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The He's Variational Iteration Method for Solving the Integro-differential Parabolic Problem with Integral Conditions

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Abstract

In this paper, the variational iteration method is applied for finding the solution of an Integro-differential parabolic problem with integral conditions. Convergence of the proposed method is also discussed. Finally, some numerical examples are given to show the effectiveness of the proposed method.

Keywords: Integro-differential equations, non-local condition, variational iteration method, convergence.

MSC (2010) No.: 35A15; 45K05

1. Introduction

In modeling of many physical systems in various fields of physics, ecology, biology, etc, an integral term over the spatial domain is appeared in some part or in the whole boundary see Bouziani (2002), Carlson (1972), Cushman et al. (1993, 1995), Day (1983), Kavalloris and Tzanetis (2002), Renardy et al. (1987), Samarskii (1980). Such boundary value problems are known as non-local problems. The integral term may appear in the boundary conditions. Non-local conditions appear when values of the function on the boundary are connected to values inside the domain. In recent years, several numerical techniques have been presented to solve various types of non-local boundary value problems see Beilin (2001), Cannon and Lin (1990), and Dehghan et al. (2003, 2006, 2007, and 2009).

In this paper we consider the following parabolic integro-differential equation with integral conditions,

$$\frac{\partial}{\partial t}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)+\gamma u(x,t)=K(u(x,t))+f(x,t),$$

$$(x,t)\in\Omega=(0,l)\times(0,T], \qquad (1)$$

with the initial condition

$$u(x,0)=r(x), \quad 0\le x\le l, \qquad (2)$$

the Neumann condition

$$u_x(0,t)=\alpha(t), \quad 0<t\le T, \qquad (3)$$

and the integral (non-local) condition

$$\int_0^l u(x,t)\,dx=E(t), \quad 0<t\le T, \qquad (4)$$

where f, r, α and E are given functions, γ is a given real value and K is the nonlinear Volterra operator of the form

$$K(u(x,t))=\int_0^t a(t-s)g(s,u(x,s))\,ds.$$

The study of some special types of the problem (1)-(4) is motivated by the works of Merazga and Bouziani (2003, 2005, and 2007). Recently, the existence and uniqueness of the solution of this problem with $\gamma=0$ were discussed in Guezane-Lakouda et al. (2010), and Dabas and Bahuguna (2009).

As we know, the He's variational iteration method (VIM) see He (1997, 1998, 1999, 2000), Mohyud-Din (2009), Abbasbandy (2007) is a powerful device for solving differential equations. This method have been applied successfully to solve many problems of various fields of science and engineering see Tatari and Dehghan (2007) and references therein. In Dehghan and Saadatmandi (2009), authors applied the VIM to solve wave equation with non-local condition. Recently, Salkuyeh and Roohani in Salkuyeh and Roohani (2010) used the VIM to solve telegraph equation with boundary integral condition. In this paper, we use the VIM to solve problem (1)-(4) and our emphasis is on verifying the convergence of the proposed method.

2. A Brief Description of the Variational Iteration Method

Consider the following differential equation

$$Lu(t)+Nu(t)=g(t),$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is an inhomogeneous term. In the variational iteration method, a correctional functional as

$$u_{m+1}(t)=u_m(t)+\int_0^t \lambda(Lu_m(s)+N\tilde{u}_m(s)-g(s))\,ds, \qquad m=0,1,2,\ldots,$$

is made, where λ is a general Lagrangian multiplier see Inokuti et al.(1978) which can be identified optimally via the variational theory. Obviously the successive approximations $u_j,\ j=0,1,\ldots,$ can be computed by determining λ. Here, the function $\tilde{u}_m$ is a restricted variation which means $\delta\tilde{u}_m=0$.

3. Assumptions and Reformulation of the Problem

In this section we firstly, give some basic definitions and assumptions. Throughout this paper, we let $L^2(\Omega)$ be the space of square-integrable real functions defined from Ω into R with the corresponding norm.

$$\| u\|^2=\int_{\overline{\Omega}}| u |^2 d\Omega, \qquad u\in L^2(\Omega).$$

And also for analysis, the problem (1)-(4) we assume the following conditions:

(C1) We assume that $a(t)$ is a real-valued functions defined on $[0,T]$ and $a(t)\in L^2(0,T)$.

(C2) Let $f(x,t)$ is sufficiently smooth to produce a smooth classical solution u .

(C3) We mention that the function $r(x)$ satisfy the following compatibility conditions Guezane-Lakouda et al. (2010)

$$r'(0)=\alpha(0), \qquad \int_0^l r(x)\,dx=E(0).$$

(C4) $\alpha(t)\in L^2(0,T)$ and also $E(t)\in L^2(0,T)$.

(C5) Finally, we assume that $g(t,u(x,t))$ satisfy a Lipschitz condition uniformly with respect to its second argument:

$$\| g(t,u)-g(t,v) \|_2 \le L\| u-v \|_2, \quad \forall (t,u),(t,v)\in ((0,T)\times L^2(\Omega)),$$

where L is a constant independent of t.

For the sake of simplicity, we transform problem (1)-(4) with inhomogeneous conditions (3) and (4) to an equivalent one with homogenous conditions. To do so, we use the transformation of Dehghan and Saadatmandi (2009)

$$v(x,t)=u(x,t)-z(x,t), \qquad (x,t)\in\Omega=(0,l)\times(0,T],$$

where

$$z(x,t)=\alpha(t)(x-\frac{l}{2})+\frac{E(t)}{l}.$$

In this case, by a simple manipulation, the problem is transformed to

$$\frac{\partial}{\partial t}v(x,t)-\frac{\partial^2}{\partial x^2}v(x,t)+\gamma\, v(x,t)=\bar{F}(x,t), \qquad (x,t)\in\Omega=(0,l)\times(0,T], \tag{5}$$

with the initial condition

$$v(x,0)=\bar{r}(x), \quad 0\le x\le l, \tag{6}$$

the Neumann condition

$$v_x(0,t)=0, \quad 0<t\le T. \tag{7}$$

And the integral (non-local) condition

$$\int_0^l v(x,t)\,dx=0, \quad 0<t\le T, \tag{8}$$

where

$$\bar{F}(x,t)=K((v+z)(x,t))+f(x,t)-\frac{\partial}{\partial t}z(x,t)-\gamma\, z(x,t),$$

$$\bar{r}(x)=r(x)-z(x,0).$$

As we observe, the Neumann and integral conditions are now homogeneous. Hence, instead of looking for $u(x,t)$ we simply look for $v(x,t)$, after computing $v(x,t)$, the solution of problem (1)-(4) will be directly obtained by the relation $u(x,t)=v(x,t)+z(x,t)$.

4. Convergence of the VIM for the Equation

In this section, the application of the VIM is discussed for solving problem (5)-(8). According to the VIM, we consider the correction functional in t direction for equation (5) in the following form:

$$v_{m+1}(x,t)=v_m(x,t)+\int_0^t \lambda(s)(\frac{\partial}{\partial s}v_m(x,s)+\gamma v_m(x,s)-\tilde{F}(v_m(x,s)))\,ds,$$

where

$$\tilde{F}(v_m(x,s))=\frac{\partial^2}{\partial x^2}v_m(x,s)+f(x,s)-\frac{\partial}{\partial s}z(x,s)-\gamma z(x,s)$$
$$+\int_0^s a(s-\xi)g(\xi,[v_m(x,\xi)+z(x,\xi)])\,d\xi,$$

so, λ(s) being the Lagrange multipliers and $\tilde{F}[v_m(x,s)]$ being the restricted variation, i.e., $\delta\tilde{F}[v_m(x,s)]=0$. The variation of above equation is then

$$\delta v_{m+1}(x,t)=\delta v_m(x,t)+\delta\int_0^t \lambda(s)(\frac{\partial}{\partial s}v_m(x,s)+\gamma v_m(x,s)-\tilde{F}(v_m(x,s)))\,ds.$$

By using integration by parts and constructing the correction functional

$$\begin{aligned}\delta v_{m+1}(x,t)&=\delta v_m(x,t)+\lambda(s)\delta v_m(x,s)\big|_{s=t}\\&\quad-\delta\int_0^t(\lambda'(s)v_m(x,s)-\lambda(s)\gamma v_m(x,s)+\lambda(s)\tilde{F}[v_m(x,s)])ds\\&=(1+\lambda(s)\big|_{s=t})\delta v_m(x,t)-\delta\int_0^t(\lambda'(s)-\lambda(s)\gamma)v_m(x,s)\\&\quad+\lambda(s)\tilde{F}[v_m(x,s)]ds\end{aligned}$$

the stationary conditions would be as follows

$$1+\lambda(s)\big|_{s=t}=0,$$
$$\lambda'(s)-\gamma\lambda(s)=0.$$

Thus, we have $\lambda(s) = -e^{\gamma(s-t)}$ and the following iteration formula for computing $v_m(x,t)$ may be obtained

$$v_{m+1}(x,t) = v_m(x,t) - \int_0^t e^{\gamma(s-t)} \left(\frac{\partial}{\partial s} v_m(x,s) + \gamma\, v_m(x,s) - \tilde{F}(v_m(x,s))\right) ds. \tag{9}$$

Now, we show that the sequence $v_m(x,t)$ defined by (9) with suitable initial approximation converges to the solution of (5). To do this, we state and prove the following theorem.

Theorem 1.

Let $\overline{\Omega} = [0,l] \times [0,T]$ and $v(x,t) \in C^2(\overline{\Omega})$ be the exact solution of (5) and $v_m(x,t) \in C^2(\overline{\Omega})$ be the obtained solutions of the sequence defined by (9) with $v_0(x,t) = \bar{r}(x).$ If $E_m(x,t) = v_m(x,t) - v(x,t)$ and $\| \frac{\partial^2}{\partial x^2} E_m(x,t) \|_2 \le \| E_m(x,t) \|_2$, then the functional sequence defined by (9) converges to $v(x,t)$.

Proof: We first mention that the initial approximation $v_0(x,t)$ satisfies equations (6)-(8). Since *v(x,t)* is the exact solution of (9), it is obvious that

$$v(x,t) = v(x,t) - \int_0^t e^{\gamma(s-t)} \left(\frac{\partial}{\partial s} v(x,s) + \gamma\, v(x,s) - \tilde{F}(v(x,s))\right) ds. \tag{10}$$

Now from (9), (10) and after some simplifications, we get

$$\begin{aligned} E_{m+1}(x,t) = E_m(x,t) - \int_0^t e^{\gamma(s-t)} \Big(&\frac{\partial}{\partial s} E_m(x,s) + \gamma\, E_m(x,s) - \frac{\partial^2}{\partial x^2} E_m(x,s) \\ &- K[v(x,s) + z(x,s)] + K[v_m(x,s) + z(x,s)]\Big)\, ds. \end{aligned}$$

By using integration by parts, we conclude that

$$\begin{aligned} E_{m+1}(x,t) = E_m(x,t) - \Big[e^{\gamma(s-t)} E_m(x,s)\Big|_0^t + \int_0^t e^{\gamma(s-t)} \Big(&-\frac{\partial^2}{\partial x^2} E_m(x,s) \\ &- K[v(x,s) + z(x,s)] + K[v_m(x,s) + z(x,s)]\Big)\, ds\Big]. \end{aligned}$$

Obviously $E_m(x,0) = 0,\ m = 0,1,\ldots$. Hence,

$$E_{m+1}(x,t)=\int_0^t e^{\gamma(s-t)}\Big(\frac{\partial^2}{\partial x^2}E_m(x,s)\ +K[v(x,s)+z(x,s)]-K[v_m(x,s)+z(x,s)]\Big)\,ds.$$

Taking 2-norm of both sides of the latter equation gives

$$\|\ E_{m+1}(x,t)\|_2\le\int_0^t\|\ e^{\gamma(s-t)}\|_2\Big(\|\ \frac{\partial^2}{\partial x^2}E_m(x,s)\|_2 \\ +\|\ K[v(x,s)+z(x,s)]-K[v_m(x,s)+z(x,s)]\|_2\Big)\,ds.$$

Now from the assumption $\|\ \frac{\partial^2}{\partial x^2}E_m(x,t)\|_2\le\|\ E_m(x,t)\|_2$, we obtain

$$\|E_{m+1}(x,t)\|_2\le\int_0^t\|e^{\gamma(s-t)}\|_2[\|E_m(x,s)\|_2+\|K[v(x,s)+z(x,s)]-K[v_m(x,s)+z(x,s)]\|_2]ds.$$

It is easy to see that from $s\le t\le T$, we obtain

$$\|\ e^{\gamma(s-t)}\|_2\le e^{\|\ \gamma(s-t)\|_2}=e^{|\ \gamma|\ \|\ s-t\|_2}\le e^{2|\ \gamma|\ t}\le e^{2|\ \gamma|\ T}$$

and, also from assumption we have:

$$\|K[v(x,s)+z(x,s)]-K[v_m(x,s)+z(x,s)]\|_2\le\int_0^s\|a(s-\xi)\|_2\|g(\xi,v+z)-g(\xi,v_m+z)\|_2\,d\xi \\ \le\int_0^s\|a(s-\xi)\|_2\,L\|E_m(x,\xi)\|_2\,d\xi.$$

Therefore, it follows from two above relations that

$$\|\ E_{m+1}(x,t)\|_2\le e^{2|\ \gamma|\ T}\int_0^t[\|\ E_m(x,s)\|_2+LT\max_{t\in(0,T)}(\|\ a(t)\|_2)\max_{(x,\xi)\in[0,l]\times[0,s]}(\|\ E_m(x,\xi)\|_2]ds.$$

So, we have:

$$\|\ E_{m+1}(x,t)\|_2\le M_1\int_0^t\|\ E_m(x,s)\|_2\,ds+M_2\int_0^t\max_{(x,\xi)\in[0,l]\times[0,s]}\|\ E_m(x,\xi)\|_2\,ds,$$

where $M_1=e^{2|\ \gamma|\ T}$ and $M_2=(LT\ \max_{t\in(0,T)}(\|\ a(t)\|_2))e^{2|\ \gamma|\ T}$. Also, we assume

$M = M_1 + M_2$. Now, we proceed as following

$$\begin{aligned}
\| E_1(x,t)\|_2 &\le M_1\int_0^t \| E_0(x,s)\|_2\, ds + M_2 \int_0^t \max_{(x,\xi)\in[0,l]\times[0,s]} \| E_0(x,\xi)\|_2\, ds \\
&\le M_1 \max_{(x,s)\in\overline{\Omega}} \| E_0(x,s)\|_2 \int_0^t ds + M_2 \max_{(x,s)\in\overline{\Omega}} \| E_0(x,s)\|_2 \int_0^t ds \\
&= M \max_{(x,s)\in\overline{\Omega}} \| E_0(x,s)\|_2\, t,
\end{aligned}$$

$$\begin{aligned}
\| E_2(x,t)\|_2 &\le M_1\int_0^t \| E_1(x,s)\|_2\, ds + M_2 \int_0^t \max_{(x,\xi)\in[0,l]\times[0,s]} \| E_1(x,\xi)\|_2\, ds \\
&\le M_1 M\int_0^t \max_{(x,\bar{s})\in\overline{\Omega}} \| E_0(x,\bar{s})\|_2\, s\, ds + M_2 \int_0^t \max_{(x,\xi)\in[0,l]\times[0,s]} (M \max_{(x,s)\in\overline{\Omega}} \| E_0(x,s)\|_2\, \xi)\, ds \\
&= M^2 \max_{(x,s)\in\overline{\Omega}} \| E_0(x,s)\|_2 \frac{t^2}{2!}, \\
&\vdots \\
\| E_m(x,t)\|_2 &\le M_1\int_0^t \| E_{m-1}(x,s)\|_2\, ds + M_2 \int_0^t \max_{(x,\xi)\in[0,l]\times[0,s]} \| E_{m-1}(x,\xi)\|_2\, ds \\
&\le M_1 M^{m-1}\int_0^t \max_{(x,\bar{s})\in\overline{\Omega}} \| E_0(x,\bar{s})\|_2 \frac{s^{m-1}}{(m-1)!} ds \\
&\quad + M_2 \int_0^t \max_{(x,\xi)\in[0,l]\times[0,s]} (M^{m-1} \max_{(x,s)\in\overline{\Omega}} \| E_0(x,s)\|_2 \frac{\xi^{m-1}}{(m-1)!})\, ds \\
&= \max_{(x,s)\in\overline{\Omega}} \| E_0(x,s)\|_2 \frac{(M\,t)^m}{m!}.
\end{aligned}$$

Now, we have

$$\max_{(x,s)\in\overline{\Omega}} \| E_0(x,s)\|_2 \frac{(M\,t)^m}{m!} \le \max_{(x,s)\in\overline{\Omega}} \| E_0(x,s)\|_2 \frac{(M\,T)^m}{m!} \to 0$$

as $m \to \infty$, and this completes the proof. □

5. Numerical Examples

In this section, we present some examples to show the efficiency of the proposed method for solving problem (1)-(4). All of the computations have done by the Maple software.

Example 1. For the first example we consider

$$\frac{\partial}{\partial t}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)-2\ u(x,t)=\int_0^t(t-s)\ |u(x,s)-3|\ ds+f(x,t),$$

where $(x,t)\in\Omega=(0,1)\times(0,1)$, and

$$f(x,t)=e^t\cos(\pi x)\pi^2-t\cos(\pi x)-x-t\,x-\cos(\pi x)-\frac{3}{2}t^2,$$
$$r(x)=\cos(\pi x)+x,\qquad 0\le x\le 1,$$
$$\alpha(t)=e^t,\quad E(t)=\frac{1}{2}e^t,\qquad 0<t\le 1.$$

For this problem, we obtain

$$z(x,t)=x e^t,$$
$$\bar{r}(x)=\cos(\pi x).$$

Proceeding as before, we can select $v_0(x,t)=\bar{r}(x)$. Using this selection into (9) after some simplifications and by using the Taylor expansion we obtain the following successive approximations.

$$v_1(x,t)=\cos(\pi x)(1+t+O(t^2)),$$
$$v_2(x,t)=\cos(\pi x)(1+t+\frac{t^2}{2!}+O(t^3)),$$
$$v_3(x,t)=\cos(\pi x)(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+O(t^4)).$$

Computing the other terms, for n > 0 we have

$$v_n(x,t)=\cos(\pi x)(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots+\frac{t^n}{n!}+O(t^{n+1})).$$

Thus, we get

$$v(x,t)=\lim_{n\to\infty}v_n(x,t)=\cos(\pi x)e^t.$$

Now we have $u(x,t)=v(x,t)+z(x,t)=e^t(\cos(\pi x)+x)$ which is the exact solution of the problem.

Example 2. For the second example we consider

$$\frac{\partial}{\partial t}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)-\pi^2 u(x,t)=\int_0^t (t-s)\ \left|u(x,s)-4\right|\ ds+f(x,t),$$

where $(x,t)\in\Omega=(0,1)\times(0,1)$, and

$$f(x,t)=t\sin(\pi x)-\sin(\pi x)-2t^2,$$
$$r(x)=\sin(\pi x),\qquad 0\le x\le 1,$$
$$\alpha(t)=\pi e^{-t},\quad E(t)=\frac{2}{\pi}e^{-t},\qquad 0<t\le 1.$$

For this problem, we obtain

$$z(x,t)=e^t(\pi(x-\frac{1}{2})+\frac{2}{\pi}),$$
$$\bar{r}(x)=\sin(\pi x)-\pi(x-\frac{1}{2})-\frac{2}{\pi}.$$

Proceeding as before, we can select $v_0(x,t)=\bar{r}(x)$. Using this selection into (9) after some simplifications and by using the Taylor expansion we obtain the following successive approximations.

$$v_1(x,t)=(\sin(\pi x)-\pi(x-\frac{1}{2})-\frac{2}{\pi})(1-t+O(t^2)),$$
$$v_2(x,t)=(\sin(\pi x)-\pi(x-\frac{1}{2})-\frac{2}{\pi})(1-t+\frac{t^2}{2!}+O(t^3)),$$
$$v_3(x,t)=(\sin(\pi x)-\pi(x-\frac{1}{2})-\frac{2}{\pi})(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+O(t^2)).$$

Computing the other terms, for *n > 0* we have

$$v_n(x,t)=(\sin(\pi x)-\pi(x-\frac{1}{2})-\frac{2}{\pi})(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots+\frac{(-1)^n t^n}{n!}+O(t^{n+1})).$$

Thus, we get

$$v(x,t)=\lim_{n\to\infty}v_n(x,t)=(\sin(\pi x)-\pi(x-\frac{1}{2})-\frac{2}{\pi})e^{-t}.$$

Now, we have $u(x,t)=v(x,t)+z(x,t)=e^{-t}\sin(\pi x)$ which is tha exact solution of the problem.

6. Conclusions

In this paper, we applied the well-know He's variational iteration method for solve the integro-differential parabolic problem with an integral condition. We also shown that under some conditions the VIM is convergent for this problem. Numerical results presented in this paper show that the proposed method is very effective.

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