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Exact Soliton Solutions for Second-Order Benjamin-Ono Equation

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Abstract

The homogeneous balance method is proposed for seeking the travelling wave solutions of the second-order Benjamin-Ono equation. Many exact traveling wave solutions of second-order Benjamin-Ono equation, which contain soliton like and periodic-like solutions are successfully obtained. This method is straightforward and concise, and it may also be applied to other nonlinear evolution equations.

Keywords: Homogeneous balance method; second-order Benjamin-Ono equation; Riccati equation; Soliton-like solution; Periodic-like solution.

MSC 2000: 34B15; 47E05; 35G25.

1. Introduction

It is well known that the nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various fields of sciences, such as physics, biology, chemistry, etc. Exact solutions of these equations are therefore very important and significant in the nonlinear sciences.

In recent years, Wang [(1995), (1996)] and Khalafallah (2009) presented a useful homogeneous balance method for finding exact solutions of certain nonlinear partial differential equations. Fan (2000) used the homogeneous balance method to search for the Backlund transformation and similarity reductions of nonlinear partial differential equations.
The aim of this paper is to find exact soliton solutions of the second-order Benjamin-Ono Equation, using the homogeneous balance method.

2. Second-Order Benjamin-Ono Equation

For the second-order Benjamin-Ono equation [Hereman et al. (1986)]:

$$u_{tt} + \alpha(u^2)_{xx} + \beta u_{xxxx} = 0, \tag{1}$$

where α and β are nonnegative constants,
Let us consider the traveling wave solutions

$$u(x,t) = u(\xi), \quad \xi = k(x + lt) + \xi_0, \tag{2}$$

Where k, l and ξ_0 are constants. Then Eq. (1) becomes

$$l^2 u'' + 2\alpha(u')^2 + 2\alpha u u'' + \beta k^2 u'''' = 0. \tag{3}$$

We now seek the solutions of Eq. (3) in the form

$$u = \sum_{i=0}^{m} q_i \phi^i, \tag{4}$$

where q_i are constants to be determined later and ϕ satisfy the following Riccati equation

$$\phi' = a\phi^2 + b\phi + c \tag{5}$$

where a, b and c are constants.
Balancing the highest order derivative term with nonlinear term in Eq. (3) gives leading order $m = 2$. We may therefore choose

$$u = q_0 + q_1\phi + q_2\phi^2, \tag{6}$$

where q_0, q_1 and q_2 are constants to be determined and ϕ satisfy Eq. (5).
Substituting (6) and (5) into Eq. (3), we have

$$
\begin{aligned}
&l^2u''+2\alpha(u')^2+2\alpha uu''+\beta k^2u''''=l^2(q_0+q_1\phi+q_2\phi^2)'' \\
&+2\alpha((q_0+q_1\phi+q_2\phi^2)')^2+2\alpha(q_0+q_1\phi+q_2\phi^2)(q_0+q_1\phi \\
&+q_2\phi^2)''+\beta k^2(q_0+q_1\phi+q_2\phi^2)''''=\{20\alpha a^2q_2^2+120\beta k^2a^4q_2\}\phi^6 \\
&+\{24\alpha a^2q_1q_2+24\beta k^2a^4q_1+336\beta k^2a^3bq_2+36\alpha abq_2^2\}\phi^5 \\
&+\{6\alpha a^2q_1^2+60\beta k^2a^3bq_1+(240\beta k^2ca^3+330\beta k^2b^2a^2+6l^2a^2)q_2 \\
&+12\alpha a^2q_0q_2+42\alpha abq_1q_2+(32\alpha ac+16\alpha b^2)q_2^2\}\phi^4+\{10\alpha abq_1^2+ \\
&(40\beta k^2ca^3+2l^2a^2+50\beta k^2b^2a^2)q_1+4\alpha a^2q_0q_1+(36\alpha ca+18\alpha b^2)q_1q_2 \\
&+(10l^2ab+440\beta k^2bca^2+130\beta k^2b^3a)q_2+20\alpha abq_0q_2+28\alpha bcq_2^2\}\phi^3 \\
&+\{(8\alpha ac+4\alpha b^2)q_1^2+(60\beta k^2bca^2+15\beta k^2b^3a+3l^2ba)q_1+6\alpha baq_0q_1 \\
&+30\alpha bcq_1q_2+(8l^2ac+4l^2b^2+136\beta k^2c^2a^2+232\beta k^2cb^2a+16\beta k^2b^4)q_2 \\
&+(16\alpha ac+8b^2\alpha)q_0q_2+12ac^2q_2^2\}\phi^2+\{6\alpha cbq_1^2+(16\beta k^2c^2a^2+2al^2c \\
&+22\beta ak^2b^2c+\beta k^2b^4+l^2b^2)q_1+12\alpha c^2q_1q_2+(4\alpha ac+2\alpha b^2)q_0q_1 \\
&+12\alpha bcq_0q_2+(6l^2bc+120\beta k^2bc^2a+30\beta k^2b^3c)q_2\}\phi+2\alpha c^2q_1^2 \\
&+2\alpha bcq_0q_1+(8\beta k^2ac^2b+\beta k^2b^3c+l^2bc)q_1+(16\beta k^2c^3a \\
&+2l^2c^2+14\beta k^2b^2c^2)q_2+4\alpha c^2q_0q_2=0.
\end{aligned}
$$

Setting the coefficients ϕ^i $(i=0,1,2,3,4,5,6)$ to zero yields the following set of algebraic equations:

$$20\alpha a^2q_2^2+120\beta k^2a^4q_2=0,$$

$$24\alpha a^2q_1q_2+24\beta k^2a^4q_1+336\beta k^2a^3bq_2+36\alpha abq_2^2=0,$$

$$
\begin{aligned}
&6\alpha a^2q_1^2+60\beta k^2a^3bq_1+(240\beta k^2ca^3+330\beta k^2b^2a^2+6l^2a^2)q_2 \\
&+12\alpha a^2q_0q_2+42\alpha abq_1q_2+(32\alpha ac+16\alpha b^2)q_2^2=0,
\end{aligned}
$$

$$
\begin{aligned}
&10\alpha abq_1^2+(40\beta k^2ca^3+2l^2a^2+50\beta k^2b^2a^2)q_1 \\
&+4\alpha a^2q_0q_1+(36\alpha ca+18\alpha b^2)q_1q_2+(10l^2ab+ \\
&440\beta k^2bca^2+130\beta k^2b^3a)q_2+20\alpha abq_0q_2 \\
&+28\alpha bcq_2^2=0,
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&(8\alpha ac+4\alpha b^2)q_1^2+(60\beta k^2bca^2+15\beta k^2b^3a+3l^2ba)q_1 \\
&+6\alpha baq_0q_1+30\alpha bcq_1q_2+(8l^2ac+4l^2b^2+136\beta k^2c^2a^2 \\
&+232\beta k^2cb^2a+16\beta k^2b^4)q_2+(16\alpha ac+8b^2\alpha)q_0q_2 \\
&+12ac^2q_2^2=0,
\end{aligned}
$$

$$
\begin{aligned}
&6\alpha cbq_1^2+(16\beta k^2c^2a^2+2al^2c \\
&+22\beta ak^2b^2c+\beta k^2b^4+l^2b^2)q_1+12\alpha c^2q_1q_2+(4\alpha ac+2\alpha b^2)q_0q_1 \\
&+12\alpha bcq_0q_2+(6l^2bc+120\beta k^2bc^2a+30\beta k^2b^3c)q_2=0,
\end{aligned}
$$

$$
\begin{aligned}
&2\alpha c^2q_1^2+2\alpha bcq_0q_1+(8\beta k^2ac^2b+\beta k^2b^3c+l^2bc)q_1 \\
&+(16\beta k^2c^3a+2l^2c^2+14\beta k^2b^2c^2)q_2+4\alpha c^2q_0q_2=0.
\end{aligned}
$$

For which, with the aid of Maple, we obtain the following solution of the above set of algebraic equations:

$$
q_0=-\frac{\beta k^2b^2+8\beta k^2ca+l^2}{2\alpha},\qquad q_1=-\frac{6\beta k^2ab}{\alpha},\qquad q_2=-\frac{6\beta k^2a^2}{\alpha}. \tag{8}
$$

For the Riccati equation (5), we can solve it by using the homogeneous balance method as follows:

Case: I. Let $\phi=\sum_{i=0}^{m}b_i\tanh^i\xi$. Balancing ϕ' with ϕ^2 leads to

$$
\phi=b_0+b_1\tanh\xi. \tag{9}
$$

Substituting (9) into (5), we obtain the following solution of (5):

$$
\phi=-\frac{1}{2a}(b+2\tanh\xi),\quad ac=\frac{b^2}{4}-1. \tag{10}
$$

From (8), (10) and (6), we have the following traveling wave solution of second-order Benjamin-Ono equation (1):

$$u(x,t)=\frac{2\beta k^2b^2-8\beta k^2ca-l^2}{2\alpha}-\frac{6k^2\beta}{\alpha}\tanh^2[k(x+lt)+\xi_0], \tag{11}$$

where

$$ac=\frac{b^2}{4}-1.$$

Similarly, let $\phi=\sum_{i=0}^{m}b_i\coth^i\xi,$ then we obtain the following new traveling wave soliton solutions of second-order Benjamin-Ono equation (1):

$$u(x,t)=\frac{2\beta k^2b^2-8\beta k^2ca-l^2}{2\alpha}-\frac{6k^2\beta}{\alpha}\coth^2[k(x+lt)+\xi_0], \tag{12}$$

where

$$ac=\frac{b^2}{4}-1.$$

Case: II. From (Zhao, X. Q.; Tang, D. B.; 2002), when a=1, b=0, the Riccati Eq. (5) has the following solutions:

$$\phi=-\sqrt{-c}\tanh(\sqrt{-c}\,\xi),\ c<0,$$

$$\phi=-\frac{1}{\xi},\ c=0, \tag{13}$$

$$\phi=\sqrt{c}\tan(\sqrt{c}\,\xi),\ \ c>0.$$

From (6), (8) and (13), we have the following traveling wave solutions of second-order Benjamin-Ono equation (1) :

When $c<0,$ we have

$$\begin{aligned}u(x,t)=&-\frac{\beta k^2b^2+8\beta k^2ca+l^2}{2\alpha}+\frac{6\beta k^2ab\sqrt{-c}}{\alpha}\tanh[\sqrt{-c}(k(x+lt)+\xi_0)]\\&+\frac{6\beta k^2a^2c}{\alpha}\tanh^2[\sqrt{-c}(k(x+lt)+\xi_0)].\end{aligned} \tag{14}$$

When $c=0,$ we have

$$u(x,t) = -\frac{\beta k^2 b^2 + 8\beta k^2 ca + l^2}{2\alpha} + \frac{6\beta k^2 ab}{\alpha(k(x+lt)+\xi_0)} - \frac{6\beta k^2 a^2}{\alpha(k(x+lt)+\xi_0)^2}. \tag{15}$$

When $c > 0$, we have

$$u(x,t) = -\frac{\beta k^2 b^2 + 8\beta k^2 ca + l^2}{2\alpha} - \frac{6\beta k^2 ab\sqrt{c}}{\alpha}\tan[\sqrt{c}(k(x+lt)+\xi_0)] - \frac{6\beta k^2 a^2 c}{\alpha}\tan^2[\sqrt{c}(k(x+lt)+\xi_0)]. \tag{16}$$

Case: III. We suppose that the Riccati equation (5) has the following solutions of the form

$$\phi = A_0 + \sum_{i=1}^{m}(A_i f^i + B_i f^{i-1} g), \tag{17}$$

with

$$f = \frac{1}{\cosh\xi + r}, \quad g = \frac{\sinh\xi}{\cosh\xi + r},$$

which satisfy

$$f'(\xi) = -f(\xi)g(\xi), \quad g'(\xi) = 1 - g^2(\xi) - rf(\xi),$$

$$g^2(\xi) = 1 - 2rf(\xi) + (r^2-1)f^2(\xi).$$

Balancing ϕ' with ϕ^2 leads to

$$\phi = A_0 + A_1 f + B_1 g. \tag{18}$$

Substituting (18) into (5), collecting the coefficient of the same power $f^i(\xi)g^i(\xi)(i = 0,1,2; j = 0,1)$ and setting each of the obtained coefficients to zero yield the following set of algebra equations

$$aA_0^2 + aB_1^2 + bA_0 + c = 0,$$

$$2aA_0A_1 - 2arB_1^2 - rB_1 + bA_1 = 0,$$

$$aA_1^2 + a(r^2 - 1)B_1^2 + (r^2 - 1)B_1 = 0,$$

$$2aA_0B_1 + bB_1 = 0,$$

$$2aA_1B_1 + A_1 = 0,$$

which have solutions

$$A_0 = -\frac{b}{2a}, \qquad A_1 = \pm\sqrt{\frac{(r^2-1)}{4a^2}}, \qquad B_1 = -\frac{1}{2a}, \qquad c = \frac{b^2-1}{4a}. \tag{19}$$

From (18), (19) we obtain

$$\phi = -\frac{1}{2a}\left(b + \frac{\sinh\xi \mp \sqrt{(r^2-1)}}{\cosh\xi + r}\right). \tag{20}$$

Also from (6), (8) and (20), we obtain the new solutions of second-order Benjamin-Ono equation (1):

$$u(x,t) = \frac{2\beta k^2 b^2 - 8\beta k^2 ca - l^2}{2\alpha} - \frac{3k^2\beta}{2\alpha}\left(\frac{\sinh(k(x+lt)+\xi_0) \mp \sqrt{(r^2-1)}}{\cosh(k(x+lt)+\xi_0)+r}\right)^2. \tag{21}$$

Case: IV. We take ϕ in the Riccati equation (5) being of the form

$$\phi = e^{p_1\xi}\rho(z) + p_4(\xi), \tag{22}$$

where

$$z = e^{p_2\xi} + p_3,$$

where p_1, p_2 and p_3 are constants to be determined.

Substituting (22) into (5) we find that when $c = \frac{-p_1^2 + b^2}{4a}$, we have

$$\phi = -\frac{p_1 e^{p_1\xi}}{a(e^{p_1\xi} + p_3)} + \frac{p_1 - b}{2a}. \tag{23}$$

If $p_3 = 1$ in (23), we have

$$\phi = -\frac{p_1}{2a}\tanh(\frac{1}{2}p_1\xi) - \frac{b}{2a}. \tag{24}$$

If $p_3 = -1$ in (23), we have

$$\phi = -\frac{p_1}{2a}\coth(\frac{1}{2}p_1\xi) - \frac{b}{2a}. \tag{25}$$

From (6), (8) and (23), we obtain the following new traveling wave solutions of second-order Benjamin-Ono equation (1):

$$\begin{aligned} u(x,t) = &-\frac{\beta k^2 b^2 + 8\beta k^2 ca + l^2}{2\alpha} \\ &-\frac{6\beta k^2}{\alpha}\left(\frac{p_1 e^{p_1(k(x+lt)+\xi_0)}}{e^{p_1(k(x+lt)+\xi_0)} + p_3} - \frac{p_1 - b}{2}\right)\left(\frac{p_1 e^{p_1(k(x+lt)+\xi_0)}}{e^{p_1(k(x+lt)+\xi_0)} + p_3} - \frac{p_1 + b}{2}\right). \end{aligned} \tag{26}$$

When p3 = 1, we obtain the following traveling wave (soliton-like) solutions of second-order Benjamin-Ono equation (1):

$$u(x,t) = \frac{2\beta k^2 b^2 - 8\beta k^2 ca - l^2}{2\alpha} - \frac{3k^2\beta p_1^2}{2\alpha}\tanh^2[\frac{p_1}{2}(k(x+lt)+\xi_0)]. \tag{27}$$

When $p_3 = -1$, we have the following traveling wave (periodic-like) solutions of equal width wave equation (1):

$$u(x,t) = \frac{2\beta k^2 b^2 - 8\beta k^2 ca - l^2}{2\alpha} - \frac{3k^2\beta p_1^2}{2\alpha}\coth^2[\frac{p_1}{2}(k(x+lt)+\xi_0)]. \tag{28}$$

Case: V. We suppose that the Riccati equation (5) have the following solutions of the form:

$$\phi = A_0 + \sum_{i=1}^{m}\sinh^{i-1}(A_i \sinh w + B_i \cosh w), \tag{29}$$

where $\frac{dw}{d\xi} = \sinh w$ or $\frac{dw}{d\xi} = \cosh w$. It is easy to find that $m = 1$ by balancing ϕ' and ϕ^2. So we choose

$$\phi = A_0 + A_1 \sinh w + B_1 \cosh w, \tag{30}$$

when $\frac{dw}{d\xi} = \sinh w$, we substitute (30) and $\frac{dw}{d\xi} = \sinh w$, into (5) and set the coefficients of $\sinh^i w \cosh^i w \ (i = 0,1,2; j = 0,1)$ to zero. A set of algebraic equations is obtained as follows:

$$aA_0^2 + aB_1^2 + bA_0 + c = 0,$$

$$2aA_0A_1 + bA_1 = 0,$$

$$aA_1^2 + aB_1^2 - B_1 = 0,$$

$$2aA_0B_1 + bB_1 = 0,$$

$$2aA_1B_1 + A_1 = 0,$$

for which, we have the following solutions:

$$A_0 = -\frac{b}{2a}, \qquad A_1 = 0, \qquad B_1 = \frac{1}{a}, \tag{31}$$

where

$$c = \frac{b^2 - 4}{4a},$$

and

$$A_0 = -\frac{b}{2a}, \qquad A_1 = \pm\sqrt{\frac{1}{2a}}, \qquad B_1 = \frac{1}{2a}, \tag{32}$$

where

$$c = \frac{b^2 - 1}{4a}.$$

From $\dfrac{dw}{d\xi} = \sinh w$, we have

$$\sinh w = -\operatorname{csc} h\xi, \ \cosh w = -\coth\xi. \tag{33}$$

Also (31) − (33), give

$$\phi = -\frac{b + 2\coth\xi}{2a}, \tag{34}$$

where

$$c=\frac{b^2-4}{4a},$$

and

$$\phi=-\frac{b\pm\csc h\xi+\coth\xi}{2a}, \tag{35}$$

where

$$c=\frac{b^2-1}{4a}.$$

From (6), (8), (34) and (35) we get the following traveling wave solutions of second-order Benjamin-Ono equation (1):

$$u(x,t)=\frac{2\beta k^2b^2-8\beta k^2ca-l^2}{2\alpha}-\frac{6k^2\beta}{\alpha}\coth^2[k(x+lt)+\xi_0], \tag{36}$$

where

$$ac=\frac{b^2}{4}-1.$$

and

$$u(x,t)=\frac{2\beta k^2b^2-8\beta k^2ca-l^2}{2\alpha} -\frac{3k^2\beta}{2\alpha}(\coth(k(x+lt)+\xi_0)\pm\csc h(k(x+lt)+\xi_0))^2, \tag{37}$$

where

$$c=\frac{b^2-1}{4a}.$$

Similarly, when $\frac{dw}{d\xi}=\cosh w$, we obtain the following traveling wave (periodic-like) solutions of second-order Benjamin-Ono equation (1):

$$u(x,t)=\frac{2\beta k^2b^2-8\beta k^2ca-l^2}{2\alpha}-\frac{6k^2\beta}{\alpha}\cot^2[k(x+lt)+\xi_0], \tag{38}$$

where

$$ac = \frac{b^2}{4} - 1.$$

and

$$u(x,t) = \frac{2\beta k^2 b^2 - 8\beta k^2 ca - l^2}{2\alpha} - \frac{3k^2\beta}{2\alpha}(\cot(k(x+lt)+\xi_0) \pm \csc(k(x+lt)+\xi_0))^2, \tag{39}$$

where

$$c = \frac{b^2 - 1}{4a}.$$

In summary we have used the homogeneous balance method to obtain many traveling wave solutions of second-order Benjamin-Ono equation.

We now summarize the key steps as follows:

Step 1: For a given nonlinear evolution equation

$$F(u, u_t, u_x, u_{xt}, u_{tt}, \ldots) = 0, \tag{40}$$

we consider its traveling wave solutions $u(x,t) = u(\xi)$, $\xi = k(x+lt) + \xi_0$ then Eq. (40) is reduced to an nonlinear ordinary differential equation

$$Q(u, u', u'', u''', \ldots) = 0, \tag{41}$$

where a prime denotes $\dfrac{d}{d\xi}$.

Step 2: For a given ansatz equation (for example, the ansatz equation is $\phi' = a\phi^2 + b\phi + c$ in this paper), the form of u is decided and the homogeneous balance method is used on Eq. (41) to find the coefficients of u .

Step 3: The homogeneous balance method is used to solve the ansatz equation.

Step 4: Finally, the traveling wave solutions of Eq. (40) are obtained by combining steps 2 and 3.

3. Conclusion

The second-order Benjamin-Ono equation is soluble using the homogeneous balance method. The efficiency of this method was demonstrated. New exact solution of the second-order Benjamin-Ono equation was obtained. The solutions obtained may be significant and important for the explanation of some practical physical problems. The method may also be applied to other nonlinear partial differential equations.

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