



Modeling the Effect of Environmental Factors on the Spread of Bacterial Disease in an Economically Structured Population

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Abstract

We have proposed and analyzed a nonlinear mathematical model for the spread of bacterial disease in an economically structured population (rich and poor) including the role of vaccination. It is assumed that rich susceptible get infected through direct contact with infectives in the same class and with infectives from the poor class who work as service providers in the houses of rich people, living in much cleaner environment. The susceptible in the poor class are assumed to become infected through direct contact with infectives in the same class as well as by bacteria present in their own environment, degraded due to unhygienic environmental conditions. It is further assumed that the bacteria population affects only the population in the degraded environment of the poor class but does not survive in the clean environment of rich people. The density of bacteria population is assumed to be governed by a logistic model and is dependent on environmental discharges conducive to the growth of bacteria population. The cumulative density of environmental discharges depends upon the human population related factors of the poor class. The model analysis shows that the increased growth rate of environmental discharges increases the bacteria population density in the poor class due to unhygienic environmental conditions leading to increase the infectives in the poor class i.e., service providers. As a consequence, due to interaction with these service providers the spread of disease increases in the rich class. The improved environmental conditions of the region inhabited by service providers along with suitable vaccination strategy can be helpful in reducing the spread of the disease.

Keywords: Bacterial disease, economically structured population, service providers, environmental factors, vaccination, stability, simulation

MSC 2010: 92D30, 92D25

1. Introduction

Infectious diseases are the world's biggest killer of people and accounts for millions of deaths per year. There are many infectious diseases in which infection is transmitted by direct contact of susceptible with infectives, while there are some diseases like tuberculosis, which are also transmitted indirectly by the flow of bacteria from infectives into the environment. The poor environmental conditions existing in the densely populated cities of the third world countries have the greatest impact on the spread of bacterial diseases. If the environment is conducive to the growth of bacterial population, then it further helps in the spread of infectious diseases, Ghosh et al. (2005, 2006), Naresh et al. (2008, 2009). Also, the migration of population from environmentally degraded region to a cleaner region plays a vital role in the spread of infectious diseases as infected persons act as carrier/reservoir of infection. Various modeling studies of infectious diseases have been done, May and Anderson (1979), Anderson and May (1981, 1983), Struchiner et al. (1989), Mena-Lorca and Hethcote (1992), Kribs-Zaleta and Velasco-Hernandez (2000), Feng et al. (2002), Murphy et al. (2002), Moghadas and Alexander (2004), Bowman et al. (2005), Naresh and Tripathi (2005), White and Comiskey (2007), Naresh et al. (2009), Pedro and Tchuenche (2010), Mushayabasa et al. (2011). In particular, Ghosh et al. (2005) modeled the effect of service providers from an environmentally degraded region on the spread of bacterial disease and concluded that the spread of the infectious disease increases when the growth of bacteria caused by conducive environmental discharge due to human sources increases. Also the spread of the infectious disease in rich class increases due to the interaction with service providers. Thus, unhygienic environmental conditions in the habitat caused by service providers become responsible for the fast spread of an infectious disease. They (2006) also formulated an SIS model for the spread of a bacterial disease assuming logistically growing human population and concluded that the disease spread is faster when bacterial growth increases due to conducive environmental discharges.

Li and Jin (2005) analyzed a SEIR model having infectious force in latent, infected and immune period. They derived basic reproduction number, R_0 , and concluded that if $R_0 \leq 1$, the disease-free equilibrium is globally stable so that the disease always dies out and if $R_0 > 1$, the disease-free equilibrium becomes unstable while the endemic equilibrium emerges as the unique positive equilibrium and is locally and globally stable when disease induced death rate is zero. McCluskey (2006) proposed and analyzed models for the spread of TB, which included fast and slow progression to the infected class and showed that when basic reproduction number is less than or equal to one, the disease-free equilibrium is globally asymptotically stable and when it is greater than one there is an endemic equilibrium which is globally asymptotically stable. Martcheva et al. (2007) formulated an epidemic model to investigate the complexities of the effect of vaccination on a multi strain disease in the presence of mutation. Naresh et al. (2009) analyzed a nonlinear model for the spread of HIV/AIDS in a population of varying size with immigration of infectives (also assumed infectious) and all infectives ultimately developing AIDS. They concluded that the spread of infection can be slowed down if direct inflow of infectives is restricted into the population. Pedro and Tchuenche (2010) studied an HIV/AIDS model by taking into account the social structure of population (rich and poor) and found that the prevalence of HIV in rich communities is higher than that in the poor, but the disease develops faster in impoverished individuals.

It is pointed out here that in some of the above models, vaccination has been studied without considering the effective role of variable bacteria population density which depends on cumulative density of environmental discharges and indirectly on population density of service providers and is responsible for spreading the bacterial disease. In big cities of third world countries where rich and poor people live in nearby neighborhoods, then the poor people work as service providers in the houses of rich people but do not settle in the habitat of rich people. These service providers play vital role in the spread of infectious diseases as they carry pathogens in or on their bodies and may also transport disease vectors. With increase in the population density of service providers, the effects of human population related factors like discharge of household wastes; open sewage drainage, open water storage tanks, ponds etc. lead to further growth in the density of bacteria population, thereby increasing the fast spread of bacterial disease. It is, therefore, reasonable to assume the growth rate and the carrying capacity of bacteria population density to be the functions of cumulative density of environmental discharges. The growth rate of environmental discharges is also assumed to be the function of total population of service providers.

In this paper, we therefore propose a nonlinear model to study the spread of bacterial disease in an economically structured population (rich and poor) including the role of vaccination. We have taken into account the growth rate and carrying capacity of bacteria population to be function of cumulative density of environmental discharges. The growth rate of environmental discharges is also taken to be dependent on the total population of service providers, living in a degraded environment. Since our objective is to study explicitly the role of environmental factors conducive to the growth of bacteria population on the spread of disease with the above population structure, we have considered bilinear interaction to model the transmission dynamics. The model, however, can further be generalized by assuming standard incidence or other nonlinear interactions, Mena-Lorca and Hethcote (1992), Naresh et al. (2008).

2. Mathematical Model

We consider the spread of bacterial infectious disease in an economically structured population (rich and poor) living in two adjoining habitats or neighborhoods with different environmental conditions. The environment where the rich people live is much cleaner, whereas the environment where poor people live is not so clean and is very conducive to the growth of bacteria population due to unhygienic household discharges. Here the total population N_1 of rich class is divided into susceptible X_1 , infectives Y_1 and vaccinated individuals V_1 . The total population N_2 of poor class is divided into susceptible X_2 , infectives Y_2 and vaccinated individuals V_2 . It is assumed that rich susceptible get infected through direct contacts with infectives in the same class and with infectives from the poor class who work as service providers in the houses of the rich people. These service providers interact with people in the rich class during work and then they return back to their homes. The service providers do not settle in the habitat of rich people.

Thus, we have considered interaction between rich and poor population but not the migration of poor population into rich population. It is further assumed that the bacteria population affects only the population in the degraded environment of the poor class but does not survive in the

clean environment of rich people. It is also assumed that susceptible in the poor class get infected through direct contacts with infectives in the same class and indirectly by bacteria present in their degraded environment. The susceptible in the poor class may also get infected by the infected rich people and the heterogeneity of the mixing patterns between the rich and poor may also influence the infection dynamics of bacterial disease, Bhunu et al. (2012). However, in the modeling process we have considered the interaction of infected rich people only with susceptible in the same class but not with the susceptible of the poor class as our purpose is to study the spread of bacterial disease through service providers (poor people) living in unhygienic environmental conditions conducive to the growth of bacteria, Ghosh et al (2005). The density of bacteria population B is assumed to grow logistically with the growth rate and carrying capacity dependent on the cumulative density of environmental discharges in the poor class, which further depends on the human population related factors in the poor class. The bacteria population density is also enhanced by the release of bacteria from infected poor population. We have also assumed that the rich and poor susceptibles X_1 and X_2 respectively, are vaccinated at a constant rate and some of them may again become susceptible due to inefficacy of vaccines. Also, a fraction of infectives in both the classes, after recovery, may join the respective susceptible classes.

In view of the above assumptions and considerations, the model dynamics is governed by the following system of nonlinear ordinary differential equations:

$$\frac{dX_1}{dt} = A_1 - \beta_1 X_1 Y_1 - \lambda_1 X_1 Y_2 - (d_1 + \phi_1) X_1 + \nu_1 Y_1 + \theta_1 V_1 \quad (1)$$

$$\frac{dY_1}{dt} = \beta_1 X_1 Y_1 + \lambda_1 X_1 Y_2 - (\nu_1 + \alpha_1 + d_1) Y_1 \quad (2)$$

$$\frac{dV_1}{dt} = \phi_1 X_1 - (d_1 + \theta_1) V_1 \quad (3)$$

$$\frac{dX_2}{dt} = A_2 - \beta_2 X_2 Y_2 - \lambda_2 X_2 B - (d_2 + \phi_2) X_2 + \nu_2 Y_2 + \theta_2 V_2 \quad (4)$$

$$\frac{dY_2}{dt} = \beta_2 X_2 Y_2 + \lambda_2 X_2 B - (\nu_2 + \alpha_2 + d_2) Y_2 \quad (5)$$

$$\frac{dV_2}{dt} = \phi_2 X_2 - (d_2 + \theta_2) V_2 \quad (6)$$

$$\frac{dB}{dt} = s(E) \left(1 - \frac{B}{L(E)} \right) B + s_2 Y_2 - s_{20} B \quad (7)$$

$$\frac{dE}{dt} = Q(N_2) - \delta_0 E \quad (8)$$

$$X_1 + Y_1 + V_1 = N_1, \quad X_2 + Y_2 + V_2 = N_2$$

$$X_1(0) > 0, Y_1(0) \geq 0, V_1(0) \geq 0, N_1(0) > 0, X_2(0) > 0, Y_2(0) \geq 0, \\ V_2(0) \geq 0, N_2(0) > 0, B(0) \geq 0, E(0) \geq 0,$$

where A_1 and A_2 are the constant immigration rates of human population into the rich and poor populations respectively, β_1 and λ_1 are transmission coefficients in the rich population due to infectives of the rich and poor class respectively, β_2 and λ_2 are transmission coefficients in the poor population due to infectives of the poor class and bacteria respectively, d_1 and d_2 are the natural death rates corresponding to rich and poor classes, α_1 and α_2 are the disease induced death rates corresponding to rich and poor classes respectively. The parameters ϕ_1 and ϕ_2 represent the vaccination coverage (of susceptible) of rich and poor population respectively, ν_1 and ν_2 are therapeutic treatment coverage (of infected individuals) of rich and poor classes respectively, θ_1 and θ_2 denotes the rate at which vaccinated individuals of rich and poor population again become susceptible due to inefficacy of vaccines. Here, $E(t)$ denotes the cumulative density of environmental discharges conducive to the growth of bacteria population and $s(E)$ is the intrinsic growth rate of the bacteria population density, $L(E)$ is the carrying capacity of the bacteria population in the natural environment, s_{20} is the decay rate of bacteria population density due to natural factors as well as by control measures, s_2 is the rate of release of bacteria from the infective poor population, $Q(N_2)$ is the rate of cumulative environmental discharges conducive to the growth of bacteria into the poor population which depends on the density N_2 of the poor population and δ_0 is the depletion rate coefficient of the cumulative environmental discharges.

As the human population increases, the effects of human population related factors/activities enhance the cumulative density of environmental discharges which further intensify the growth of bacteria population. Thus, in the model, $s(E)$ and $L(E)$ are taken to be functions of cumulative density of environmental discharges. Since we assume that the growth rate per capita increases as the cumulative density of environmental discharges, we have

$$s(0) = s_0 \text{ and } s'(E) \geq 0 \quad (9)$$

where s_0 is the value of $s(E)$ at $E = 0$ and $()'$ denotes the derivative of the function with respect to its argument. We assume that the modified carrying capacity increases with the cumulative density of environmental discharges, so that

$$L(0) = L_0 > 0 \text{ and } L'(E) \geq 0 \quad (10)$$

where L_0 is the value of $L(E)$ when $E = 0$.

We also assume that rate of cumulative environmental discharges increases with the human population density N_2 , so that

$$Q(0) = Q_0 > 0 \text{ and } Q'(N_2) \geq 0 \quad (11)$$

where Q_0 is the value of $Q(N_2)$ when $N_2 = 0$.

3. Equilibrium Analysis

It is sufficient to consider the following reduced system of model (1-8) (since $X_1 + Y_1 + V_1 = N_1$ and $X_2 + Y_2 + V_2 = N_2$) as follows:

$$\frac{dY_1}{dt} = \beta_1(N_1 - Y_1 - V_1)Y_1 + \lambda_1(N_1 - Y_1 - V_1)Y_2 - (v_1 + \alpha_1 + d_1)Y_1 \tag{12}$$

$$\frac{dV_1}{dt} = \phi_1(N_1 - Y_1 - V_1) - (d_1 + \theta_1)V_1 \tag{13}$$

$$\frac{dN_1}{dt} = A_1 - d_1N_1 - \alpha_1Y_1 \tag{14}$$

$$\frac{dY_2}{dt} = \beta_2(N_2 - Y_2 - V_2)Y_2 + \lambda_2(N_2 - Y_2 - V_2)B - (v_2 + \alpha_2 + d_2)Y_2 \tag{15}$$

$$\frac{dV_2}{dt} = \phi_2(N_2 - Y_2 - V_2) - (d_2 + \theta_2)V_2 \tag{16}$$

$$\frac{dN_2}{dt} = A_2 - d_2N_2 - \alpha_2Y_2 \tag{17}$$

$$\frac{dB}{dt} = s(E) \left(1 - \frac{B}{L(E)} \right) B + s_2Y_2 - s_{20}B \tag{18}$$

$$\frac{dE}{dt} = Q(N_2) - \delta_0E \tag{19}$$

Lemma. The region of attraction for the system (12-19) is given by,

$$\Omega = \left\{ \begin{array}{l} (Y_1, V_1, N_1, Y_2, V_2, N_2, B, E) : 0 \leq Y_1 \leq N_1 \leq A_1 / d_1, 0 \leq V_1 \leq \phi_1 A_1 / d_1 (\phi_1 + d_1 + \theta_1), \\ 0 \leq Y_2 \leq N_2 \leq A_2 / d_2, 0 \leq V_2 \leq \phi_2 A_2 / d_2 (\phi_2 + d_2 + \theta_2), \\ 0 \leq B \leq B_m, 0 \leq E \leq E_m \end{array} \right\} \tag{20}$$

is positively invariant and all solutions starting in the region stay in Ω , where,

$$B_m = \frac{L(E_m)}{2s(E_m)} \left\{ [s(E_m) - s_{20}] + \sqrt{[s(E_m) - s_{20}]^2 + 4 \frac{s_2 A_2 s(E_m)}{d_2 L(E_m)}} \right\} \text{ and } E_m = \frac{Q(A_2 / d_2)}{\delta_0}.$$

The equilibrium analysis of the model system (12-19) has been carried out and the results are given as follows. There exist following three non-negative equilibria of the system (12-19),

$$(1) W_0 \left(0, \frac{\phi_1 A_1}{d_1 (\phi_1 + d_1 + \theta_1)}, \frac{A_1}{d_1}, 0, \frac{\phi_2 A_2}{d_2 (\phi_2 + d_2 + \theta_2)}, \frac{A_2}{d_2}, 0, \frac{Q(A_2 / d_2)}{\delta_0} \right).$$

This is the disease-free equilibrium (DFE) which exists without any condition. The existence of W_0 is obvious.

$$(2) W_1 (\bar{Y}_1, \bar{V}_1, \bar{N}_1, 0, \bar{V}_2, \bar{N}_2, 0, \bar{E}).$$

This is the bacteria-free equilibrium (BFE) and it exists under the following conditions,

$$\frac{\beta_1 A_1}{d_1} > \frac{(\phi_1 + d_1 + \theta_1)(\nu_1 + \alpha_1 + d_1)}{(d_1 + \theta_1)},$$

where

$$\bar{N}_2 = \frac{A_2}{d_2}, \quad \bar{V}_2 = \frac{\phi_2 A_2}{d_2(\phi_2 + d_2 + \theta_2)}, \quad \bar{E} = \frac{Q(A_2/d_2)}{\delta_0},$$

$$\bar{N}_1 = \frac{A_1}{\alpha_1 + d_1} + \frac{\alpha_1(\phi_1 + d_1 + \theta_1)(\nu_1 + \alpha_1 + d_1)}{\beta_1(\alpha_1 + d_1)(d_1 + \theta_1)},$$

$$\bar{Y}_1 = \frac{A_1}{\alpha_1 + d_1} - \frac{d_1(\phi_1 + d_1 + \theta_1)(\nu_1 + \alpha_1 + d_1)}{\beta_1(\alpha_1 + d_1)(d_1 + \theta_1)},$$

$$\bar{V}_1 = \frac{\phi_1(\nu_1 + \alpha_1 + d_1)}{\beta_1(d_1 + \theta_1)}.$$

In this case disease only spreads through direct contacts of susceptible with infectives.

$$(3) \text{ Endemic equilibrium, } W_2 (Y_1^*, V_1^*, N_1^*, Y_2^*, V_2^*, N_2^*, B^*, E^*)$$

We prove the existence of endemic equilibrium by the isocline method. Setting the right hand side of the equations in model (12-19) to zero, we get following algebraic equations,

$$N_1 = \frac{A_1 - \alpha_1 Y_1}{d_1}, \quad V_1 = \frac{\phi_1 [A_1 - (\alpha_1 + d_1) Y_1]}{d_1(\phi_1 + d_1 + \theta_1)}, \quad (21)$$

$$\beta_1 \left(1 + \frac{\alpha_1}{d_1}\right) Y_1^2 - \left\{ \beta_1 \frac{A_1}{d_1} - \frac{(\nu_1 + \alpha_1 + d_1)(\phi_1 + d_1 + \theta_1)}{(d_1 + \theta_1)} \right\} Y_1 + \lambda_1 \left(1 + \frac{\alpha_1}{d_1}\right) Y_1 Y_2 - \lambda_1 \frac{A_1}{d_1} Y_2 = 0 \quad (22)$$

$$N_2 = \frac{A_2 - \alpha_2 Y_2}{d_2}, \quad V_2 = \frac{\phi_2 [A_2 - (\alpha_2 + d_2) Y_2]}{d_2(\phi_2 + d_2 + \theta_2)}, \quad E = \frac{Q(N_2)}{\delta_0}, \quad (23)$$

$$\beta_2 \left(1 + \frac{\alpha_2}{d_2} \right) Y_2^2 - \left\{ \beta_2 \frac{A_2}{d_2} - \frac{(v_2 + \alpha_2 + d_2)(\phi_2 + d_2 + \theta_2)}{(d_2 + \theta_2)} \right\} Y_2 + \lambda_2 \left(1 + \frac{\alpha_2}{d_2} \right) Y_2 B - \lambda_2 \frac{A_2}{d_2} B = 0 \tag{24}$$

$$Y_2 = \frac{1}{s_2} \left\{ \frac{s(E)}{L(E)} B^2 - [s(E) - s_{20}] B \right\} . \tag{25}$$

Now we show the existence of Y_2^* and B^* from Equations (24) and (25), and the corresponding values of Y_1^* , V_1^* , N_1^* , V_2^* , N_2^* and E^* can be obtained from Equations (21), (22) and (23).

From Equation (24), we have

(i) For $B = 0$,

$$Y_2 = 0 \text{ and } Y_2 = \frac{d_2}{\beta_2(\alpha_2 + d_2)} \left\{ \beta_2 \frac{A_2}{d_2} - \frac{(v_2 + \alpha_2 + d_2)(\phi_2 + d_2 + \theta_2)}{(d_2 + \theta_2)} \right\} = \tilde{Y}_2,$$

which is positive, if $\beta_2 \frac{A_2}{d_2} > \frac{(v_2 + \alpha_2 + d_2)(\phi_2 + d_2 + \theta_2)}{(d_2 + \theta_2)}$ and negative otherwise.

(ii) At $Y_2 = \frac{A_2}{(\alpha_2 + d_2)}$, there is a horizontal asymptote.

(iii) At $(0, 0)$, the slope of eq. (24) is given by,

$$\frac{dY_2}{dB} = - \frac{\lambda_2 A_2}{d_2 \left\{ \beta_2 \frac{A_2}{d_2} - \frac{(v_2 + \alpha_2 + d_2)(\phi_2 + d_2 + \theta_2)}{(d_2 + \theta_2)} \right\}},$$

which is positive or negative depending upon \tilde{Y}_2 being negative or positive, respectively.

(iv) At $(0, \tilde{Y}_2)$, the slope of Equation (24) is given by,

$$\frac{dY_2}{dB} = \frac{\lambda_2(v_2 + \alpha_2 + d_2)(\phi_2 + d_2 + \theta_2)}{\beta_2(d_2 + \theta_2) \left\{ \beta_2 \frac{A_2}{d_2} - \frac{(v_2 + \alpha_2 + d_2)(\phi_2 + d_2 + \theta_2)}{(d_2 + \theta_2)} \right\}},$$

which is positive or negative depending upon \tilde{Y}_2 being positive or negative, respectively.

From Equation (25), we observe the following points,

(i) When $Y_2 = 0$,

$$B = 0 \quad \text{and} \quad B = \frac{L(E)[s(E) - s_{20}]}{s(E)} = \tilde{B}.$$

(ii) At $(0, 0)$, the slope of Equation (25) is given by,

$$\frac{dY_2}{dB} = -\frac{[s(E) - s_{20}]}{s_2} < 0.$$

(iii) At $(\tilde{B}, 0)$, the slope of Equation (25) is given by,

$$\frac{dY_2}{dB} = \frac{d_2 \delta_0 L^2(E)[s(E) - s_{20}]}{s_2 d_2 \delta_0 L^2(E) + \alpha_2 Q'(N_2)[s'(E)L(E) - L'(E)s(E)]\tilde{B}},$$

which is positive if

$$\frac{s'(E)}{s(E)} > \frac{L'(E)}{L(E)}.$$

Thus, after plotting Y_2 and B corresponding to Equations (24) and (25) in Figure 1, we see that there are two intersecting points $(0, 0)$ and (Y_2^*, B^*) . After finding Y_2^* and B^* , we can calculate Y_1^* , N_1^* , V_1^* , N_2^* , V_2^* and E^* using eqs. (21), (22) and (23).

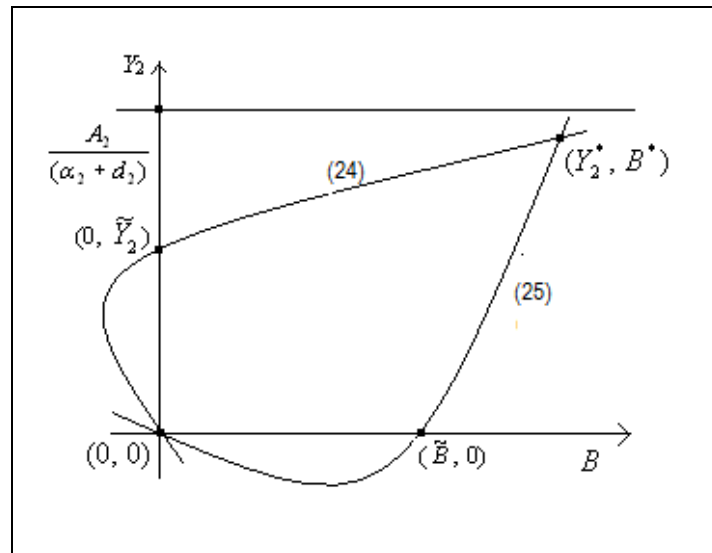


Figure 1 (a). Existence of endemic equilibrium when $Y_2 > 0$.

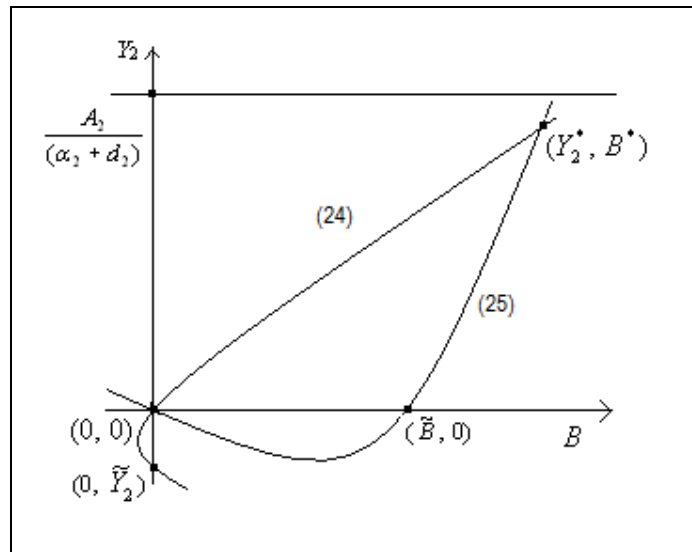


Figure 1 (b). Existence of endemic equilibrium when $Y_2 < 0$.

4. Stability Analysis

Now we analyze the stability of equilibria W_0 , W_1 and W_2 . The local stability results of these equilibria are stated in the following theorem.

Theorem 4.1. The equilibrium W_0 and W_1 are unstable and the endemic equilibrium W_2 is locally asymptotically stable under the following conditions:

$$\lambda_1^2 (N_1^* - Y_1^* - V_1^*)^2 < \frac{1}{3} \left[\beta_1 Y_1^* + \lambda_1 (N_1^* - V_1^*) \frac{Y_2^*}{Y_1^*} \right] \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right]. \tag{26}$$

$$s \lambda_2 (N_2^* - V_2^*)^2 < \frac{1}{2} Y_2^* \left[\frac{s_2 Y_2^*}{B^*} + \frac{s(E^*)}{L(E^*)} \right] \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right]. \tag{27}$$

$$\varphi_1^2 (\beta_1 Y_1^* + \lambda_1 Y_2^*)^2 < \frac{16}{27} (\varphi_1 + d_1 + \theta_1)^2 \left[\beta_1 Y_1^* + \lambda_1 (N_1^* - V_1^*) \frac{Y_2^*}{Y_1^*} \right] \min \left\{ \left[\beta_1 Y_1^* + \lambda_1 (N_1^* - V_1^*) \frac{Y_2^*}{Y_1^*} \right], \frac{d_1 (\beta_1 Y_1^* + \lambda_1 Y_2^*)}{\alpha_1} \right\}. \tag{28}$$

$$\varphi_2^2 (\beta_2 Y_2^* + \lambda_2 Y_2^*)^2 < \frac{1}{9} (\varphi_2 + d_2 + \theta_2)^2 \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right] \min \left\{ \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right], \frac{d_2 (\beta_2 Y_2^* + \lambda_2 B^*)}{\alpha_2} \right\}. \tag{29}$$

(For proof see Appendix - I)

Theorem 4.2. In addition to assumptions (9), (10) and (11), let $s(E)$, $L(E)$ and $Q(N_2)$ satisfy $0 \leq s'(E) \leq p$, $0 \leq L'(E) \leq q$ and $0 \leq Q'(N_2) \leq r$ for some positive constant p , q and r in Ω , then endemic equilibrium W_2 is nonlinearly asymptotically stable in the region Ω provided the following inequalities are satisfied,

$$\alpha_1 \lambda_1^2 A_2^2 < \frac{1}{2} d_1 d_2^2 \beta_1^2 Y_1^{*2} \quad (30)$$

$$\lambda_1^2 (N_1^* - Y_1^* - V_1^*)^2 < \frac{1}{5} \beta_1 \beta_2 Y_1^* \quad (31)$$

$$\alpha_2 \lambda_2^2 B_m^2 < \frac{4}{15} d_2 \beta_2^2 Y_2^{*2} \quad (32)$$

$$\lambda_2 L(E^*) (N_2^* - V_2^*)^2 < \frac{2}{5} \beta_2 s(E^*) B^* Y_2^{*2} \quad (33)$$

$$\alpha_2 \lambda_2 L(E^*) B^* r^2 \left[p \left(1 - \frac{B_m}{L(E^*)} \right) + q \frac{s(E_m) B_m}{L_0^2} \right]^2 < \frac{8}{3} d_2 \beta_2 \delta_0^2 s(E^*) \quad (34)$$

$$\phi_1^2 (\beta_1 Y_1^* + \lambda_1 A_2 / d_2)^2 < \frac{1}{9} \beta_1^2 (\phi_1 + d_1 + \theta_1)^2 Y_1^{*2} \min.(1, 2d_1 / \alpha_1) \quad (35)$$

$$\phi_2^2 (\beta_2 Y_2^* + \lambda_2 B_m)^2 < \frac{16}{45} \beta_2^2 (\phi_2 + d_2 + \theta_2)^2 Y_2^{*2} \min.(1/5, d_2 / 3\alpha_2) \quad (36)$$

(For proof see Appendix-II)

Remark. If the contact rate of the susceptible of the rich class with the infectives of the poor class is very small i.e., $\lambda_1 \rightarrow 0$, then inequalities (26, 30 and 31) are automatically satisfied. If the indirect contact rate of the susceptible of the poor class with bacteria is very small i.e., $\lambda_2 \rightarrow 0$, then inequalities (27, 32 and 33) are automatically satisfied. It is clear from inequality (34) that in the absence of environmental and human population related factors i.e., $p = q = r = 0$, the inequality is automatically satisfied. This implies that the environmental and human population related factors, conducive to the growth of bacteria population, have a destabilizing effect on the system. We also note that due to the presence of vaccinated class, conditions (28), (29) and (35) and (36) are required for the local and nonlinear stability which further destabilize the system.

5. Numerical Simulation

In order to study the dynamical behavior of the model (12-19) and to prove feasibility of stability conditions, we have conducted numerical simulation for the set of parameters given in Table 1, using MAPLE 7.0, Feng et al. (2002), Bowman et. al. (2005), Ghosh et al. (2005).

In the model, $s(E)$ and $L(E)$ are the growth rates and modified carrying capacity of the bacteria population and are functions of the cumulative density of the environmental discharge E . The rate of the cumulative environmental discharges is also a function of the population density N_2 of

the poor class. Thus, for numerical simulation it is assumed that $s(E)$ and $L(E)$ are linear functions of E , i.e., $s(E) = s_0 + aE$ and $L(E) = L_0 + bE$, satisfying conditions (9) and (10). We have also assumed $Q(N_2)$ to be a linear function of N_2 such as $Q(N_2) = Q_0 + lN_2$, satisfying condition (11).

Table 1. Parameter Values

Parameters	Symbol	Parameter value
Recruitment rate of susceptibles in rich class	A_1	100
Recruitment rate of susceptibles in poor class	A_2	100
Transmission coefficient due to contacts of susceptibles with infectives in rich class	β_1	0.002
Transmission coefficient due to contacts of susceptibles with infectives in poor class	β_2	0.003
Transmission coefficient due to contacts of susceptibles in rich class with infectives in poor class	λ_1	0.000005
Transmission coefficient due to contacts of susceptibles with bacteria in poor class	λ_2	0.00001
Recovery rate of infected individuals in rich class	ν_1	0.02
Recovery rate of infected individuals in poor class	ν_2	0.01
Natural death rate of individuals in rich class	d_1	0.15
Natural death rate of individuals in poor class	d_2	0.13
Disease-induced death rate in rich class	α_1	0.2
Disease-induced death rate in poor class	α_2	0.25
Vaccination coverage (of susceptibles) of rich population	ϕ_1	0.05
Vaccination coverage (of susceptibles) of poor population	ϕ_2	0.04
Rate at which vaccinated individuals of rich population again become susceptible	θ_1	0.001
Rate at which vaccinated individuals of poor population again become susceptible	θ_2	0.0011
Growth rate of bacteria population ($s(E) = s_0 + aE$)	s_0	0.85
	a	0.001
Rate of release of bacteria from infected individuals of poor class	s_2	0.0001
Decay rate of bacteria in the environment	s_{20}	0.3
Carrying capacity of the bacterial population in the natural environment ($L(E) = L_0 + bE$)	L_0	10000
	b	0.01
Rate of cumulative environmental discharges conducive to the growth of bacteria into the poor population ($Q(N_2) = Q_0 + lN_2$)	Q_0	25
	l	0.002
Depletion rate coefficient of the cumulative environmental discharges	δ_0	0.1

The equilibrium values for the model system (12-19) are computed as follows,

$$Y_1^* = 180.4827434, V_1^* = 61.07966804, N_1^* = 426.0230089, Y_2^* = 211.1081722,$$

$$V_2^* = 35.56875341, N_2^* = 363.2535149, B^* = 7292.523432, E^* = 257.2650703.$$

The eigenvalues of variational matrix corresponding to the endemic equilibrium for the model system (12-19) are

$$\begin{aligned} & -0.2814919115 + i0.2181458923, \quad -0.2814919115 - i0.2181458923, \quad -0.1511160098, \\ & -0.582490291 + i0.2552674044, \quad -0.582490291 - i0.2552674044, \quad -0.1311104566, \\ & -0.1000192890, \quad -0.8072630391. \end{aligned}$$

Since all the eigenvalues are negative or have negative real parts, it implies that the endemic equilibrium W_2 is locally asymptotically stable for the above set of parameter values.

The computer simulation is performed for different initial starts in the following four cases and displayed graphically in figs. 2 and 3. In these figures, the variation of infectives with the total population of rich and poor classes is shown respectively. The trajectories starting with different initial starts reach the equilibrium point. Thus, the system (12-19) is nonlinearly asymptotically stable for the above set of parameter values.

1. $Y_1(0) = 100, V_1(0) = 60, N_1(0) = 300, Y_2(0) = 150, V_2(0) = 20, N_2(0) = 200, B(0) = 7000,$
 $E(0) = 250.$

2. $Y_1(0) = 300, V_1(0) = 60, N_1(0) = 600, Y_2(0) = 250, V_2(0) = 20, N_2(0) = 600, B(0) = 7000,$
 $E(0) = 250.$

3. $Y_1(0) = 300, V_1(0) = 60, N_1(0) = 350, Y_2(0) = 150, V_2(0) = 20, N_2(0) = 500, B(0) = 7000,$
 $E(0) = 250.$

4. $Y_1(0) = 100, V_1(0) = 60, N_1(0) = 550, Y_2(0) = 270, V_2(0) = 20, N_2(0) = 300, B(0) = 7000,$
 $E(0) = 250.$

The results of numerical simulation are displayed graphically in Figures 4-9. In Figures 4 and 5, the variation of bacteria population and infective population of the rich and poor classes is shown with time, respectively, for different growth rates of cumulative density of environmental discharges. It is seen that as the growth rate of environmental discharges increases, bacteria population increases. With the increase in bacteria population, the spread of disease also increases in infective population of poor class. When this infective population of the poor class, i.e., service providers interacts with the susceptible of the rich population at a higher rate, their infective population also increases. This implies that the increased bacteria density in poor class due to growth of unhygienic environmental discharges results in increasing the infectives in poor class and as such disease spread is faster in this class.

Consequently, the population in rich class, living in much cleaner environment, is directly affected by the higher number of infectives available from poor class who work as service providers. In Figure 6, we have shown the decay of bacterial population density due to natural

factors or control measures s_{20} with time. It is observed that as the impact of control measures decreases, the number of infectives in poor class increases leading to higher number of infectives in rich class. Figure 7 depicts the role of bacteria released from the infectives of poor class (s_2), who work as service providers. This additional load of bacteria population density further increases the infective population in the poor class which ultimately leads to enhance the disease spread in rich population with increased interaction rate with service providers. Here it may be noted that since the environment of rich class is comparatively clean and hygienic, the increased disease spread in the class of service providers will impact the rich class only if the higher number of infected service providers deliver service to them.

It is, therefore, speculated that not only the growth of the bacteria population due to environmental considerations or its release from the infectives need to be curbed using effective control mechanism but the direct interaction of the susceptible of the rich class with infected service providers should also be restricted. In Figures 8-9, the effect of vaccination is shown on the vaccinated and the infective population of the poor class. It is found that as the vaccination rate ϕ_2 increases, the vaccinated population increases tremendously and consequently the infective population declines. A similar observation is made for vaccination rate ϕ_1 in the rich class.

From the above discussion, we infer that in order to control the spread of bacterial infection in a population where servants and maids working as service providers act as carrier of infection, apart from the introduction of proper vaccination strategy in the population, the environmental conditions in which the service providers live be improved so that the bacteria do not get a conducive environment to grow or accumulate in the atmosphere. Moreover, people from the poor class who are infected with the disease be restricted to act as service providers in the houses of rich people in order to keep the disease spread at minimum.

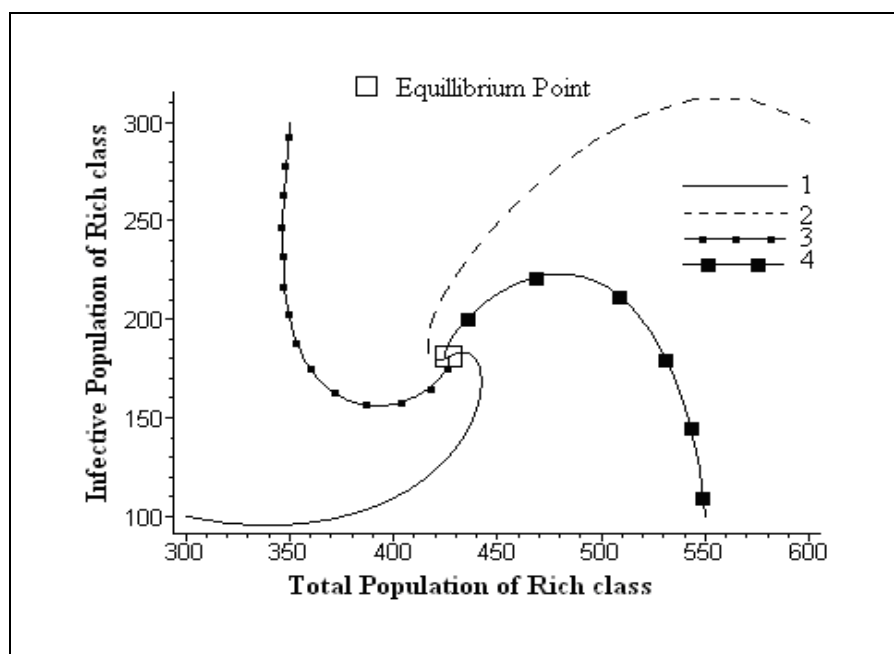


Figure 2. Variation of total population with infective population of rich class

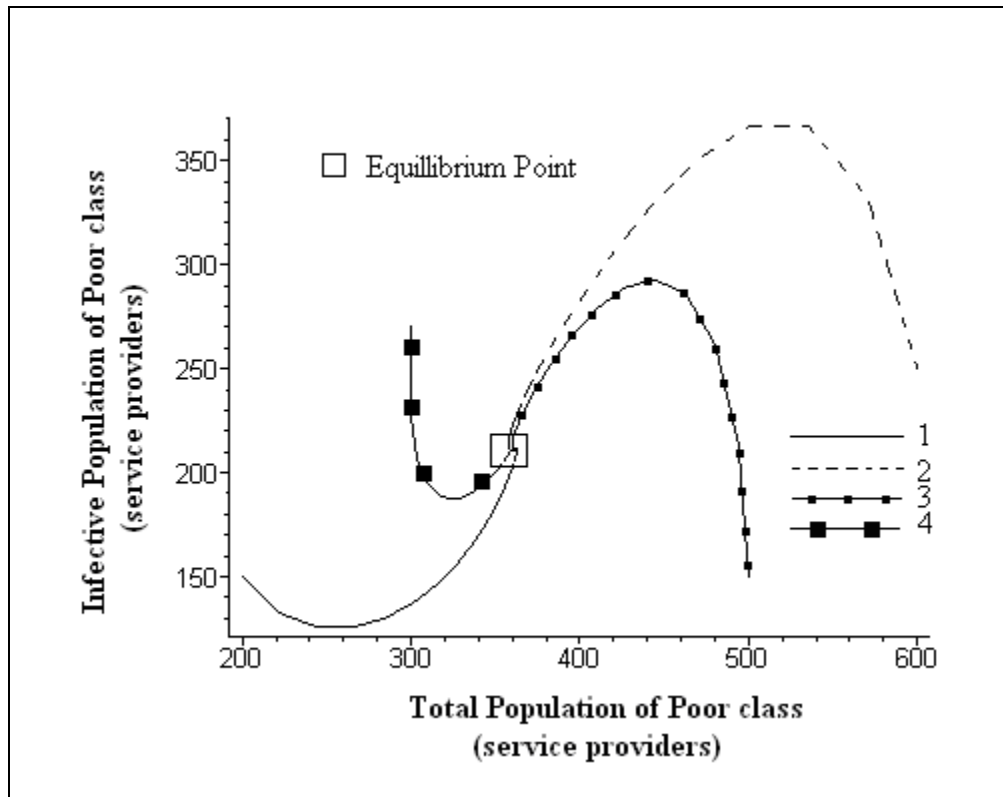


Figure 3. Variation of total population with infective population of poor class

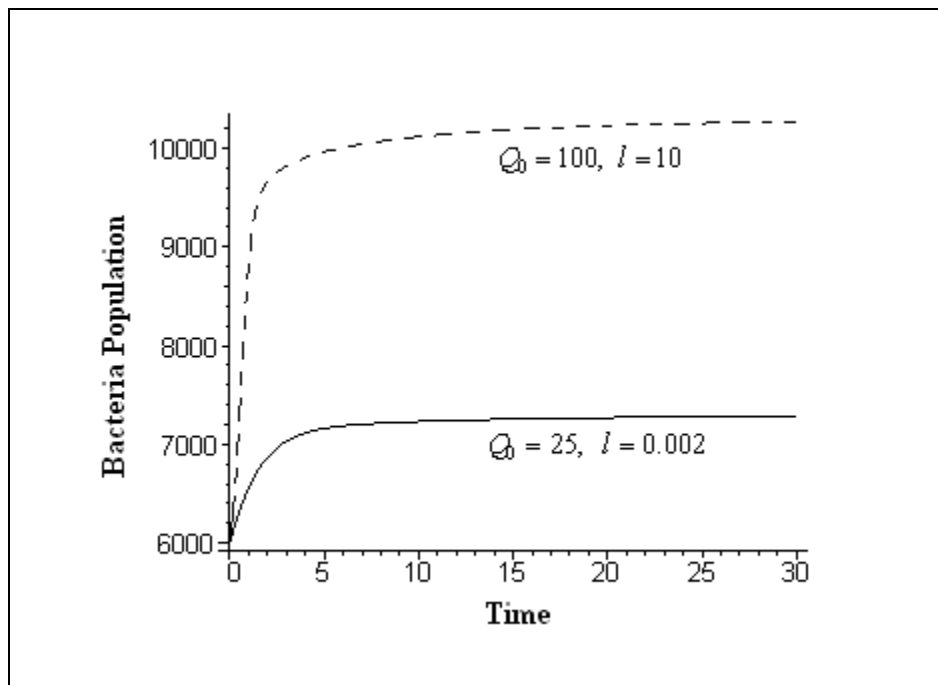


Figure 4. Variation of bacteria population with time for different values of $Q(N_2)$ i.e., $Q_0 + lN_2$

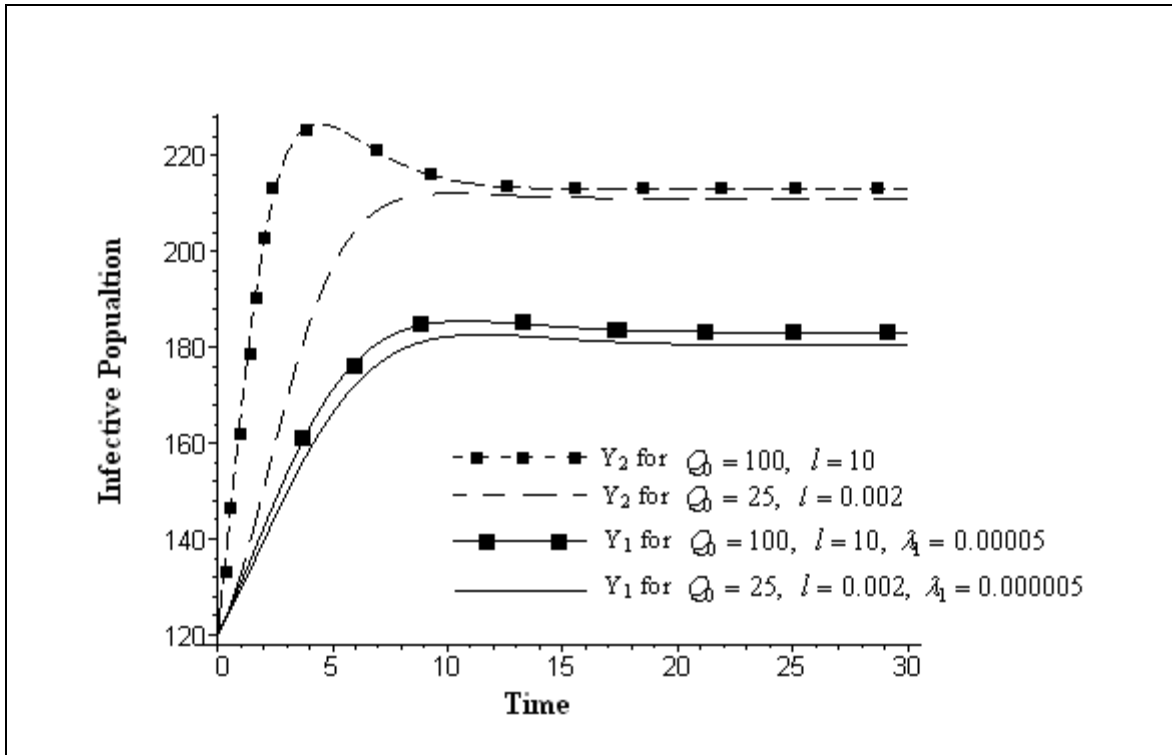


Figure 5. Variation of infective population with time for different values of $Q(N_2)$ i.e., $Q_0 + IN_2$

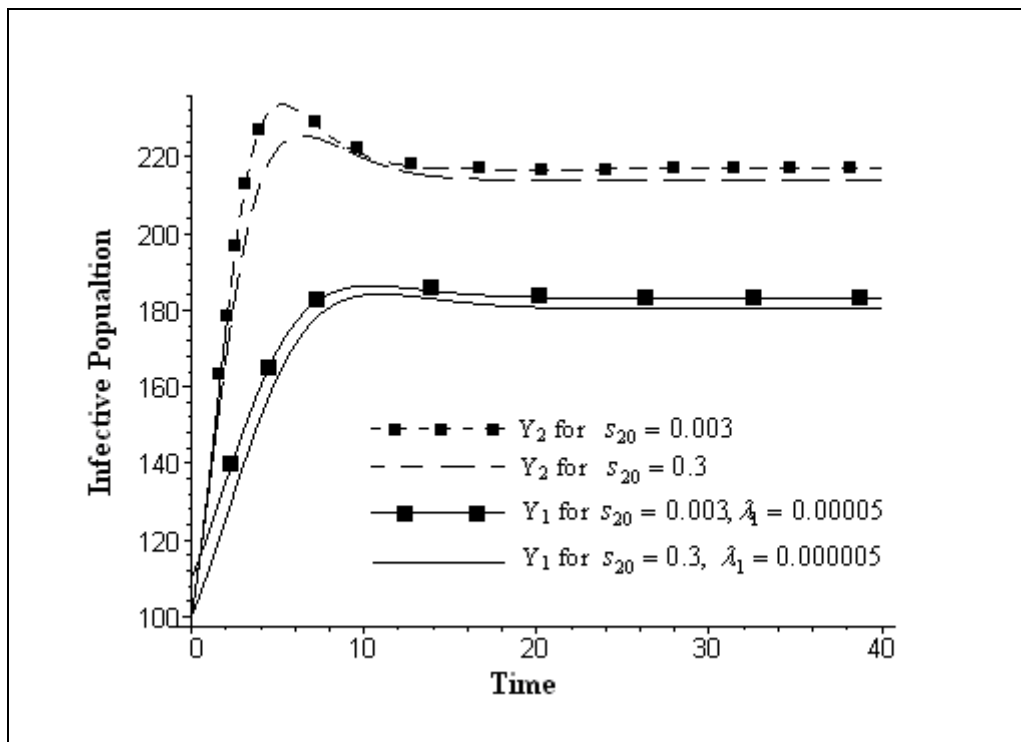


Figure 6. Variation of infective population with time for different values of s_{20}

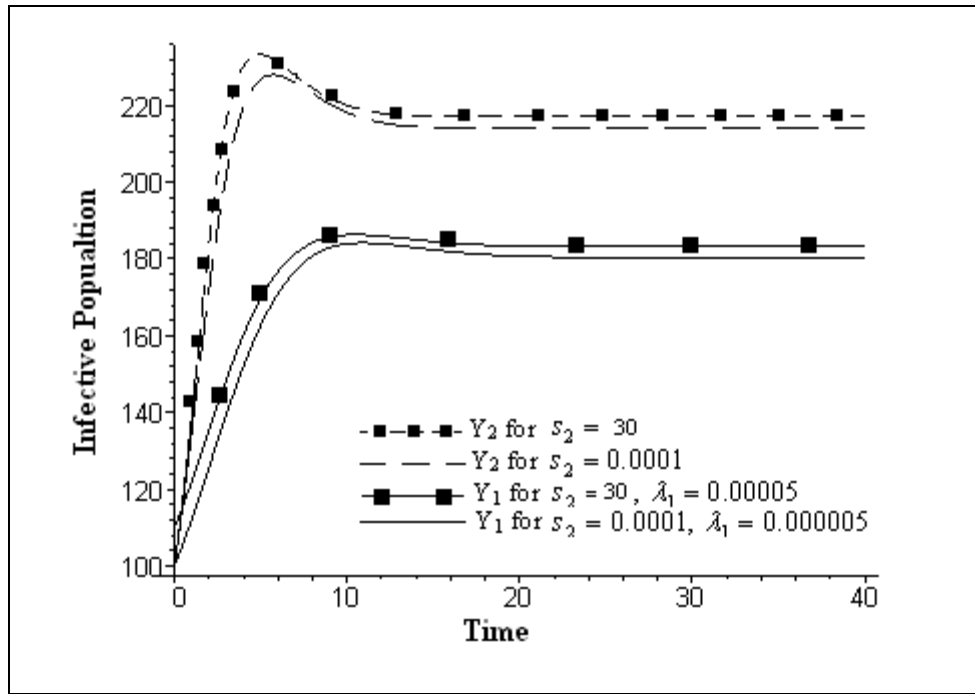


Figure 7. Variation of infective population with time for different values of s_2

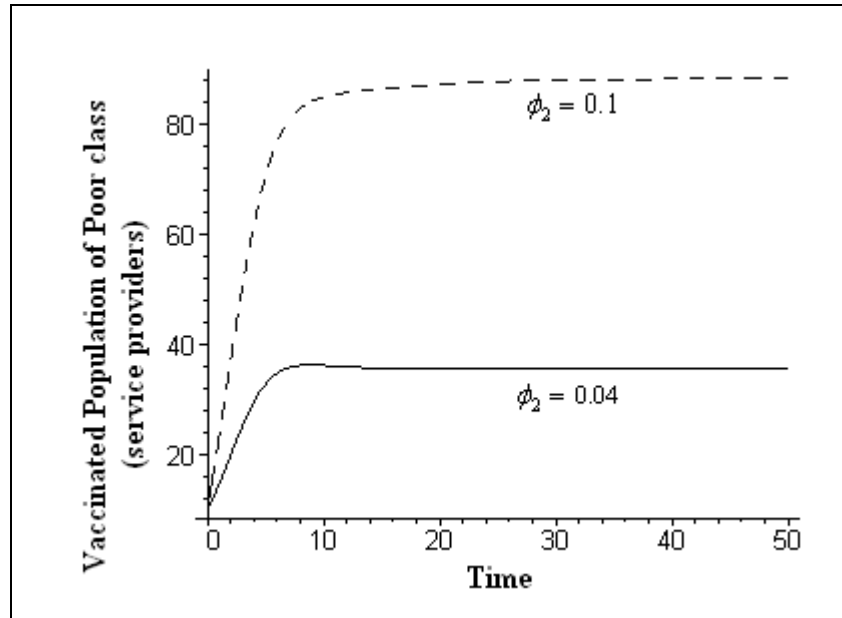


Figure 8. Variation of vaccinated poor population with time for different values of ϕ_2

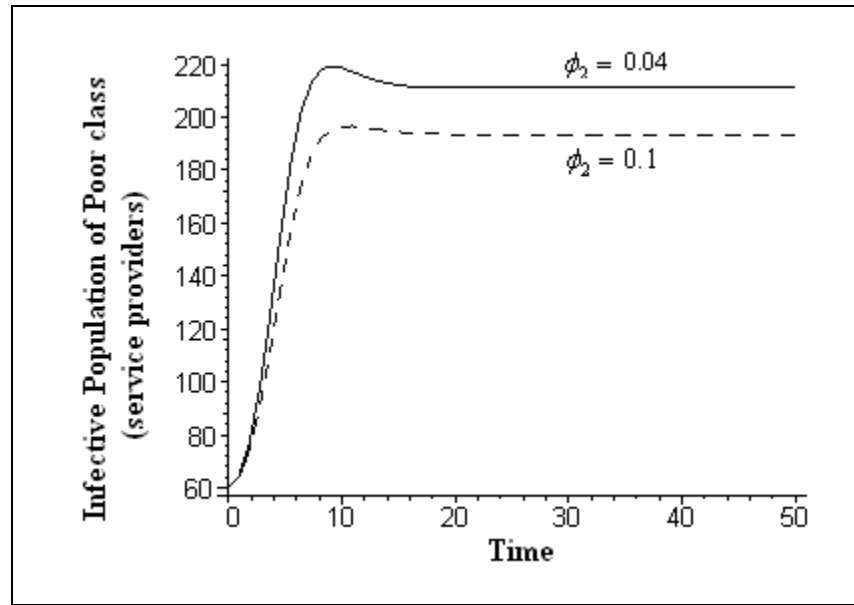


Figure 9. Variation of infective poor population with time for different values of ϕ_2

6. Conclusions

In this paper, a nonlinear mathematical model is proposed and analyzed to study the effect of environmental factors on the spread of bacterial disease in an economically structured population (rich and poor) where people from the poor population work as service providers in the houses of the rich people. It is assumed that the susceptible from the rich class get infected through direct contacts with infectives in same class and with infectives from the poor class who work as service providers. The susceptible in the poor class get infected through direct contacts with the infectives in their own class as well as by bacteria present in their unhygienic environment. The density of the bacteria population is assumed to be governed by a logistic model and is dependent on the environmental factors which are conducive to the growth of the bacteria population. The cumulative density of the environmental discharges depends upon the human population density related factors in the poor class. The model is analyzed using stability theory of differential equations and numerical simulation. The analysis shows that a disease-free equilibrium (DFE) and bacteria-free equilibrium (BFE) are always unstable whereas endemic equilibrium is locally as well as nonlinearly asymptotically stable under certain conditions. Further, the environmental as well as human population related factors conducive to the growth of bacteria population have a destabilizing effect on the system. It is found that increased growth rates of environmental discharges increase the bacteria population density. This increase of bacteria density in the poor class due to unhygienic environmental discharges results in increasing the infectives in the poor class i.e., service providers. As a consequence, these service providers further escalate the disease in the rich class, living in much cleaner environment.

It is suggested that along with a suitable vaccination strategy, the environmental condition of the region inhabited by service providers be improved so that the bacteria do not get a conducive environment to grow and in order to significantly reduce the disease spread.

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APPENDIX - I

Proof of the Theorem 4.1.

(i) The variational matrix M_0 of model (12-19) corresponding to equilibrium W_0 is given by,

$$M_0 = \begin{bmatrix} m_{11} & 0 & 0 & \frac{\lambda_1 A_1 (d_1 + \theta_1)}{d_1 (\phi_1 + d_1 + \theta_1)} & 0 & 0 & 0 & 0 \\ -\phi_1 & -(\phi_1 + d_1 + \theta_1) & \phi_1 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & -d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 & 0 & \frac{\lambda_1 A_1 (d_1 + \theta_1)}{d_1 (\phi_1 + d_1 + \theta_1)} & 0 \\ 0 & 0 & 0 & -\phi_2 & -(\phi_2 + d_2 + \theta_2) & \phi_2 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_2 & 0 & -d_2 & 0 & 0 \\ 0 & 0 & 0 & s_2 & 0 & 0 & s \left(\frac{Q(N_2)}{\delta_0} \right) & 0 \\ 0 & 0 & 0 & 0 & 0 & Q' \left(\frac{A_2}{d_2} \right) & 0 & -\delta_0 \end{bmatrix},$$

where

$$m_{11} = \frac{\beta_1 A_1 (d_1 + \theta_1)}{d_1 (\phi_1 + d_1 + \theta_1)} - (v_1 + \alpha_1 + d_1), \quad m_{44} = \frac{\beta_2 A_2 (d_2 + \theta_2)}{d_2 (\phi_2 + d_2 + \theta_2)} - (v_2 + \alpha_2 + d_2).$$

The characteristic equation corresponding to above matrix is given by,

$$(d_1 + \psi)(\phi_1 + d_1 + \theta_1 + \psi)(\delta_0 + \psi)(d_2 + \psi)(\phi_2 + d_2 + \theta_2 + \psi)(m_{11} - \psi)(\psi^2 + h_1\psi + h_2) = 0,$$

where

$$h_1 = - \left\{ m_{44} + s \left(\frac{Q(A_2/d_2)}{\delta_0} \right) - s_{20} \right\}$$

$$h_2 = m_{44} \left[s \left(\frac{Q(A_2/d_2)}{\delta_0} \right) - s_{20} \right] - \frac{s_2 \lambda_2 A_2 (d_2 + \theta_2)}{d_2 (\phi_2 + d_2 + \theta_2)}$$

Using Routh-Hurwitz criteria, this equilibrium is unstable because in the above quadratic, the coefficient of ψ and the constant term are not positive simultaneously.

(ii) The variational matrix M_1 of model (12-19) corresponding to equilibrium W_1 is given by,

$$M_1 = \begin{bmatrix} M_{11} & M_{12} \\ O & M_{44} \end{bmatrix},$$

where

$$M_{11} = \begin{bmatrix} -\beta_1 \bar{Y}_1 - \lambda_1 (\bar{N}_1 - \bar{Y}_1 - \bar{V}_1) \frac{\bar{Y}_2}{\bar{Y}_1} & -\beta_1 \bar{Y} & -\beta_1 \bar{Y} \\ -\phi_1 & -(\phi_1 + d_1 + \theta_1) & \phi_1 \\ -\alpha_1 & 0 & -d_1 \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} \frac{\lambda_1 A_1 (d_1 + \theta_1)}{d_1 (\phi_1 + d_1 + \theta_1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$M_{44} = \begin{bmatrix} m_{44} & 0 & 0 & \frac{\lambda_2 A_2 (d_2 + \theta_2)}{d_2 (\phi_2 + d_2 + \theta_2)} & 0 \\ -\phi_2 & -(\phi_2 + d_2 + \theta_2) & \phi_2 & 0 & 0 \\ -\alpha_2 & 0 & -d_2 & 0 & 0 \\ s_2 & 0 & 0 & s \left(\frac{Q(A_2/d_2)}{\delta_0} \right) - s_{20} & 0 \\ 0 & 0 & Q' \left(\frac{A_2}{d_2} \right) & 0 & -\delta_0 \end{bmatrix}.$$

The partition matrix will give rise to following characteristic equation,

$$\begin{vmatrix} -\left(\beta_1 \bar{Y}_1 + \lambda_1(\bar{N}_1 - \bar{Y}_1 - \bar{V}_1) \frac{\bar{Y}_2}{\bar{Y}_1} + \psi\right) & -\beta_1 \bar{Y} & -\beta_1 \bar{Y} & & & \\ & -\phi_1 & -(\phi_1 + d_1 + \theta_1 + \psi) & \phi_1 & & \\ & -\alpha_1 & 0 & -(d_1 + \psi) & & \\ & & & & & \\ m_{44} - \psi & 0 & 0 & \frac{\lambda_2 A_2 (d_2 + \theta_2)}{d_2 (\phi_2 + d_2 + \theta_2)} & 0 & \\ -\phi_2 & -(\phi_2 + d_2 + \theta_2 + \psi) & \phi_2 & 0 & 0 & \\ -\alpha_2 & 0 & -(d_2 + \psi) & 0 & 0 & \\ s_2 & 0 & 0 & s\left(\frac{Q(A_2 / d_2)}{\delta_0}\right) - s_{20} - \psi & 0 & \\ 0 & 0 & Q\left(\frac{A_2}{d_2}\right) & 0 & 0 & -(\delta_0 + \psi) \end{vmatrix} = 0$$

The characteristic equation corresponding to M_1 is given by,

$$(\delta_0 + \psi)(d_2 + \psi)(\phi_2 + d_2 + \theta_2 + \psi)(\psi^2 - h_1\psi + h_2)(\psi^3 + a_1\psi^2 + a_2\psi + a_3) = 0,$$

where

$$a_1 = \phi_1 + 2d_1 + \theta_1 + \beta_1 \bar{Y}_1 + \lambda_1(\bar{N}_1 - \bar{Y}_1 - \bar{V}_1) \frac{\bar{Y}_2}{\bar{Y}_1} > 0,$$

$$a_2 = d_1(\phi_1 + d_1 + \theta_1) + (\alpha_1 + 2d_1 + \theta_1) \left\{ \beta_1 \bar{Y}_1 + \lambda_1(\bar{N}_1 - \bar{Y}_1 - \bar{V}_1) \frac{\bar{Y}_2}{\bar{Y}_1} \right\} > 0,$$

$$a_3 = (\alpha_1 + d_1)(d_1 + \theta_1) \left\{ \beta_1 \bar{Y}_1 + \lambda_1(\bar{N}_1 - \bar{Y}_1 - \bar{V}_1) \frac{\bar{Y}_2}{\bar{Y}_1} \right\} > 0$$

$$h_1 = \left\{ m_{44} + s\left(\frac{Q(A_2 / d_2)}{\delta_0}\right) - s_{20} \right\},$$

$$h_2 = m_{44} \left[s\left(\frac{Q(A_2 / d_2)}{\delta_0}\right) - s_{20} \right] - \frac{s_2 \lambda_2 A_2 (d_2 + \theta_2)}{d_2 (\phi_2 + d_2 + \theta_2)}.$$

Using Routh-Hurwitz criteria, this equilibrium is unstable because in the above quadratic, the coefficient of ψ and constant term are not positive simultaneously although $a_1 a_2 - a_3 > 0$.

To establish the local stability of endemic equilibrium W_2 , we consider the following positive definite function,

$$U_1 = \frac{1}{2}(k_0 y_1^2 + k_1 v_1^2 + k_2 n_1^2 + k_3 y_2^2 + k_4 v_2^2 + k_5 n_2^2 + k_6 b^2 + k_7 e^2),$$

where k_i ($i = 0 - 7$) are positive constants to be chosen appropriately and $y_1, v_1, n_1, y_2, v_2, n_2, b$ and e are small perturbations about W_2 , defined as follows

$$Y_1 = Y_1^* + y_1, V_1 = V_1^* + v_1, N_1 = N_1^* + n_1, Y_2 = Y_2^* + y_2, V_2 = V_2^* + v_2, N_2 = N_2^* + n_2, B = B^* + b$$

and $E = E^* + e$.

Differentiating above equation, with respect to 't' and using the linearized system of model equations (12-19) corresponding to W_2 , we get,

$$\begin{aligned} \frac{dU_1}{dt} = & -k_0 \left[\beta_1 Y_1^* + \lambda_1 (N_1^* - V_1^*) \frac{Y_2^*}{Y_1^*} \right] y_1^2 - k_1 (\phi_1 + d_1 + \theta_1) v_1^2 - k_2 d_1 n_1^2 - k_3 \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right] y_2^2 \\ & - k_4 (\phi_2 + d_2 + \theta_2) v_2^2 - k_5 d_2 n_2^2 - k_6 \left[\frac{s_2 Y_2^*}{B^*} + \frac{s(E^*)}{L(E^*)} \right] b^2 - k_7 \delta_0 e^2 + [k_0 (\beta_1 Y_1^* + \lambda_1 Y_2^*) - k_2 \alpha_1] y_1 n_1 \\ & + [-k_0 (\beta_1 Y_1^* + \lambda_1 Y_2^*) - k_1 \phi_1] y_1 v_1 + k_0 \lambda_1 (N_1^* - Y_1^* - V_1^*) y_1 y_2 + k_1 \phi_1 v_1 n_1 + [k_3 (\beta_2 Y_2^* + \lambda_2 B^*) - k_5 \alpha_2] y_2 n_2 \\ & + [-k_3 (\beta_2 Y_2^* + \lambda_2 B^*) - k_4 \phi_2] y_2 v_2 + [k_3 \lambda_2 (N_2^* - Y_2^* - V_2^*) + k_6 s] y_2 b + k_4 \phi_2 v_2 n_2 + k_7 Q'(N_2^*) n_2 e \\ & + k_6 \left[s'(E^*) B^* - \frac{(s'(E^*) L(E^*) - L'(E^*) s(E^*)) B^{*2}}{L^2(E^*)} \right] b e \end{aligned}$$

Assuming $k_0 = 1$, $k_2 = \frac{(\beta_1 Y_1^* + \lambda_1 Y_2^*)}{\alpha_1}$, $k_3 = 1$, $k_5 = \frac{(\beta_2 Y_2^* + \lambda_2 B^*)}{\alpha_2}$ and $k_6 = \frac{\lambda_2 Y_2^*}{s}$, the above equation reduces to the form,

$$\begin{aligned} \frac{dU_1}{dt} = & - \left[\beta_1 Y_1^* + \lambda_1 (N_1^* - V_1^*) \frac{Y_2^*}{Y_1^*} \right] y_1^2 - k_1 (\phi_1 + d_1 + \theta_1) v_1^2 - \frac{(\beta_1 Y_1^* + \lambda_1 Y_2^*) d_1}{\alpha_1} n_1^2 \\ & - \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right] y_2^2 - k_4 (\phi_2 + d_2 + \theta_2) v_2^2 - \frac{(\beta_2 Y_2^* + \lambda_2 Y_2^*) d_2}{\alpha_2} n_2^2 \\ & - \frac{\lambda_2 Y_2^*}{s} \left[\frac{s_2 Y_2^*}{B^*} + \frac{s(E^*)}{L(E^*)} \right] b^2 - k_7 \delta_0 e^2 + [-(\beta_1 Y_1^* + \lambda_1 Y_2^*) - k_1 \phi_1] y_1 v_1 \\ & + \lambda_1 (N_1^* - Y_1^* - V_1^*) y_1 y_2 + k_1 \phi_1 v_1 n_1 + [-(\beta_2 Y_2^* + \lambda_2 B^*) - k_4 \phi_2] y_2 v_2 \\ & + \lambda_2 (N_2^* - V_2^*) y_2 b + k_4 \phi_2 v_2 n_2 + k_7 Q'(N_2^*) n_2 e + k_6 \left[s'(E^*) B^* - \frac{(s'(E^*) L(E^*) - L'(E^*) s(E^*)) B^{*2}}{L^2(E^*)} \right] b e \end{aligned}$$

Now we choose k_1, k_4, k_7 , such that,

$$\frac{9(\beta_1 Y_1^* + \lambda_1 Y_2^*)^2}{4(\phi_1 + d_1 + \theta_1) \left[\beta_1 Y_1^* + \lambda_1 (N_1^* - V_1^*) \frac{Y_2^*}{Y_1^*} \right]} < k_1 < \frac{4(\phi_1 + d_1 + \theta_1)}{3\phi_1^2} \min \left\{ \frac{1}{3} \left[\beta_1 Y_1^* + \lambda_1 (N_1^* - V_1^*) \frac{Y_2^*}{Y_1^*} \right], \frac{d_1 (\beta_1 Y_1^* + \lambda_1 Y_2^*)}{\alpha_1} \right\},$$

$$\frac{3(\beta_2 Y_2^* + \lambda_2 B^*)^2}{(\phi_2 + d_2 + \theta_2) \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right]} < k_4 < \frac{(\phi_2 + d_2 + \theta_2)}{3\phi_2^2} \min \left\{ \frac{1}{3} \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right], \frac{2d_2 (\beta_2 Y_2^* + \lambda_2 B^*)}{\alpha_2} \right\},$$

And

$$\frac{\lambda_2 Y_2^* \left[s'(E^*) B^* - \frac{(s'(E^*) L(E^*) - L'(E^*) s(E^*)) B^{*2}}{L^2(E^*)} \right]^2}{s \delta_0 \left[\frac{s_2 Y_2^*}{B^*} + \frac{s(E^*)}{L(E^*)} \right]} < k_7 < \frac{\delta_0 d_2 (\beta_2 Y_2^* + \lambda_2 B^*)}{3\alpha_2}.$$

Thus $\frac{dU_1}{dt}$ is negative definite function under the following conditions,

$$\lambda_1^2 (N_1^* - Y_1^* - V_1^*)^2 < \frac{1}{3} \left[\beta_1 Y_1^* + \lambda_1 (N_1^* - V_1^*) \frac{Y_2^*}{Y_1^*} \right] \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right]$$

$$s \lambda_2 (N_2^* - V_2^*)^2 < \frac{1}{2} Y_2^* \left[\frac{s_2 Y_2^*}{B^*} + \frac{s(E^*)}{L(E^*)} \right] \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right]$$

$$\begin{aligned}
 (\beta_1 Y_1^* + \lambda_1 Y_2^*)^2 &< \frac{16 (\varphi_1 + d_1 + \theta_1)^2}{27 \varphi_1^2} \left[\beta_1 Y_1^* + \lambda_1 (N_1^* - V_1^*) \frac{Y_2^*}{Y_1^*} \right] \times \\
 &\quad \min \left\{ \left[\beta_1 Y_1^* + \lambda_1 (N_1^* - V_1^*) \frac{Y_2^*}{Y_1^*} \right], \frac{d_1 (\beta_1 Y_1^* + \lambda_1 Y_2^*)}{\alpha_1} \right\}, \\
 (\beta_2 Y_2^* + \lambda_2 Y_2^*)^2 &< \frac{1 (\varphi_2 + d_2 + \theta_2)^2}{9 \varphi_2^2} \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right] \times \\
 &\quad \min \left\{ \left[\beta_2 Y_2^* + \lambda_2 (N_2^* - V_2^*) \frac{B^*}{Y_2^*} \right], \frac{d_2 (\beta_2 Y_2^* + \lambda_2 B^*)}{\alpha_2} \right\},
 \end{aligned}$$

showing that U_1 is a Lyapunov function with respect to W_2 , proving the theorem.

APPENDIX – II

Proof of the Theorem 4.2

Consider the following positive definite function, corresponding to the model system (12-19) about W_2 ,

$$\begin{aligned}
 U_2 = k_0 \left(Y_1 - Y_1^* - Y_1^* \ln \frac{Y_1}{Y_1^*} \right) &+ \frac{k_1}{2} (V_1 - V_1^*)^2 + \frac{k_2}{2} (N_1 - N_1^*)^2 + k_3 \left(Y_2 - Y_2^* - Y_2^* \ln \frac{Y_2}{Y_2^*} \right) \\
 &+ \frac{k_4}{2} (V_2 - V_2^*)^2 + \frac{k_5}{2} (N_2 - N_2^*)^2 + \frac{k_6}{2} (B - B^*)^2 + k_7 \left(E - E^* - E^* \ln \frac{E}{E^*} \right)
 \end{aligned}$$

where the coefficients k_i ($i = 0 - 7$) are positive constants to be chosen appropriately.

Differentiating the above equation with respect to 't' and using (12-19) we get,

$$\begin{aligned}
\frac{dU_2}{dt} = & -\frac{k_0\lambda_1Y_2(N_1-V_1)(Y_1-Y_1^*)^2}{Y_1Y_1^*} - \frac{k_3\lambda_2B(N_2-V_2)(Y_2-Y_2^*)^2}{Y_2Y_2^*} - \frac{k_6s_2Y_2(B-B^*)^2}{BB^*} \\
& -k_0\beta_1(Y_1-Y_1^*)^2 - k_1(\varphi_1+d_1+\theta_1)(V_1-V_1^*)^2 - k_2d_1(N_1-N_1^*)^2 - k_3\beta_2(Y_2-Y_2^*)^2 \\
& -k_4(\varphi_2+d_2+\theta_2)(V_2-V_2^*)^2 - k_5d_2(N_2-N_2^*)^2 - k_6\frac{s(E^*)}{L(E^*)}(B-B^*)^2 - k_7\delta_0(E-E^*)^2 \\
& + \left\{ k_0 \left(\beta_1 + \lambda_1 \frac{Y_2}{Y_1^*} \right) - k_2\alpha_1 \right\} (Y_1 - Y_1^*)(N_1 - N_1^*) - \left\{ k_0 \left(\beta_1 + \lambda_1 \frac{Y_2}{Y_1^*} \right) + k_1\varphi_1 \right\} (Y_1 - Y_1^*)(V_1 - V_1^*) \\
& + \frac{k_0\lambda_1(N_1^* - Y_1^* - V_1^*)}{Y_1^*} (Y_1 - Y_1^*)(Y_2 - Y_2^*) + k_1\varphi_1(V_1 - V_1^*)(N_1 - N_1^*) \\
& + \left\{ k_3 \left(\beta_2 + \lambda_2 \frac{B}{Y_2^*} \right) - k_5\alpha_2 \right\} (Y_2 - Y_2^*)(N_2 - N_2^*) - \left\{ k_3 \left(\beta_2 + \lambda_2 \frac{B}{Y_2^*} \right) + k_4\varphi_2 \right\} (Y_2 - Y_2^*)(V_2 - V_2^*) \\
& + \left\{ \frac{k_3\lambda_2(N_2^* - Y_2^* - V_2^*)}{Y_2^*} + \frac{k_6}{B^*} \right\} (Y_2 - Y_2^*)(B - B^*) + k_4\varphi_2(V_2 - V_2^*)(N_2 - N_2^*) \\
& + \left\{ k_6[f(E) + g(E)s(E)B] - k_6\frac{f(E)B}{L(E^*)} \right\} (B - B^*)(E - E^*) + k_7h(N_2)(E - E^*)(N_2 - N_2^*),
\end{aligned}$$

where $f(E)$, $g(E)$ and $h(N_2)$ are defined as follows,

$$f(E) = \begin{cases} \frac{s(E) - s(E^*)}{(E - E^*)}, & E \neq E^* \\ \frac{ds}{dE}, & E = E^* \end{cases} \quad (a)$$

$$g(E) = \begin{cases} \frac{L(E) - L(E^*)}{(E - E^*)} \cdot \frac{1}{L(E)L(E^*)}, & E \neq E^* \\ \frac{1}{L^2(E^*)} \cdot \frac{dL}{dE}, & E = E^* \end{cases} \quad (b)$$

$$h(N_2) = \begin{cases} \frac{Q(N_2) - Q(N_2^*)}{(N_2 - N_2^*)}, & N_2 \neq N_2^* \\ \frac{dQ}{dN_2}, & N_2 = N_2^*. \end{cases} \quad (c)$$

Then by considering the assumptions of the theorem and the mean value theorem, we have,

$$|f(E)| \leq p, \quad |g(E)| \leq \frac{q}{L_0^2} \quad \text{and} \quad |h(N_2)| \leq r. \tag{d}$$

Now assuming $k_0 = 1, k_2 = \frac{\beta_1}{\alpha_1}, k_3 = 1, k_5 = \frac{\beta_2}{\alpha_2}$ and $k_6 = \lambda_2 B^*$, the above equation reduces to the form,

$$\begin{aligned} \frac{dU_2}{dt} = & -\frac{\lambda_1 Y_2 (N_1 - V_1)(Y_1 - Y_1^*)^2}{Y_1 Y_1^*} - \frac{\lambda_2 B (N_2 - V_2)(Y_2 - Y_2^*)^2}{Y_2 Y_2^*} - \frac{s_2 \lambda_2 Y_2 (B - B^*)^2}{B} \\ & - \beta_1 (Y_1 - Y_1^*)^2 - k_1 (\phi_1 + d_1 + \theta_1)(V_1 - V_1^*)^2 - \frac{\beta_1 d_1}{\alpha_1} (N_1 - N_1^*)^2 - \beta_2 (Y_2 - Y_2^*)^2 \\ & - k_4 (\phi_2 + d_2 + \theta_2)(V_2 - V_2^*)^2 - \frac{\beta_2 d_2}{\alpha_2} (N_2 - N_2^*)^2 - \frac{\lambda_2 B^* s(E^*)}{L(E^*)} (B - B^*)^2 - k_7 \delta_0 (E - E^*)^2 \\ & + \lambda_1 \frac{Y_2}{Y_1^*} (Y_1 - Y_1^*)(N_1 - N_1^*) - \left\{ \left(\beta_1 + \lambda_1 \frac{Y_2}{Y_1^*} \right) + k_1 \phi_1 \right\} (Y_1 - Y_1^*)(V_1 - V_1^*) \\ & + \frac{\lambda_1 (N_1^* - Y_1^* - V_1^*)}{Y_1^*} (Y_1 - Y_1^*)(Y_2 - Y_2^*) + k_1 \phi_1 (V_1 - V_1^*)(N_1 - N_1^*) \\ & + \lambda_2 \frac{B}{Y_2^*} (Y_2 - Y_2^*)(N_2 - N_2^*) - \left\{ \left(\beta_2 + \lambda_2 \frac{B}{Y_2^*} \right) + k_4 \phi_2 \right\} (Y_2 - Y_2^*)(V_2 - V_2^*) \\ & + \frac{\lambda_2 (N_2^* - V_2^*)}{Y_2^*} (Y_2 - Y_2^*)(B - B^*) + k_4 \phi_2 (V_2 - V_2^*)(N_2 - N_2^*) \\ & + \left\{ \lambda_2 B^* \left[p \left(1 - \frac{B}{L(E^*)} \right) + q \frac{s(E)B}{L_0^2} \right] \right\} (B - B^*)(E - E^*) + k_7 r (E - E^*)(N_2 - N_2^*). \end{aligned}$$

Now we choose k_1, k_4, k_7 , such that,

$$\begin{aligned} \frac{3(\beta_1 Y_1^* + \lambda_1 Y_2)^2}{\beta_1 (\phi_1 + d_1 + \theta_1) Y_1^{*2}} < k_1 < \frac{\beta_1 (\phi_1 + d_1 + \theta_1)}{\phi_1^2} \min \left\{ 1, \frac{2d_1}{\alpha_1} \right\} \\ \frac{15(\beta_2 Y_2^* + \lambda_2 B_m)^2}{4\beta_2 (\phi_2 + d_2 + \theta_2) Y_2^{*2}} < k_4 < \frac{\beta_2 (\phi_2 + d_2 + \theta_2)}{\phi_2^2} \min \left\{ \frac{1}{5}, \frac{d_2}{3\alpha_2} \right\} \\ \frac{\lambda_2 B^* L(E^*)}{2\delta_0 s(E^*)} \left[p \left(1 - \frac{B_m}{L(E^*)} \right) + q \frac{s(E_m) B_m}{L_0^2} \right]^2 < k_7 < \frac{4\delta_0 \beta_2 d_2}{3\alpha_2 r^2} \end{aligned}$$

Thus, $\frac{dU_2}{dt}$ is negative definite function inside the region of attraction Ω , under following conditions,

$$\alpha_1 \lambda_1^2 A_2^2 < \frac{1}{2} d_1 d_2^2 \beta_1^2 Y_1^{*2}$$

$$\lambda_1^2 (N_1^* - Y_1^* - V_1^*)^2 < \frac{1}{5} \beta_1 \beta_2 Y_1^*$$

$$\alpha_2 \lambda_2^2 B_m^2 < \frac{4}{15} d_2 \beta_2^2 Y_2^{*2}$$

$$\lambda_2 L(E^*) (N_2^* - V_2^*)^2 < \frac{2}{5} \beta_2 s(E^*) B^* Y_2^{*2}$$

$$\alpha_2 \lambda_2 L(E^*) B^* r^2 \left[p \left(1 - \frac{B_m}{L(E^*)} \right) + q \frac{s(E_m) B_m}{L_0^2} \right]^2 < \frac{8}{3} d_2 \beta_2 \delta_0^2 s(E^*)$$

$$(\beta_1 Y_1^* + \lambda_1 A_2 / d_2)^2 < \frac{1}{9} \frac{\beta_1^2 (\varphi_1 + d_1 + \theta_1)^2 Y_1^{*2}}{\varphi_1^2} \min(1, 2d_1 / \alpha_1)$$

$$(\beta_2 Y_2^* + \lambda_2 B_m)^2 < \frac{16}{45} \frac{\beta_2^2 (\varphi_2 + d_2 + \theta_2)^2 Y_2^{*2}}{\varphi_2^2} \min(1/5, d_2 / 3\alpha_2),$$

showing that U_2 is a Lyapunov function with respect to W_2 , proving the theorem.