Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 9, Issue 1 (June 2014), pp. 428 – 434

Applications and Applied
Mathematics:
An International Journal
(AAM)

Isotopic Form of M-Rings

M.R. Molaei and A. Keyhaninejad

Department of Mathematics
Islamic Azad Unibersity, Bardsir Branch
Bardsir, Iran
mrmolaei@uk.ac.ir

Received: October 11, 2013; Accepted: March 3, 2014

Abstract

The aim of this work is to generalize the notion of isofields by presenting the notion of M-isorings. A method for constructing new M-isorings is presented. It is proved that an M-isoring for which its isounit is the fixed point of its identity function is an M-ring. Two methods for constructing new M-rings are presented.

Keywords: Isounit; Isotopy; Isoring; M-isoring; M-ring

MSC 2010 No.: 16Y99; 16D30; 12F05

1. Introduction

Isotheory has been presented first in (Santilli, 1993) and (Santilli, 1996), as a new theory in the area of mathematics and physics. For some of its applications in geometry and dynamical systems see (Molaei, 2001) and (Molaei, 2006). Isotheory or axiom preserving theory is based on Santilli's isounits. In fact the system of Santilli's isounits is a set S with a binary operation $. : S \times S \to S$ which satisfies the following conditions:

(1) There is $1 \in S$ such that $I.1 = 1.I = I$ for all $I \in S$.

(2) For $I \in S$ there is $I^{-1} \in S$ such that $I.I^{-1} = I^{-1}.I = 1$.

(Santilli, 1996) presented an algebraic method to change a left handed system to a right handed one via isounits. Isofields play an essential role in this algebraic method (see section 2).

In this paper we are going to present M-isorings as a generalization of the notion of isofields. We prove that under a special condition on an isounit of an M-isoring, it can be an invariant object under an axiom preserving map or an isotopy (see Theorem 2.3). We also present methods for constructing M-rings and M-isorings (M-rings) which are invariant under special isotopies (see Theorem 2.4, Theorem 2.5, and Theorem 2.6).

2. Isotopic form of M-rings

We begin this section by definition of generalized isounits. A generalized isounit is a member of a nonempty set Ω with a binary operation

$$* : \Omega \times \Omega \to \Omega,$$

with the following conditions:

(1) If $I \in \Omega$, then there exists a unique $1(I) \in \Omega$ such that $I * 1(I) = 1(I) * I = I$;
(2) For given $I \in \Omega$ there exists $I^{-1} \in \Omega$ such that $I * I^{-1} = I^{-1} * I = 1(I)$.

Let $(E, +, \times)$ be a ring, and let $I \in \Omega$ be given. Then $E(I) = \{e \times I \ : \ e \in E\}$ with the addition

$$\hat{+} : E(I) \times E(I) \to E(I),$$

$$(e_1 \times I, e_2 \times I) \mapsto (e_1 + e_2) \times I,$$

and the multiplication

$$\hat{\times} : E(I) \times E(I) \to E(I),$$

$$(e_1 \times I, e_2 \times I) \mapsto (e_1 \times e_2) \times I,$$

is a ring. Two arbitrary elements $e_1 \times I$, and $e_2 \times I$ in $E(I)$ are called equal if $e_1 = e_2$.
Since the mapping

$$\varphi : E \to E(I)$$

$$e \mapsto e \times I$$

is a ring isomorphism, then mathematically and physically E and $E(I)$ are the same. But the difference will appear if E is a ring with the identity 1. In this case the identity of $E(I)$ is $1 \times I$, and physically it may present a different perspective. For example if E is the real numbers ring R, and Ω is the set of rational numbers, then $R(-2)$ has different direction with R. So by an algebraic isomorphism we deduce two different geometrical perspectives, and this is one of the main points of isotheory. For details see (Santilli, 1993) and (Santilli, 2003). If fact we can change an Euclidean right-handed system to an Euclidean left-handed system only with an algebraic isomorphism. For example $R(1) \times R(1)$ is right-handed system and $R(-1) \times R(1)$ is a left-handed system. So with axiom preserving maps we can deduce two different physical structures. Axiom preserving maps are those maps which preserve the axioms of a mathematical theory which we use for describing a physical fact. For example in group theory axiom preserving maps are group isomorphisms and in ring theory they are ring isomorphisms.

When E is a field, then $E(I)$ is called an isofield. This structure and its application in unified theory has been considered in (Santilli, 1993) and (Santilli, 1996).

M-ring is one of the extensions of ring which was presented in (Molaei, 2002). Now we would like to make use of this algebraic structure to present the notion of M-isorings.

An M-ring with an identity is a mathematical structure which its additive identity element is a mapping. When this mapping is a constant mapping then an M-ring will be a ring, but it is not a ring when its additive identity mapping is not a constant map. More precisely an M-ring with an identity is a nonempty set G with two different operations:

$$(x, y) \mapsto x + y \quad \text{and} \quad (x, y) \mapsto xy$$

admitting the following axioms:

(1) $x + (y + z) = (x + y) + z$, where $x, y, z \in G$;
(2) For all $x \in G$ there exists a unique $e(x) \in G$ such that $x + e(x) = e(x) + x = x$;
(3) For all $x \in G$ there exists $-x \in G$ such that $x + (-x) = (-x) + x = e(x)$;
(4) $x(yz) = (xy)z$, where $x, y, z \in G$;
(5) For all $x, y, z \in G$, $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
(6) There is $1 \in G$ such that $x1 = 1x = x$ for all $x \in G$;
(7) $e(xy) = e(x)e(y)$ and $e(x + y) = e(x) + e(y)$ for all $x, y \in G$.

Axioms (1), (2) and (3) mean that $(G, +)$ is a completely simple semigroup. This structure has been considered and applied in (Ebrahimi, 2012), (Farhangdoost and Molaei, 2009), (Farhangdoost and Nasirzade, 2013), (Ghane and Hamed, 2010), (Molaei, 2005), and (Molaei, 2009). If we remove axioms (6), and (7), then G is called a generalized ring.

Theorem 2.1. If G is an M-ring with identity, then $e(x)$ is a fixed point of the mapping $e(\ )$, and $e(x)x = xe(x) = (e(x))^2$, for all $x \in G$.

Proof. Let $x \in G$ be given. Then $e(x)x = (e(x)+e(x))x = e(x)x+e(x)x$. So $e(e(x)x) = e(x)x$. Hence $e(e(x))e(x) = e(x)x$. Since $e(x) + e(x) = e(x)$, then $e(e(x)) = e(x)$. Thus $e(x)e(x) = e(x)x$. The similar calculations imply $e(x)e(x) = xe(x)$. □

Example 2.1. R^2 with the operations:

$$(a_1, b_1) + (a_2, b_2) := (a_1, b_2) \quad \text{and} \quad (a_1, b_1)(a_2, b_2) := (a_1a_2, b_1b_2)$$

is an M-ring with the identity $(1, 1)$.

Theorem 2.2. Let G be an M-ring with identity and let $x + y = y + x$ for all $x, y \in G$. Then G is a ring with identity.

Proof. Let $x, y \in G$ be given. Then $e(x) + x + y = x + y = y + x = y + x + e(x)$. So $e(x + y) = e(x)$. Similarly $e(x + y) = e(y)$. Thus $e(x) = e(y)$ for all $x, y \in G$. So G is a commutative group. Thus G is a ring with identity. □

Now let us to define an M-isoring. For this purpose we assume that $(G, +, .)$ is an M-ring with identity, and Ω is a set of generalized isounits. If $I \in \Omega$, then the set $G(I) = \{x.I \ : \ x \in G\}$ with the operations:

$$\hat{+} : (x.I, y.I) \mapsto (x + y).I,$$

and

$$\hat{.} : (x.I, y.I) \mapsto (x.y).I,$$

is called an M-isoring.

Two arbitrary elements $x.I$, and $y.I$ of $G(I)$ are called equal if $x = y$.

Theorem 2.3. If $G(I)$ is an M-isoring, and $I \in G$ is a fixed point of the mapping $e(.)$, then $(G(I), \hat{+}, \hat{.})$ is an M-ring with the identity $1.I$.

Proof. We only prove the axiom (7) of M-ring axioms. If $x.I, y.I \in G(I)$, then

$$e((x.I)\hat{.}(y.I)) = e((x.y).I) = e(x.y).e(I) =$$
$$(e(x).e(y)).I = (e(x).I)\hat{.}(e(y).I)$$

and

$$\begin{aligned} e((x.I)\hat{+}(y.I)) &= e((x + y).I) = e(x + y).e(I) \\ &= (e(x) + e(y)).I = (e(x).I)\hat{+}(e(y).I). \end{aligned}$$

Theorem 2.4. Let $(G, +, .)$ be a ring and let
$f : G \to G$ be a ring homomorphism, such that $f^2 = f$. Then $G(I)$ with the operations:

$$a.I\hat{\oplus}b.I := (a + f(b)).I \quad \text{and} \quad a.I\hat{\odot}b.I = (f(a).f(b)).I$$

is an M-ring.

Proof. We first prove that G with the operations $a \oplus b = a + f(b)$ and $a \odot b = f(a).f(b)$ is an M-ring.
Let $x, y, z \in G$ be given. Then

$$\begin{aligned} (x \oplus y) \oplus z &= (x + f(y)) + f(z) = x + (f(y) + f(z)) \\ &= x + (f(y) + f^2(z)) = x + f(y + f(z)) \\ &= x \oplus (y + f(z)) = x \oplus (y \oplus z) \end{aligned}$$

So $\oplus$ is an associative operation.
Let $x \in G$ be given, and let $-x$ be its inverse in G. Then

$$\begin{aligned} x \oplus (x + f(-x)) &= x + f(x + f(-x)) = x + (f(x) + f^2(-x)) \\ &= x + (f(x) + f(-x)) = x + f(e) = x, \quad and \\ (x + f(-x)) \oplus x &= (x + f(-x)) + f(x) \\ &= x + f(e) = x + e = x \end{aligned}$$

Moreover if $y \oplus x = x \oplus y = x$, then $x + f(y) = y + f(x) = x$. So $y = x + f(-x)$. Thus $e(x) = x + f(-x)$.
If $x \in G$, then

$$x \oplus (x + f(-2x)) = x + f(x + f(-2x)) = x + f(-x) = (x + f(-2x)) \oplus x.$$

So $x + f(-2x)$ is the inverse of x in $(G, \oplus)$.

If $x, y, z \in G$, then

$$\begin{array}{llcl} a) & (x \odot y) \odot z & = & f(x.y) \odot z = f(x.y).f(z) = f((x.y).z) \\ & & = & f(x.(y.z)) = f(x).f(y.z) = x \odot f(y.z) \\ & & = & x \odot (y \odot z) \\ b) & x \odot (y \oplus z) & = & f(x).f(y + f(z)) = f(x).(f(y) + f(z)) \\ & & = & f(x).f(y) + f(x).f(z) = x \odot y + f^2(x).f^2(z) \\ & & = & x \odot y + f(x \odot z) = (x \odot y) \oplus (x \odot z) \end{array}$$

and

$$\begin{aligned} (y \oplus z) \odot x &= f(y + f(z)).f(x) = (f(y) + f(z)).f(x) \\ &= f(y).f(x) + f(z).f(x) = (y \odot x) + f(z \odot x) \\ &= (y \odot x) \oplus (z \odot x) \ . \end{aligned}$$

Moreover

$$\begin{aligned} e(x \oplus y) &= e(x + f(y)) = (x + f(y)) + f(-(x + f(y))) \\ &= x + f(y) + f(-x - f(y)) = x + f(y) - f(x) - f^2(y) \\ &= x - f(x) + f(y) - f^2(y) \\ &= (x - f(x)) \oplus (y + f(-y)) = e(x) \oplus e(y), \end{aligned}$$

and

$$\begin{aligned} e(x \odot y) &= e(f(x).f(y)) = e(f(x)).e(f(y)) \\ &= f(e(x)).f(e(y)) = e(x) \odot e(y). \end{aligned}$$

The straightforward calculations imply $\hat{e}(x.I) = e(x).I$ and the inverse of $x.I$ is $x^{-1}.I$. Moreover $\hat{e}$ is a homomorphism. So $(G(I), \hat{\oplus}, \hat{\odot})$ is an M-ring. □

Theorem 2.5. Let $(H, +, .)$ be a ring with an identity and let
$f : H \to H$ be a ring homomorphism, such that $f^2 = f$. Moreover let $G = f(H)$. Then $G(I)$ with the operations:

$$a.I \hat{\oplus} b.I := (a + f(b)).I \quad \text{and} \quad a.I \hat{\odot} b.I = (f(a).f(b)).I$$

is an M-isoring with the identity $1.I$.

Proof. Theorem 2.4. implies G is an M-ring. If $a \in G$, then $a = f(h)$ for some $h \in H$. We have $f(a) = f(f(h)) = f(h) = a$. So f is the identity function on G. Thus $a.I \hat{\odot} 1.I = (f(a).f(1)).I = a.I$. □

Theorem 2.6. Let G be a ring with the identity 1, and let
$f : G \to G$ be a ring homomorphism such that $f(1) = 1$. Moreover let $K = \{g \in G \ : \ f^2(g) = f(g)\}$. Then $K(I)$ with the operations

$$a.I \hat{\oplus} b.I := (a + f(b)).I \quad \text{and} \quad a.I \hat{\odot} b.I = (f(a).f(b)).I$$

is an M-ring and $f(K)(I)$ with the above operation is an M-isoring with the identity $1.I$.

Proof. We prove that K is a subring of G with the identity 1. Let $r, s \in K$. Then $f^2(r + s) = f(f(r) + f(s)) = f^2(r) + f^2(s) = f(r) + f(s) = f(r + s)$. So $r + s \in K$. Moreover $f^2(r.s) = f(f(r).f(s)) = f^2(r).f^2(s) = f(r).f(s) = f(r.s)$. Thus $r.s \in K$. So K is a subring of G. The condition $f(1) = 1$ implies $1 \in K$. Thus K is a subring of G with the identity 1. If $r \in K$, then $f^2(f(r)) = f(f^2(r)) = f(f(r))$. Thus $f(r) \in K$. So $f : K \to K$ is a ring homomorphism. As the same as the proof of Theorem 2.4 we can show that $K(I)$ is an M-ring. Moreover Theorem 2.5 implies $f(K)(I)$ is an M-isoring with the identity $1.I$. □

Example 2.2. The mapping $f : Z_5 \to Z_5$ defined by $f(x) = x^5$ is a ring homomorphism and $f(1) = 1$. With the notations of Theorem 2.6, $K = \{0, 1, 4\}$. So $(K(4), \hat{\oplus}, \hat{\odot})$ is an M-isoring, and it is also an M-ring with the identity 1.4.

3. Conclusion

We considered the notion of M-isoring as an extension of the notion of isofields. We proved that an M-isoring is also a ring if its generalized isounit is a fixed point of its identity function. We also presented a method for constructing new M-rings and M-isoring. Consideration of M-isoring from topological point of view can be a topic for further research.

References

Ebrahimi, N. (2012). *A Generalization of Lie Theory, A Generalization of Main Theorems in Lie Theory*. Lap Lambert Academic Publishing.

Farhangdoost, M. and Molaei, M. (2009). Characterization of top spaces by diffeomorphic lie groups. *Differential Geometr - Dynamical Systems*, 11:130–134.

Farhangdoost, M. and Nasirzade, T. (2013). Geometrical categories of generalized lie groups and lie group-groupoids. *Iranian Journal of Science & Technology*, A1:69–73.

Ghane, F. and Hamed, Z. (2010). Upper topological generalized groups. *Iranian Journal of Science & Technology*, 27:321–332.

Molaei, M. (2001). Isotopic form of complex manifolds. *Hadronic Journal*, 24:291–300.

Molaei, M. (2002). Generalized rings. *Italian Journal of Pure and Applied Mathematics*, 12:105–111.

Molaei, M. (2005). Generalized structures based on completely simple semigroups. *Hadronic Press*, pages 1–94.

Molaei, M. (2006). Dynamical and geometrical structures based on santilli's isotheory. *Hadronic Journal*, 29:259–284.

Molaei, M. (2009). Farhangdoost m.r., lie algebras of a class of top spaces. *Balkan Journal of Geometry and Its Applications*, 14(1):46–51.

Santilli, R. (1993). Isonumbers and genonumbers of dimension 1, 2, 4, 8 their isoduals and pseudoduals and hidden numbers of dimension 3, 5, 6, 7. *Algebras, Groups and Geometries*, 10:273–322.

Santilli, R. (1996). Nonlocal-integral isotopies of differential calculus. *Algebras, Groups and Geometries*, 42:7–82.

Santilli, R. (2003). Iso-, geno-, hyper-mechanics for matter, their isoduals for antimatter, and their novel applications in physics, chemistry and biology. *Journal of Dynamical Systems and Geometric Theories*, 1(2):121–193.