Illustrative Examples

To give a clear overview of the multistage homotopy analysis method, we present the following examples. We apply the multistage homotopy analysis method and compare the results with the homotopy analysis method.

Example 2. Consider the following nonlinear Volterra-Fredholm integral equation Yousefi, and Yousefi (2005):

$$y(x) = -\frac{x^6}{30} + \frac{x^4}{3} - x^2 + \frac{5x}{3} - \frac{5}{4} + \int_0^x (x-t)y^2(t)dt + \int_0^1 (x+t)y(t)dt\,, \qquad (23)$$

with the exact solution $y(x) = x^2 - 2$.

For utilize of multi stage HAM we divide interval [0, 1] into subinterval [0, 0.3] and [0.3,1]

In first interval we chose $y_{10}(x) = -\frac{x^6}{30} + \frac{x^4}{3} - x^2 + \frac{5x}{3} - \frac{5}{4}$ and using *h*-curve we find *h* =-1.047 and obtained:

$$y_{11}(x) = -0.375278 - 0.963333\ x + 1.09375\ x^2 - 0.972\ x^3 + ...$$
$$y_{12}(x) = -0.349437 - 0.520494\ x + 0.21923611\ 1111\ x^2 + 0.658950x^3 - 0.85237152\ x^4$$
$$+ 0.6535472\ x^5 - 0.3399\ x^6 + ...$$
$$y_1(x) = y_{10}(x) + y_{11}(x) + y_{12}(x) + y_{13}(x)$$

For the second interval, we choose and use *h*-curve we select *h* = -1.059 and obtain:

$$y(x) = \chi[0,.3]y_1(x) + \chi[.3,1]y_2(x)\,.$$

Yousefi et al. (2005) have solved this example using the He's variational iteration method .We draw below graphs of HAM and MHAM solution and compare the exact solution. Figures (5) and (6) show comparison of exact solutions, HAM and MHAM. In this example we extend the interval [0,1] in Yousefi et al. (2005) to [0,2] and observe that rapid convergence for the solutions is very good.

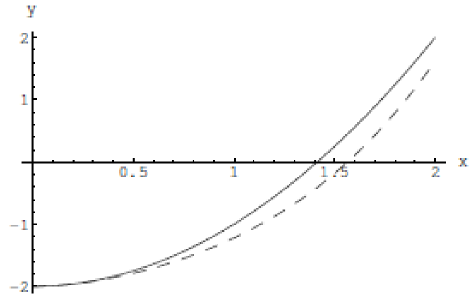

Figure 5. Comparison of exact solution and HAM (dash line) (order 14) by h = -Ĩ11

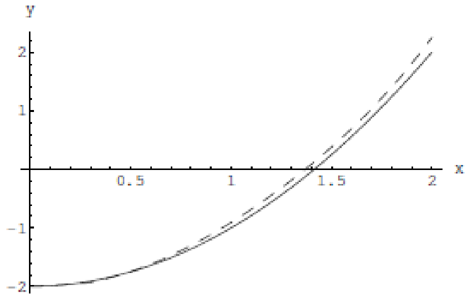

Figure 6. Comparison of exact solution (solid line) and multi-stage HAM (dash line) (order 3)

Example 3. Consider the Fredholm integro-differential problem 16.

$$y'''(x) = -e^x + \int_{-1}^{1} e^{x-2t} y(t)^2 \, dt, \qquad (24)$$

with initial condition:

$$y(0) = y'(0) = 1, \; y''(0) = e.$$

The multistage HAM procedure would lead to:

Divide the interval [0,3] into subintervals [0,1], [1,2] and [2,3]. In the first interval we choose $y_{10}(x) = (-2+e)x^2 + x + 1$ and using h-curve we get h =-0.3 and:

$$y_{11}(x) = h\left(\frac{e^4x^2}{8} - \frac{3e^3x^2}{8} + \frac{3e^2x^2}{8} - \frac{55x^2}{8e} + \frac{99x^2}{8e^2} - \frac{39x^2}{8e^2} + ...\right).$$

Therefore,

$$y_1(x) = y_{10}(x) + y_{11}(x) = 0.47588\; x^2 + 0.5151x - 2.025e^{x-2} + 4.125e^{x-1} - 1.875e^x + .2249\,e^{x+2} - ....$$

which is the solution in the first interval.

For the second interval, we choose $y_{20}(x) = y_1(x)$ and using h-curve, we have h =-0.19. So,

$$y_{21}(x) = h\left(-\frac{4641h}{32e^4} - \frac{111h}{16} - \frac{e^{x+4}}{4} + \frac{39e^{x-2}}{4} - \frac{55e^{x-1}}{4} + \frac{25e^x}{4} - \frac{3e^{x+2}}{4} + ...\right).$$

Therefore,

$$y_2(x) = y_{20}(x) + y_{21}(x) = 0.4553 - 5.338e^{x-6} + 22.229e^{x-5} - 31.151e^{x-4} + 10.766e^{x-3} + 12.404e^{x-2}$$

and in interval [2,3], we chose $y_{30}(x) = y_2(x)$ and by h-curve we chose $h = -0.197$,

$$y_{31}(x) = h(-\frac{25}{4} - \frac{39}{4e^2} + \frac{55}{4e} + \frac{3e^2}{4} - \frac{3e^3}{4} + \frac{e^4}{4} + \frac{39e^{x-2}}{4} - \frac{55e^{x-1}}{4} + ...) .$$

Therefore,

$$y_3(x) = y_{30}(x) + y_{31}(x) = 0.25 - 57.113e^{x-14} + 552.373e^{x-13} - 2285.8891e^{x-12} + 5129.352e^{x-11} - 6219.486e^{x-10}$$

We obtained the solution MHAM:

$$y(x) = \chi_{[0,1]} y_1(x) + \chi_{[1,2]} y_2(x) + \chi_{[2,3]} y_3(x)$$

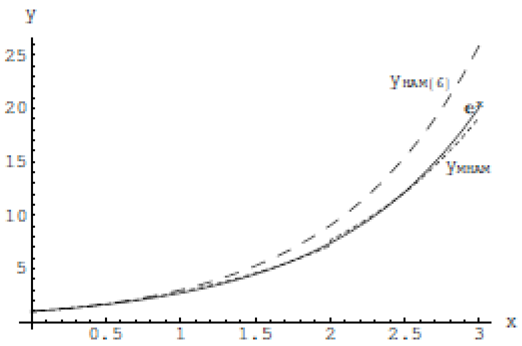

Figure 7. Comparison of MHAM solution, HAM solution (order6) and the exact solution $y(x) = e^x$.

Example 4. Consider the Fredholm integro-differential problem Shidfar et al. (in press)

$$y''(x) = e^x - 2 + \int_{-1}^{1} e^{-4t} y^2(t)(y'(t))^2 dt , \qquad (25)$$

with inital condition: $y(0) = y'(0) = 1$.

To utilize MHAM, we divide the interval [0,1] into subintervals [0,0.1] and [0.1,1] and Figure 8 shows that the convergence is faster than in the series obtained by standard HAM Shidfar et al. (in press).

In the first interval, we choose $y_{10}(x) = x + 1$ and using h-curve choose h =-1.44,

$$y_{11}(x) = -0.206x^2 - 1.3x + 1.3e^x - 1.3.$$

Therefore,

$$y_1(x) = y_{10}(x) + y_{11}(x) = -0.2062\,\mathrm{x}^2 - 0.3x + 1.3\mathrm{e}^x - 0.3$$

and in the interval [.1,1], choose $y_{20}(x) = y_1(x)$ and by *h*-curve we choose *h* =-0.89 and obtain:

$$y_{21}(x) = 0.153\mathrm{x}^2 - 0.089x + 0.09\mathrm{e}^x - 0.09\,.$$

Therefore,

$$y_2(x) = y_{20}(x) + y_{21}(x) = 0.01\mathrm{x}^2 + 0.01x + 0.99\mathrm{e}^x + 0.01$$

$$y(x) = \chi_{[0,1]} y_1(x) + \chi_{[1,3]} y_2(x)\,.$$

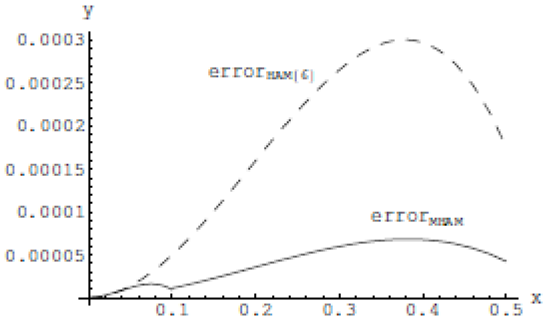

Figure 8. Comparison error of MHAM and HAM (order 6) by *h* = -Õ44

Remark. In the above examples we have solved nonlinear integral equations by using MHAM through few iterations and faster convergence respect to HAM.

6. Conclusion

Homotopy analysis method is a powerful method which yields a convergent series solution for linear/nonlinear problems. This method is better than numerical methods, as it is free from rounding off errors, and does not require large computer power.

In this paper we have suggested a modification of this method which is called 'multistage HAM'. Here, we have applied multistage HAM for solving a nonlinear mixed Volterra-Fredholm integral equations. The multistage HAM yields a series solution which converges faster than the series obtained by HAM and VIM and Legendre wavelet. Illustrative examples presented clear support for this claim.

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