Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 10, Issue 1 (June 2015), pp. 571 - 587

Optimal Solution of a Fully Fuzzy Linear Fractional Programming Problem by Using Graded Mean Integration Representation Method

Moumita Deb and P. K. De

Department of Mathematics
National Institute of Technology, Silchar
Silchar-788010, Assam, India
moumitad75@gmail.com; pijusde@gmail.com

Received: September 4, 2014; Accepted: April 7, 2015

Abstract

In the present paper, the study of fully fuzzy linear fractional programming problem (FFLFPP) using graded mean integration representation method is discussed where all the parameters and variables are characterized by trapezoidal fuzzy numbers. A computational algorithm has been presented to obtain an optimal solution by applying simplex method. To demonstrate the applicability of the proposed approach, one numerical example is solved. Also to check the efficiency and feasibility of the proposed approach, we compare the results of examples by applying crisp numbers, triangular fuzzy numbers and trapezoidal fuzzy numbers.

Keywords: Fully fuzzy linear fractional programming problem; Trapezoidal fuzzy number; Graded mean integration representation

MSC 2010 No.: 90C32, 90C70, 97N60

1. Introduction

Fractional programming (FP) was studied extensively in the middle of 1960s and early 1970s of the last century in the literature (1962, 1970). In many practical applications like stock problems, ore blending problems, shipping schedules problems, optimal policy for a Markovian chains, sensitivity of linear programming problem (LPP), optimization of ratios of criterion gives more insight into the situations than the optimization of each criterion (1988).

To study the efficiency in different fields such as education, hospital administration, court systems, air force maintenance units, bank branches, etc., fractional programming solves more efficiently the above type of problems (1978). FP problem, which has been used as an important planning tool for the past decades, is applied to different disciplines such as engineering, business, finance, economics, etc. FP is generally used for modelling real life problems with one or more objective(s) such as profit/cost, inventory/sales, actual cost/standard cost, output/employee, etc.

Charnes and Cooper (1962) showed that the linear fractional programming problem (LFPP) can be optimized by solving a LPP. Bellman and Zadeh (1970) has proposed decision making in a fuzzy environment. S. H. Chen (1985) has discussed operations of fuzzy numbers with function principle. Chen and Hsieh (1999) has studied the graded mean integration representation (GMIR) of generalized fuzzy numbers while Chen et al. (2006) has used some concepts of GMIR of L-R type fuzzy numbers. Nachammi et al. (2012) have studied fuzzy linear fractional programming problem (FLFPP) by metric distance ranking. The present authors (2013) have discussed FLFPP by signed distance ranking where variables and parameters are characterized by triangular fuzzy numbers (TFN).

The objective of this paper is to deal with a kind of FLFPP where all the variables and parameters are trapezoidal fuzzy numbers (TrFN). In this paper an attempt has been made to develop an algorithm for solving FLFPP based on GMIR of fuzzy numbers.

The paper is organized as follows: Section 2 outlines the definitions and preliminaries of TrFN and its arithmetic operations. In section 3 we have discussed the methodology of GMIR of TrFN based on fuzzy set theory. In section 4, we first recall some basic notions about FLPP and FLFPP in which a theorem based on conditions of fuzzy optimality is presented.

An algorithm is developed which gives the computational procedure for optimal solution of FLFPP. Section 5 demonstrates two numerical examples to illustrate the above algorithm. Section 6 discusses the main results and conclusions of this paper.

2. Definitions and Preliminaries

We review the fundamental notation of fuzzy set theory initiated by Bellman and Zadeh (1970). Below, we give definitions and notations taken from Zimmerman (1983).

Definition 2.1.

If X is a collection of objects denoted generally by x, then a fuzzy set $\tilde{A}$ in X is defined as a set of ordered pairs:

$$\tilde{A}=\{(x,\mu_{\tilde{A}}(x)): x \in X\},$$

where $\mu_{\tilde{A}}(x)$ is called the membership function of x in $\tilde{A}$. The membership function maps each element of X to a membership grade (or membership value) between 0 and 1.

Definition 2.2.

A fuzzy number $\tilde{A}$ is a convex normalized fuzzy set on the real line **R** such that:

(i) it exists at least one x_0 in **R** with $\mu_{\tilde{A}}(x_0)$=1, and

(ii) $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Definition 2.3. Trapezoidal Fuzzy Number

A fuzzy number $\tilde{A}$= (a, b, c, d), $a \le b \le c \le d$ (a, b, c, d > 0) is said to be a trapezoidal fuzzy number if its membership function is given by:

$$\mu_{\tilde{A}}(x)=\begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a < x \le b, \\ 1, & b < x < c, \\ \frac{d-x}{d-c}, & c \le x < d, \\ 0, & x > d. \end{cases}$$

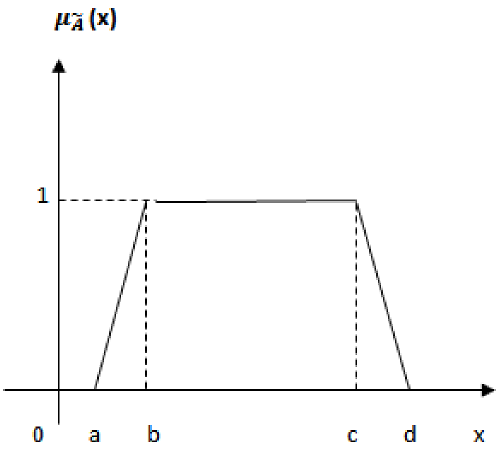

Figure1: Trapezoidal Fuzzy Number $\tilde{A} = (a, b, c, d)$

Remark:

If b = c, the trapezoidal fuzzy number reduces to a triangular fuzzy number as shown in Figure 2. It can be written by a triplet (a, b, d).

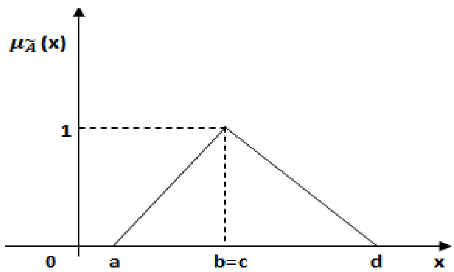

Figure2: Triangular Fuzzy Number $\tilde{A}$=(a,b,d)

3. Methodology

3.1. Graded Mean Integration Representation Method

Chen and Hsieh (1999) introduced graded mean integration representation method based on the integral value of graded mean α-level of generalized fuzzy number.
First, we describe generalized fuzzy number as follows:

Suppose, $\tilde{A} = (a_1, a_2, a_3, a_4; w_A)$ is a generalized fuzzy number. It is described as any fuzzy subset of the real line **R**, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

(a) $\mu_{\tilde{A}}(x)$ is a continuous mapping from **R** to the closed interval [0,1],

(b) $\mu_{\tilde{A}}(x) = 0, \quad -\infty < x \le a_1,$

(c) $\mu_{\tilde{A}}(x) = L(x)$, is strictly increasing on $[a_1, a_2]$,

(d) $\mu_{\tilde{A}}(x) = w_A, \;\; a_2 \le x \le a_3,$

(e) $\mu_{\tilde{A}}(x) = R(x)$, is strictly decreasing on $[a_3, a_4]$,

(f) $\mu_{\tilde{A}}(x)$=0, $a_4 \le x < \infty$,

where $0 < w_A \le 1$ and a_1, a_2, a_3 and a_4 are real number. Also this type of generalized fuzzy number be denoted as $\tilde{A} = (a_1, a_2, a_3, a_4; w_A)_{LR}$.

When w_A=1, it can be written as $\tilde{A} = (a_1, a_2, a_3, a_4)_{LR}$.

Secondly, by graded mean integration representation method L^{-1} and $R^{-1}$are the inverse functions of L and R, respectively and the graded mean α-level value of generalized fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4; w_A)_{LR}$:

$$\frac{\alpha\left(L^{-1}(\alpha)+R^{-1}(\alpha)\right)}{2}.$$

Then the graded mean integration representation of $\tilde{A}$ is $P(\tilde{A})$ with grade w_A , where

$$P(\tilde{A}) = \frac{\int_0^{w_A} \frac{\alpha(L^{-1}(\alpha)+R^{-1}(\alpha))}{2} d\alpha}{\int_0^{w_A} \alpha \, d\alpha} \quad \text{with } 0 < \alpha \le w_A \text{ and } 0 < w_A \le 1.$$

When $w_A = 1$, the above generalized fuzzy number $\tilde{A}$ is considered as trapezoidal fuzzy number and is written as $\tilde{A} = (a_1, a_2, a_3, a_4)_{LR}$.

Therefore,

$$P(\tilde{A}) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6}.$$

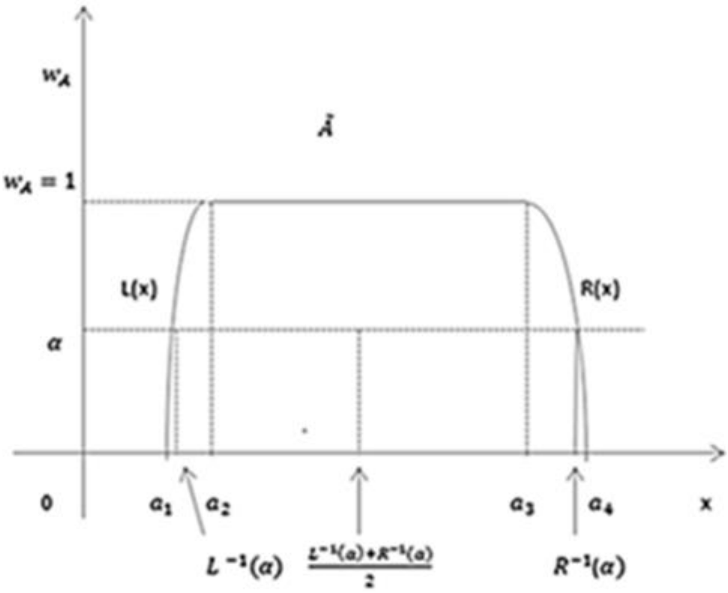

Figure 3. The graded mean $\alpha - level$ value of generalized fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4; w_A)_{LR}$

3.2. The fuzzy mathematical operations under function principle

Function principle is proposed by Chen (1985) to be as the fuzzy arithmetical operations by trapezoidal fuzzy numbers. This method is more useful than the extension principle (1999, 2006) for the fuzzy numbers with the trapezoidal membership functions.

We describe some fuzzy arithmetical operations under function principle as follows:

Suppose, $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are two trapezoidal fuzzy numbers. Then,

1. $\tilde{A} \oplus \tilde{B} = \left(a_1 + b_1, a_2 + b_{2,} a_3 + b_3, a_4 + b_4\right)$.

2. $\tilde{A} \otimes \tilde{B} = \left(a_1 b_1, a_2 b_{2,} a_3 b_3, a_4 b_4\right)$.

3. $-\tilde{B} = -(b_1, b_2, b_3, b_4) = (-b_4, -b_3, -b_2, -b_1)$.

4. $\tilde{A} \ominus \tilde{B} = (a_1 - b_4, a_2 - b_3, a_4 - b_2, a_4 - b_1)$.

5. $\frac{1}{\tilde{B}} = \widetilde{B^{-1}} = \left(\frac{1}{b_4}, \frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1}\right)$.

6. $\tilde{A} \oslash \tilde{B} == (\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1})$.

7. Let, $\lambda \in R$,

$$\text{For} > 0, \quad \lambda \otimes \tilde{A} = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4),$$

$$\text{For} < 0, \quad \lambda \otimes \tilde{A} = (\lambda a_4, \lambda a_3, \lambda a_2, \lambda a_1),$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ *and* b_4 are all non-zero positive real numbers.

3.3. Signed distance ranking of Triangular fuzzy number

Yao and Wu (2000) proposed a signed distance ranking to rank fuzzy numbers.

Let, $\tilde{A}=(a_1, a_2, a_3)$ be a triangular fuzzy number of L-R type then

$$R(\tilde{A}) = \frac{a_1+2a_2+a_3}{4}.$$

4. Fuzzy linear fractional programming problem

4.1. Definition

Fuzzy linear fractional programming: The general format of fuzzy linear fractional programming problem (FLFPP) may be written as:

$$\text{Max } \tilde{Z}(x) = \frac{\tilde{c}x+\tilde{\alpha}}{\tilde{d}x+\tilde{\beta}},$$

subject to the constraints:

$$\tilde{A}x = \tilde{b},\ x \geq 0,$$

where $x \in R^n$, $\tilde{c}, \tilde{d} \in R^n, \tilde{A} \in R^{m\times n}, \tilde{b} \in R^m, \tilde{\alpha}, \tilde{\beta} \in R$.

With the additional assumption that the denominator is positive for all feasible solution.

4.2. Definition

Initial basic feasible solution:

Let $x_{\tilde{B}}$ be the initial basic feasible solution such that

$$\tilde{B}x_{\tilde{B}} = \tilde{b} \Rightarrow x_{\tilde{B}} = \tilde{B}^{-1}\tilde{b}, \qquad x_{\tilde{B}} \geq 0,$$

$$\text{where } \tilde{B} = (\widetilde{b_1}, \widetilde{b_2}, \dots, \widetilde{b_m}).$$

Further let $\tilde{z}^1 = \tilde{c}_B x_{\tilde{B}} + \tilde{\alpha}$ and $\tilde{z}^2 = \tilde{d}_B x_{\tilde{B}} + \tilde{\beta}$,

where $\tilde{c}_B$ and $\tilde{d}_B$ are the vectors having their components as the coefficients associated with the basic variables in numerator and denominator of the objective function respectively.

4.1. Theorem (Conditions of optimality). [Swarup et al. (1997) and De and Deb (2013)]

A sufficient condition for a feasible solution to a fully fuzzy linear fractional programming problem (FFLFPP) to be fuzzy optimum is that $\tilde{u}_{ij} \geq \tilde{0}$ for all j which the column vector $\tilde{a}_j$ in $\tilde{A}$ is not in the basis $\tilde{B}$.

Proof:

Let the fully fuzzy linear fractional programming be stated as follows-

$$\text{Max } \tilde{Z}(x) = \frac{\tilde{c}x+\tilde{\alpha}}{\tilde{d}x+\tilde{\beta}},$$

subject to the constraints:

$$\tilde{A}x = \tilde{b} \text{ and } \tilde{x} \geq \tilde{0},$$

where x$\in R^n$, $\tilde{c}, \tilde{d} \in R^n, \tilde{A} \in R^{m\times n}$, $\tilde{b} \in R^m$, $\tilde{\alpha}, \tilde{\beta} \in R$.

Here we show that for any $\tilde{a}_j$ and $\tilde{A}$ not in $\tilde{B}$ at least one $\tilde{u}_{ij} > \tilde{0}$. If possible let all $\tilde{u}_{ij} \leq \tilde{0}$, ($i$=1,2, ..., m). We have the fuzzy basic feasible solution to this FLFPP is

$$\sum_{i=1}^{m} \tilde{x}_{Bj}\tilde{b}_i = \tilde{b}\ . \qquad (1)$$

Suppose that we add and subtract $\bar{\theta}\tilde{a}_j$ ($\bar{\theta}$ is any scalar), then we get

$$\sum_{i=1}^{m} \tilde{x}_{Bi}\tilde{b}_i - \bar{\theta}\tilde{a}_j + \bar{\theta}\tilde{a}_j = \tilde{b}\ . \qquad (2)$$

Since

$$-\bar{\theta}\tilde{a}_j = -\theta\sum_{i=1}^{m}\tilde{u}_{ij}\tilde{b}_i\ , \qquad (3)$$

we have ,

$$\sum_{i=1}^{m}\tilde{x}_{Bi}\tilde{b}_i - \bar{\theta}\sum_{i=1}^{m}\tilde{u}_{ij}\tilde{b}_i + \tilde{\bar{\theta}}\tilde{a}_j = \tilde{b}\ ,$$

$$\sum_{i=1}^{m}(\tilde{x}_{Bi} - \bar{\theta}\tilde{u}_{ij})\tilde{b}_i + \bar{\theta}\tilde{a}_j = \tilde{b}\ .$$

When $\bar{\theta} > \tilde{0}$, we have $\tilde{x}_{Bi} - \bar{\theta}\tilde{u}_{ij} \geq \tilde{0}$. Since by assumption, $\tilde{u}_{ij} \leq \tilde{0}$, (i =1,2, ..., m). Therefore, $(\tilde{x}_{Bi} - \bar{\theta}\tilde{u}_{ij}), \cdots, (\tilde{x}_{Bm} - \bar{\theta}\tilde{u}_{mj})$ and $\bar{\theta}$ is a fuzzy feasible solution for all $\bar{\theta} > \tilde{0}$. Thus, the set S is unbounded, which is contrary to our hypothesis of regularity. Hence, $\tilde{u}_{ij} > \tilde{0}$.

4.6. Definition

A fuzzy basic feasible solution $\tilde{x}_B$ to fuzzy linear fractional programming problem is called fuzzy unbounded solution if for at least one j, for which $\tilde{u}_{ij} \leq 0$ (i =1, 2,..., m), $\tilde{z}_j - \tilde{c}_j$ and $\tilde{z}_j - \tilde{d}_j$ are negative, then there does not exist any optimum solution to this FLFPP.

Algorithm for solving fully fuzzy linear fractional programming problem by simplex method:

The steps for the computation of a fuzzy optimum solution are as follows:

Step 1:

First consider the variables and parameters of the example as fuzzy number like trapezoidal fuzzy number.

Step 2:

Check whether the objective function of the given fully fuzzy linear fractional programming problem (FLFPP) is to be maximized or minimized. If it minimized then we convert it into a problem of maximizing it by using the result,

$$\text{Min } \tilde{z} = -\text{ Max }(-\tilde{z}).$$

Step 3:

Check whether all $\tilde{b}_i$ (i = 1, 2, ..., m) are non-negative . If any one of $\tilde{b}_i$ is negative then multiply the corresponding in equation of the constraints by -1, so as to get all $\tilde{b}_i$ (i =1, 2, ..., m) non-negative.

Step 4:

Convert all the in equations of the constraints into equations by introducing slack and /or surplus variables in the constraints. The costs of these variables equal to zero.

Step 5:

Obtain an initial basic feasible solution to the given problem in the form $\tilde{x}_B = \tilde{B}^{-1}\tilde{b}$ and put it in the third column of the simplex table.

Step 6:

Compute $\tilde{z} = \dfrac{\tilde{z}^1}{\tilde{z}^2}$, where $\tilde{z}^1 = \tilde{c}_B x_{\tilde{B}} + \tilde{\alpha}$ and $\tilde{z}^2 = \tilde{d}_B x_{\tilde{B}} + \tilde{\beta}$.

Step 7:

Compute the net evaluation $\tilde{\Delta}_j$ (j =1, 2, ..., m), where

$$\tilde{\Delta}_j = \tilde{z}^2(\tilde{z}_j^{\,1} - \tilde{c}_j) - \tilde{z}^1(\tilde{z}_j^{\,2} - \tilde{d}_j).$$

Here, $(\tilde{z}_j^{\,1} - \tilde{c}_j)$ and $(\tilde{z}_j^{\,2} - \tilde{d}_j)$ are calculated by using the relation $\tilde{z}_j^{\,1} - \tilde{c}_j = \tilde{c}_{\tilde{B}} \tilde{y}_j - \tilde{c}_j$ and $\tilde{z}_j^{\,2} - \tilde{d}_j = \tilde{d}_{\tilde{B}} \tilde{y}_j - \tilde{d}_j$.

Step 8:

Examine the sign $\tilde{\Delta}_j$ (using the method of GMIR method):

(i) If all $(\tilde{z}_j - \tilde{c}_j) \geq \tilde{0}$, then the initial basic feasible solution $x_{\tilde{B}}$ is a fuzzy optimum basic feasible solution.

(ii) If at least one $(\tilde{z}_j - \tilde{c}_j) < \tilde{0}$, proceed to the next step.

Step 9:

If there are more than one negative $\tilde{\Delta}_j$, then choose the most negative of them. Let it be $\tilde{\Delta}_r$ for some $j = r$.

(i) If all $\tilde{y}_{ir} \leq \tilde{0}$, ($i$ =1, 2, ..., m) then there is an unbounded solution to the given problem.

(ii) If at least one $\tilde{y}_{ir} > \tilde{0}$ (i=1, 2, ..., m), then the corresponding vector $\tilde{y}_r$ enters the basis $\tilde{y}_B$.

Step 10:

Compute the ratios

$$\left\{ \frac{\tilde{x}_{Bi}}{\tilde{y}_{ir}}, \tilde{y}_{ir} > \tilde{0}, i = 1, 2, \cdots, m \right\}$$

and choose the minimum of them. Let the minimum of these ratios be $\frac{\tilde{x}_{Bk}}{\tilde{y}_{kr}}$. Then, the vector $\tilde{y}_k$ will leave the basis $\tilde{y}_B$. The common element $\tilde{y}_{kr}$ which is in the k^{th} row and r^{th} column is known as leading or (pivotal) element of the table.

Step 11:

Convert the leading element to unit TrFN by dividing its row by the leading TrFN itself and all other elements in its column to zero TrFN by making the use of the relations:

$$\hat{\tilde{y}}_{ij} = \tilde{y}_{ij} - \frac{\tilde{y}_{kj}}{\tilde{y}_{kr}} \tilde{y}_{ir}, \; i = 1, 2, ..., m+1; i \neq k \text{, and}$$

$$\hat{\tilde{y}}_{kj} = \frac{\tilde{y}_{kj}}{\tilde{y}_{kr}}, \; j = 0, 1, 2, \cdots, n.$$

Step 12:

Go to **step 6** and repeat the computational procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

5. Numerical Example

Example 1.

Solve the following linear fractional programming problem:

$$\text{Max } z = \frac{6x_1+5x_2}{2x_1+7}$$

subject to the constraints:

$$x_1 + 2x_2 \leq 3,$$
$$3x_1 + 2x_2 \leq 6,$$
$$x_1, x_2 \geq 0.$$

Solution:

Table1: Initial Iteration:

		c_j	6	5	0	0	
		d_j	2	0	0	0	
c_B	d_B	x_B	y_1	y_2	y_3	y_4	$\frac{x_{Bi}}{y_{ij}}$
0	0	$x_3 = 3$	1	2	1	0	3
0	0	$x_4 = 6$	3^*	2	0	1	$2 \longrightarrow$
$z^1 = 0$	$z^2 = 7$	$z_j^1 - c_j$	-6	-5	0	0	
$z = 0$		$z_j^2 - d_j$	-2	0	0	0	
		Δ_j	$-42 \uparrow$	-35	0	0	

Here $\Delta_j \leq 0$, so x_4 leaves the basis and x_1 enters the basis.

Table 2: The next iterated simplex table is:

		c_j	6	5	0	0	
		d_j	2	0	0	0	
c_B	d_B	x_B	y_1	y_2	y_3	y_4	$\frac{x_{Bi}}{y_{ij}}$
0	0	$x_3 = 1$	0	$\frac{4}{3}^*$	1	$\frac{-1}{3}$	$\frac{3}{4} \longrightarrow$
6	2	$x_1 = 2$	1	$\frac{2}{3}$	0	$\frac{1}{3}$	3
$z^1 = 12$	$z^2 = 11$	$z_j^1 - c_j$	0	-1	0	2	
$z = \frac{12}{11}$		$z_j^2 - d_j$	0	$\frac{4}{3}$	0	$\frac{2}{2}$	
		Δ_j	0	$-27 \uparrow$	0	14	

Here $\Delta_j \leq 0$, so x_3 leaves the basis and x_2 enters the basis.

Table 3: The optimum iterated simplex table is:

		c_j	6	5	0	0	
		d_j	2	0	0	0	
c_B	d_B	x_B	y_1	y_2	y_3	y_4	$\frac{x_{Bi}}{y_{ij}}$
5	0	$x_2 = \frac{3}{4}$	0	1	$\frac{3}{4}$	$\frac{-1}{4}$	
6	2	$x_1 = \frac{3}{2}$	1	0	$\frac{-2}{3}$	$\frac{1}{2}$	
$z^1 = \frac{51}{4}$	$z^2 = 10$	$z_j^1 - c_j$	0	0	$\frac{-1}{4}$	$\frac{7}{4}$	
$z = \frac{51}{40}$		$z_j^2 - d_j$	0	0	$\frac{-4}{3}$	1	
		Δ_j	0	0	$\frac{29}{2}$	$\frac{19}{4}$	

In the optimum iterated table, all $\Delta_j \geq 0$ and the problem reaches its optimality. Hence, the optimal solution is $x_1 = \frac{3}{2}$=1.5, $x_2 = \frac{3}{4} = 0.75$ and

$$\text{Max } Z^* = \frac{51}{40} = 1.28.$$

Now the above problem can be solve by fuzzified version as followed: By taking triangular fuzzy number:

$$\text{Max } \tilde{z} = \frac{(5,6,7)\tilde{x}_1 + (3,5,6)\tilde{x}_2}{(1,2,3)\tilde{x}_1 + (5.5,7,8.5)}$$

subject to the constraints:

$$(0.5, 1, 1.5)\tilde{x}_1 + (1.5, 2, 2.5)\tilde{x}_2 \leq (2, 3, 5),$$
$$(2, 3, 4)\ \tilde{x}_1 + (1, 2, 3)\tilde{x}_2 \leq (5, 6, 7),$$
$$\tilde{x}_1, \tilde{x}_2 \geq 0.$$

Solution: The initial iteration table is given below:

Table 4:Initial Iteration:

		$\tilde{c}_j$:	(5,6,7)	(3,5,6)	(0,0,0)	(0,0,0)	
		$\tilde{d}_j$:	(1,2,3)	(0,0,0)	(0,0,0)	(0,0,0)	
$\tilde{c}_B$	$\tilde{d}_B$	$\tilde{x}_B$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$	$\frac{\tilde{x}_{Bi}}{\tilde{y}_{ij}}$
0	0	$\tilde{x}_3 = (2,3,5)$	(0.5,1,1.5)	(1.5,2,2.5)	(0,1,0)	(0,0,0)	(8,3,7)
0	0	$\tilde{x}_4 = (5,6,7)$	(2,3,4)*	(1,2,3)	(0,0,0)	(0,1,0)	(4.6,2,3.8) →
$\tilde{z}^1$ =(0,0,0)	$\tilde{z}^2$ =(5.5,7,8.5)	$\tilde{z}_j^1 - \tilde{c}_j$:	(-7,-6,-5)	(-6,-5,-3)	(0,0,0)	(0,0,0)	
$\tilde{z}$ =(0,0,0)		$\tilde{z}_j^2 - \tilde{d}_j$:	(-3,-2,-1)	(0,0,0)	(0,0,0)	(0,0,0)	
		$\tilde{\Delta}_j$:	(-82,-42, -86) ↑	(-69.5,-35,-63.5)	(0,0,0)	(0,0,0)	

In Table 4, $\widetilde{\Delta}_j \leq 0$, so $\tilde{x}_1$ enters the basis and $\tilde{x}_4$ leaves the basis by using

$$\text{Min } [d(\widetilde{\Delta}_{j1}, \widetilde{\Delta}_{j2})] = d(\widetilde{\Delta}_{j1}) = \tilde{y}_1,$$

$$\text{Min } [d(\frac{\tilde{x}_{B1}}{\tilde{y}_{12}}, \frac{\tilde{x}_{B2}}{\tilde{y}_{22}})] = d(\frac{\tilde{x}_{B2}}{\tilde{y}_{22}}) = \tilde{x}_4.$$

Therefore, $(2,3,4)^*$ is the leading element. Now convert this element as unit triangular fuzzy number and all other elements in the column as zero. The new iterated simplex table is given below:

Table5: Iterated Simplex Table:

		$\tilde{c}_j$:	(5,6,7)	(3,5,6)	(0,0,0)	(0,0,0)	
		$\tilde{d}_j$:	(1,2,3)	(0,0,0)	(0,0,0)	(0,0,0)	
$\tilde{c}_B$	$\tilde{d}_B$	$\tilde{x}_B$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$	$\frac{\tilde{x}_{Bi}}{\tilde{y}_{ij}}$
0	0	$\tilde{x}_3 = (5.8,1,9.6)$	(2.55,0,3.6)	$(4,1.3,4.3)^*$	(0,1,0)	(0.25, -0.3, 0.5)	(4.71,0.77, 7.61)
(5,6,7)	(1,2,3)	$\tilde{x}_1 = (4.6,2,3.8)$	(2.1,1,2.05)	(1.8,0.7,1.5)	(0,0,0)	(0.5,0.3 ,0.25)	(7.68,2.85, 6.76)
$\tilde{z}^1$ =(37.6,12, 36.8)	$\tilde{z}^2$ =(16.7, 11,22.1)	$\tilde{z}_j^{\,1} - \tilde{c}_j$:	(24.6,0, 24.3)	(20.3, -0.8, 16.9)	(0,0,0)	(4.5,1.8 ,3.6)	
$\tilde{z}$ =		$\tilde{z}_j^{\,2} - \tilde{d}_j$:	(8.2,0,8.1)	(4.3,1.4,5.1)	(0,0,0)	(1.30.6, 1.4)	
		$\tilde{\Delta}_j$:	(367.8,0, 365.7)	(97.22, -25.6, 272.46)	(0,0,0)	(118.44 ,12.6, 117.54)	

In Table 5, $\tilde{\Delta}_j \leq 0$, so $\tilde{x}_2$ enters the basis and $\tilde{x}_3$ leaves the basis by using

$$\text{Min}[\text{d}\,(\tilde{\Delta}_{j1}, \tilde{\Delta}_{j2})] = \text{d}\,(\tilde{\Delta}_{j1}) = \tilde{y}_2,$$

$$\text{Min}[\text{d}\,(\frac{\tilde{x}_{B1}}{\tilde{y}_{21}}, \frac{\tilde{x}_{B2}}{\tilde{y}_{22}})] = \text{d}(\frac{\tilde{x}_{B1}}{\tilde{y}_{21}}) = \tilde{x}_3.$$

Therefore, $(4, 1.3, 4.3)^*$ is the leading element. Now convert this element as unit triangular fuzzy number and all other elements in the column as zero. The optimum iterated simplex table is given below

Table 6: Optimal iteration:

		$\tilde{c}_j$:	(5,6,7)	(3,5,6)	(0,0,0)	(0,0,0)	
		$\tilde{d}_j$:	(1,2,3)	(0,0,0)	(0,0,0)	(0,0,0)	
$\tilde{c}_B$	$\tilde{d}_B$	$\tilde{x}_B$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$	$\frac{\tilde{x}_{Bi}}{\tilde{y}_{ij}}$
(3,5,6)	(0,0,0)	$\tilde{x}_2 = (4.71, 0.77,7.61)$	(1.96,0, 2.77)	(3.39,1, 3.61)	(0.25,0.77, 0.23)	(0.12, -0.23,0.31)	
(5,6,7)	(1,2,3)	$\tilde{x}_1$=(9.93,1.46, 7.09)	(4.04,1, 3.42)	(4.33,0,3.87)	(0.16, -0.54,0.18)	(0.72,0.46, 0.33)	
$\tilde{z}^1$ =(92.74, 12.61, 95.43)	$\tilde{z}^2$ =(26.82, 9.92, 26.06)	$\tilde{z}_j^{\,1} - \tilde{c}_j$:	(46.04,0, 46.37)	(51.93, 0, 50.27)	(1.82, 0.61, 3.07)	(6.53,1.61, 5.37)	
$\tilde{z}$ =(9.82, 1.27,10.09)		$\tilde{z}_j^{\,2} - \tilde{d}_j$:	(12.08, 0, 10.84)	(8.66,0,7.44)	(-0.22,- 1.08,-1.26)	(1.9,0.92, 2.04)	
		$\tilde{\Delta}_j$:	(593.41,0, 612.32)	(608.97,0, 607.88)	(153.36, 19.67, 198.36)	(221.48, 4.37,206.12)	

In Table 6, $\widetilde{\Delta}_j > 0$ and, hence, d($\widetilde{\Delta}_j$)> 0. Therefore, the problem reaches its optimality. Hence, the optimal solution is

$$\widetilde{x_1} = (9.93, 1.46, 7.09) \text{ and } \widetilde{x_2} = (4.71, 0.77, 7.61).$$

Therefore, Max $\widetilde{z^*}_1 = (9.82, 1.27, 10.09)$. By taking trapezoidal fuzzy number:

$$\text{Max } \tilde{z} = \frac{(5,6,7,8)\tilde{x}_1 + (3,5,6,7)\tilde{x}_2}{(1,2,3,4)\tilde{x}_1 + (5.5,7,8.5,9)}$$

subject to the constraints:

$$(0.5, 1, 1.5, 2)\tilde{x}_1 + (1.5, 2, 2.5, 3)\tilde{x}_2 \leq (2, 3, 5, 7),$$
$$(2,3,4,5)\,\tilde{x}_1 + (1, 2, 3, 4)\tilde{x}_2 \leq (5, 6, 7, 8),$$
$$\tilde{x}_1, \tilde{x}_2 \geq 0.$$

Solution: The initial iteration table is given below:

Table7.: Initial Iteration:

		$\tilde{c}_j$:	(5,6,7,8)	(3,5,6,7)	(0,0,0,0)	(0,0,0,0)	
		$\tilde{d}_j$:	(1,2,3,4)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	
$\tilde{c}_B$	$\tilde{d}_B$	$\tilde{x}_B$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$	$\frac{\tilde{x}_{Bi}}{\tilde{y}_{ij}}$
(0,0,0,0)	(0,0,0,0)	$\tilde{x}_3 = (2,3,5,7)$	(0.5,1,1.5,2)	(1.5,2,2.5,3)	(1,0,0,0)	(0,0,0,0)	(1,2,5,14)
(0,0,0,0)	(0,0,0,0)	$\tilde{x}_4 = (5,6,7,8)$	(2,3,4,5)*	(1,2,3,4)	(0,0,0,0)	(0,1,0,0)	(1,1.5,2.3,4) →
$\tilde{z}^1$ =(0,0,0,0)	$\tilde{z}^2$ =(5.5,7, 8.5, 9)	$\tilde{z}_j^1 - \tilde{c}_j$:	(-8,-7,-6, -5)	(-7,-6,-5,-3)	(0,0,0,0)	(0,0,0,0)	
$\tilde{z}$ =(0,0,0,0)		$\tilde{z}_j^2 - \tilde{d}_j$:	(-4,-3,-2, -1)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	
		$\tilde{\Delta}_j$:	(-44,-49, -51,-45) ↑	(-38.5,-42, -42.5,-27)	(0,0,0,0)	(0,0,0,0)	

In Table 7, $\widetilde{\Delta}_j \leq 0$, so $\tilde{x}_1$ enters the basis and $\tilde{x}_4$ leaves the basis by using

$$\text{Min } [\text{P}(\widetilde{\Delta}_{j1}, \widetilde{\Delta}_{j2})] = \text{P}(\widetilde{\Delta}_{j1}) = \tilde{y}_1,$$

$$\text{Min } [\text{P}(\frac{\tilde{x}_{B1}}{\tilde{y}_{11}}), P(\frac{\tilde{x}_{B2}}{\tilde{y}_{12}})] = \text{P}(\frac{\tilde{x}_{B2}}{\tilde{y}_{12}}) = \tilde{x}_4 \ .$$

Therefore, $(2,3,4,5)^*$ is the leading element. Now convert this element as unit trapezoidal fuzzy number and all other elements in the column as zero.
The new iterated simplex table is given below:

Table8:Iterated Simplex Table:

		$\tilde{c}_j$:	(5,6,7,8)	(3,5,6,7)	(0,0,0,0)	(0,0,0,0)	
		$\tilde{d}_j$:	(1,2,3,4)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	
$\tilde{c}_B$	$\tilde{d}_B$	$\tilde{x}_B$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$	$\frac{\tilde{x}_{Bi}}{\tilde{y}_{ij}}$
(0,0,0,0)	(0,0,0,0)	$\tilde{x}_3 = (21,35.2, 61,86.5)$	(0, 9.3, 16.9, 24)	$(16.75,24, 30.75,37.3)^*$	(12.5,0,0, 0)	(0,0, -0.25,0)	(0.56,2.08, 2.54,5.16) →
(5,6,7,8)	(1,2,3,4)	$\tilde{x}_1 = (1,1.5,2.3,4)$	(1,1.8,3.2, 6.25)	(0.2,0.5,1,2)	(0,0,0,0)	(0,0.25,0,0)	(0.5,1.5, 4.6,20)
$\tilde{z}^1$=(5,9, 16.1, 32)	$\tilde{z}^2$ =(6.5,10, 15.4,25)	$\tilde{z}_j^{\,1} - \tilde{c}_j$:	(-3,3.8, 16.4,45)	(-6,-3,2,13)	(0,0,0,0)	(0,1.5,0,0)	
$\tilde{z}$ =(0.2,0.58, 1.61,4.92)		$\tilde{z}_j^{\,2} - \tilde{d}_j$:	(-3,0.6, 7.6,24)	(0.2,1,3,8)	(0,0,0,0)	(0,0.5,0,0)	
		$\tilde{\Delta}_j$:	(-787.5, -84.36, 247.16, 1140)	(-295, -78.3,21.8,324) ↑	(0,0,0,0)	(0,15, -4.5,0)	

In Table 8, $\widetilde{\Delta_j} \leq 0$, so $\tilde{x}_2$ enters the basis and $\tilde{x}_3$ leaves the basis by using

$$\text{Min } [\mathrm{P}(\widetilde{\Delta}_{j1}, \widetilde{\Delta}_{j2})] = \mathrm{P}(\widetilde{\Delta}_{j1}) = \tilde{y}_2,$$

$$\text{Min } [\mathrm{P}(\frac{\tilde{x}_{B1}}{\tilde{y}_{21}}), P(\frac{\tilde{x}_{B2}}{\tilde{y}_{22}})] = \mathrm{P}(\frac{\tilde{x}_{B1}}{\tilde{y}_{21}}) = \tilde{x}_3 \ .$$

Therefore, $(16.75, 24, 30.75, 37.3)^*$ is the leading element. Now convert this element as unit trapezoidal fuzzy number and all other elements in the column as zero. The new iterated simplex table is given below:

Table9: Optimum Iteration:

		$\tilde{c}_j$:	(5,6,7,8)	(3,5,6,7)	(0,0,0,0)	(0,0,0,0)	
		$\tilde{d}_j$:	(1,2,3,4)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	
$\tilde{c}_B$	$\tilde{d}_B$	$\tilde{x}_B$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$	$\frac{\tilde{x}_{Bi}}{\tilde{y}_{ij}}$
(3,5,6,7)	(0,0,0,0)	$\tilde{x}_2 = (1.3,2.5, 5.7,11.5)$	(0, 0.67,1.57, 3.2)	(1,1.7,2.9,5)	(0.75,0,0,0)	(0,0, -0.02,0)	
(5,6,7,8)	(1,2,3,4)	$\tilde{x}_1 = (0.5,1.27,2.2,4)$	(1, 1.7, 3.2, 6.25)	(0, 0.4, 0.9, 2)	(0,0,0, -0.03)	(0,0.25,0,0)	
$\tilde{z}^1$=(6.4, 20.12,49.6, 112.5)	$\tilde{z}^2$ =(6,9.54, 15.1,25)	$\tilde{z}_j^{\,1} - \tilde{c}_j$:	(-3,6.55, 25.82, 125.5)	(-4,4.9, 18.7,48)	(2.25,0,0, -0.24)	(0,1.5, -0.12,0)	
$\tilde{z}$ =(0.26,1.33, 5.19,18.75)		$\tilde{z}_j^{\,2} - \tilde{d}_j$:	(-3,0.4,7.6, 24)	(0,0.8,2.7,8)	(0,0,0, -0.12)	(0,0.5,0,0)	
		$\tilde{\Delta}_j$:	(-2718, -340,382, 3144.2)	(-924,-85,269, 1200)	(27,0,0,-6)	(0,15, -11.86,0)	

In the optimum iteration table, all $\widetilde{\Delta_j} > 0$ and hence $P(\widetilde{\Delta_j}) > 0$. Therefore the problem reaches its optimality. Hence the optimal solution is $\widetilde{x_1} = (0.5, 1.27, 2.2, 4)$ and $\widetilde{x_2} = (1.3, 2.5, 5.7, 11.5)$. Therefore,

$$\text{Max } \tilde{z}^*{}_2 = (0.26, 1.33, 5.19, 18.75).$$

Example 2.

Solve the following FLFPP (2013):

$$\text{Max } z = \frac{2x_1+3x_2}{x_1+x_2+7}$$
subject to the constraints:
$$3x_1 + 5x_2 \leq 15,$$
$$4x_1 + 3x_2 \leq 12,$$
$$x_1, x_2 \geq 0.$$

Solution:

The solutions of crisp number and triangular fuzzy number are taken from (2013) and that of trapezoidal fuzzy number are solved like above example 1. After solving above example, the results are given below:

By crisp number, the optimal solution is $x_1 = 0, x_2 = 3$ and Max $z^{**} = 0.9$
By triangular fuzzy number, the optimal solution is:

$$\tilde{x}_1 = (0,0,0), \tilde{x}_2 = (5.95, 3, 6.5) \text{ and}$$
$$\text{Max } \widetilde{z_1}^{**} = (3.655, 0.9, 3.9).$$

By trapezoidal fuzzy number, the optimal solution is:

$$\tilde{x}_1=(0,0,0,0),\ \tilde{x}_2= (2.8,4.2,6.8,10) \text{ and}$$
$$\text{Max } \widetilde{z_2}^{**}=(0.07,0.70,3.04,8.29).$$

The results of crisp number, triangular fuzzy number and trapezoidal fuzzy number are given in Table10.

Table 10: Comparison of optimal solution of the numerical examples

Observation	Crisp number	Triangular fuzzy number	Trapezoidal fuzzy number
Example 1	Optimal solution is $x_1 = 1.5$, $x_2 = 0.75$ and $Max\ Z^* = 1.28$	Fuzzy optimal solution is $\widetilde{x_1} = (9.93,1.46, 7.09)$, $\widetilde{x_2} = (4.71, 0.77, 7.61)$ and Max $\widetilde{z^*}_1 = (9.82, 1.27, 10.09)$.	Fuzzy optimal solution is $\widetilde{x_1} = (0.5, 1.27, 2.2, 4)$, $\widetilde{x_2} = (1.3, 2.5, 5.7, 11.5)$ and Max $\widetilde{z^*}_2 = (0.26, 1.33, 5.19, 18.75)$.
Example 2	Optimal solution is $x_1 = 0$, $x_2 = 3$ and Max $Z^{**} = 0.9$	Fuzzy optimal solution is $\tilde{x}_1 = (0,0,0), \tilde{x}_2 = (5.95, 3, 6.5)$ and Max $\widetilde{z_1}^{**} = (3.655, 0.9, 3.9)$	Fuzzy optimal solution is $\tilde{x}_1$=(0,0,0,0), $\tilde{x}_2$ = (2.8,4.2,6.8,10) and Max $\widetilde{z_2}^{**}$=(0.07,0.70,3.04,8.29).

5. Conclusions

In this paper, we have solved fully fuzzy linear fractional programming problem (FFLFPP) by using graded mean integration representation (GMIR) method where the parameters and variables are characterized by trapezoidal fuzzy number. To check the efficiency of the proposed approach one numerical example has been solved by taking parameters and variables are in crisp number, TFN and TrFN. Example 2 has been solved in paper (2013). From Table 10, the optimal solution of both the examples has been presented. From that, the optimal solutions obtained by different numbers are nearly equal. The above method can be applied to solve FLFPP to reduce the complexity in problem solving like agriculture, production planning, inventory problems, transportation problems, etc.

REFERENCES

Ananthalakshmi, S. Vijayalakshmi, C. and Ganesan,V. (2014). Modern approach for designing and solving interval estimated linear fractional programming models, *Applications and Applied Mathematics,* Vol.9, Issue 2, pp.795-810.

Bellmann, R.E. and Zadeh, L.A. (1970). Decision making in a fuzzy environment, *Management Science* , Vol.17, pp. 141-164.

Bitran, G.R. and Novaes, A.G. (1973). Linear programming with a fractional objective function, *Operational Research*, Vol.21, 22-29.

Charnes, A. and Cooper, W.W. (1962). Programming with linear fractional functions, *Naval Research Logistics Quart .,* Vol.9, pp. 181-186.

Charnes, A Cooper, W. W. and Rhodes, E. (1978). Measuring efficiency of decision making units, *European Journal Operation Research,* Vol.2, 428-449.

Chen, S.H. (1985). Operations of fuzzy numbers with function principle, *Tamkang Journal of Management Sciences,* Vol.6/1, pp. 13-26.

Chen, S.H. and Hsieh, C.H. (1999). Graded mean integration representation of generalized fuzzy number, *Journal of Chinese Fuzzy Systems,* Vol.5, No.2, pp.1-7.

Chen, S.H. Wang, S.T. and Chang, S.M. (2006). Some properties of graded mean integration representation of LR type fuzzy numbers, *Tamsui Oxford Journal of Mathematical Sciences,* 22(2), pp.185-208, Aletheia University.

Craven, B.D. (1988). Fractional programming (Sigma Series in Applied Mathematics) Heldermann Verlag, Berlin, DM-48, 4, 145PP.

De, P.K. and Deb, M. (2013). Solving fuzzy linear fractional programming problem using signed distance ranking, *3rd IEEE International Advance Computing Conference*, pp. 806-812.

Dubois, D. and Prade, P. (1988). Possibility theory, Plenum Press, New York.

Ebrahimnejad, A. (2012). A constructive proof of fundamental theory for fuzzy variable linear programming problems, *Applications and Applied Mathematics,* Vol.7, Issue2, pp. 691-707.

Nachammi, Al. and Thangaraj, P. (2012). Solving fuzzy linear fractional programming problem using metric distance ranking, *Applied Mathematical Sciences*, pp. 1275-1285.

Rezaei, J. (2005). An inventory control model with price-demand relationship in a fuzzy environment, *3rd International Management Conference*.

Swarup, K., Gupta, P.K. and Mohan, M. (1997). *Operations Research*, S.Chand and Publishers, 8^{th} edition.

Yao, Jin-Shing and Wu, Kweimei (2000). Ranking fuzzy numbers based on decomposition principle and signed distance, *Fuzzy sets and Systems,* Vol.116, pp. 275-288.

Zimmermann, H.J. (1983). Fuzzy mathematical programming, *Computers and Operations Research*, Vol.10, pp. 291-298.

Zimmermann, H.J. (1991). *Fuzzy set theory and its applications*, *Kluwer Academic*, Dordrecht, Boston.

Authors' Biographies

Ms. Moumita Deb completed her Masters degree in Mathematics from Assam University, Silchar in the year 1998. In 2000, she obtained B.Ed. degree from Assam University, Silchar. Currently she is working as a Research scholar in Department of Mathematics, NIT Silchar since 2011. Her research interest includes linear fractional programming problem, stochastic linear fractional programming problem and multi-objective linear fractional programming problem.

Dr. Pijus Kanti De obtained his M.Sc. in Mathematics and B.T. teaching training degrees from Kalyani University. Also he obtained his M. Phil and Ph.D. in Applied Mathematics degrees from Indian School of Mines Dhanbad. Presently he is working as an Associate Professor in Mathematics in National Institute of Technology Silchar since 2010. Previously, Dr. De was in Banasthali University (Raj), KIET Ghaziabad under U.P.T.U, Delhi College of Engineering (D.T.U) and Centre for Mathematical Modeling and Computer Simulation (C-MMACS), National Aerospace Laboratories (NAL), Bangalore. His research interest lies on Fuzzy Optimization and Decision Making, Operations Research, Fuzzy Mathematics, Mathematical Modeling, Uncertainty Modeling, Numerical Optimization, Numerical Analysis, Elasto-Dynamics, Wave Propagation and Applied Mathematics.