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Independent Domination in Some Wheel Related Graphs

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Abstract

A set S of vertices in a graph G is called an independent dominating set if S is both independent and dominating. The independent domination number of G is the minimum cardinality of an independent dominating set in G. In this paper, we investigate the exact value of independent domination number for some wheel related graphs.

Keywords: Graph, dominating set; independent dominating set; independent domination number

AMS-MSC (2010): 05C69, 05C76

1. Introduction

The domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it because it has the potential to solve many real life problems involving design and analysis of communication network as well as defense surveillance. Many variants of domination models are available in the existing literature. The comprehensive bibliography of papers on the concept of domination can be found in Hedetniemi and Laskar (1990). Independent sets play an important role in graph theory and discrete optimization. They appear in matching theory, coloring of graphs and in the theory of trees. The present paper is focused on independent domination in graphs.

We begin with a finite, connected and undirected graph $G=(V,E)$ without loops and multiple edges. The set $S \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.

An independent set in a graph G is a set of pairwise non-adjacent vertices of G. A set S of vertices in a graph G is called an independent dominating set if S is both an independent set and a dominating set of G. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set in G.

The theory of independent domination was formalized by Berge (1962) and Ore (1962) in 1962. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi (1974, 1977). Independent dominating sets in regular graphs and in cubic graphs in particular, are well studied in Goddard et al. (2012), Kostichka (1993) and Lam et al. (1999). Favaron (1988) initiated the quest of finding sharp upper bounds for independent domination number in general graphs, as functions of n and δ. This work was extended by Haviland (2007). Cockayne et al. (1991) found the upper bound for the product of the independent domination numbers of a graph and its complement while Shiu et al. (2010) found the upper bounds for the independent domination number of triangle-free graphs and also characterized the extremal graphs achieving these upper bounds. Allan and Laskar (1978) discussed the graphs with equal domination and independent domination numbers while Southey and Henning (2013) considered the ratio of the independent domination number versus the domination number in a cubic graph and also characterized the graphs achieving this ratio of $4/3$. Ao et al. (1996) proved that for each $k \geq 4$, there exists a connected $k-$domination critical graph with independent domination number exceeding k.

The wheel W_n is defined to be the joint $C_{n-1}+K_1$. The vertex corresponding to K_1 is known as apex vertex and the vertices corresponding to cycle are known as rim vertices. The edges corresponding to cycle are known as rim edges and the edges incident with the apex vertex are known as spoke edges.

We denote the degree of a vertex v in G by $deg(v)$ and the maximum degree among the vertices of G by $\Delta(G)$. In a graph G, a vertex of degree one is called a pendant vertex and an edge incident with a pendant vertex is called a pendant edge.

For any real number n, $\lceil n \rceil$ denotes the smallest integer not less than n and $\lfloor n \rfloor$ denotes the greatest integer not greater than n.

For the various graph theoretic notations and terminology, we follow West (2003) while the terms related to the concept of domination are used in the sense of Haynes et al. (1998).

Generally, the following types of problems are considered in the field of domination in graphs:

1) To introduce new types of domination models.
2) To determine bounds in terms of various graph parameters.
3) To obtain the exact domination number for some graphs or graph families.
4) To study the algorithmic and complexity results for particular dominating parameter.
5) To characterize the graphs with certain domination parameters.

The present work is intended to discuss the problem of the third kind in the context of independent domination in graphs. We investigate the independent domination number of some wheel related graphs.

2. Main Results

Definition 2.1.

The helm H_n is the graph obtained from wheel W_n by attaching a pendant edge to each of its rim vertices.

Theorem 2.2.

For the helm H_n $(n \geq 4)$,

$$i(H_n) = n-1.$$

Proof:

The helm H_n has $2n-1$ vertices and $3(n-1)$ edges. It contains wheel W_n and $n-1$ pendant vertices.

In order to dominate the pendant vertices of H_n, at least $n-1$ vertices of H_n are required. Moreover, it is also possible to take $n-1$ pairwise non-adjacent vertices which can dominate the pendant vertices as well as the remaining vertices of the helm H_n. Therefore, for any independent dominating set S of the helm H_n, $*S* \geq n-1$ which implies that $i(H_n) = n-1$.

Definition 2.3.

The closed helm CH_n is the graph obtained from a helm by joining each pendant vertex to form a cycle.

Illustration 2.4.

The closed helm CH_6 is shown in Figure 1.

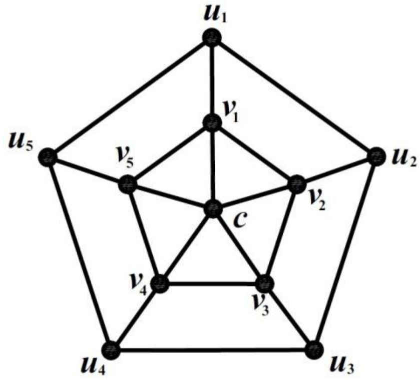

Figure 1. CH_6

Proposition 2.5. Goddard and Henning (2013)

For the cycle C_n,

$$i(C_n) = \lceil n/3 \rceil .$$

Theorem 2.6.

For the closed helm CH_n $(n \geq 4, n \neq 5)$,

$$i(CH_n) = \lceil (n+2)/3 \rceil .$$

Proof:

Let c denote the apex vertex of wheel W_n. For the closed helm CH_n,

$$\ast V(CH_n)\ast = 2n-1 \text{ and } \ast E(CH_n)\ast = 4(n-1) .$$

The closed helm CH_n contains the wheel W_n and the outer cycle C_{n-1}.

For $n = 4$, at least two non-adjacent vertices of CH_n, one from W_n and one from the outer cycle C_{n-1}, are required to dominate all vertices of CH_n. Hence,

$$i(CH_n) = 2 = \lceil (n+2)/3 \rceil \text{ for } n = 4 .$$

For $n > 5$, $deg(c) = n-1 = \Delta(CH_n)$ and the vertex c dominates W_n. Therefore, in order to attain the minimum cardinality, any independent dominating set of CH_n must contain the vertex c. Now, by Proposition 2.5, $i(C_n) = \lceil n/3 \rceil$. Hence, at least $\lceil (n-1)/3 \rceil$ pairwise non-adjacent vertices of C_{n-1} are required to dominate all vertices of outer cycle C_{n-1} of CH_n. Therefore, for any independent dominating set S of CH_n,

$$\ast S \ast \geq \lceil (n-1)/3 \rceil + 1 = \lceil (n+2)/3 \rceil$$

implying that

$$i(CH_n) = \lceil (n+2)/3 \rceil .$$

Remark 2.7.

For $n = 5$, at least two non-adjacent vertices u and v of CH_5 with $d(u,v) = 3$ are required to dominate all vertices of CH_5 where u and v are vertices of C_{n-1} and W_n respectively. Thus, $i(CH_5) = 2$.

Definition 2.8.

A gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the $(n-1)$- cycle of W_n.

Theorem 2.9.

For the gear graph G_n $(n \geq 4)$,

$$i(G_n) = \lceil 2(n-1)/3 \rceil .$$

Proof:

Let c denote the apex vertex of wheel W_n. The gear graph G_n has $2n-1$ vertices and $3(n-1)$ edges. The gear graph G_n contains the outer cycle $2(n-1)$ and

$$V(G_n) = V(C_{2(n-1)}) \bigcup \{c\} .$$

Now, by Proposition 2.5,

$$i(C_{2(n-1)}) = \lceil 2(n-1)/3 \rceil .$$

Therefore, at least $\lceil 2(n-1)/3 \rceil$ pairwise non-adjacent vertices are essential to dominate all vertices of $C_{2(n-1)}$ of G_n. Moreover, these vertices also dominate the vertex c. Hence, for any independent dominating set S of G_n,

$$*S* \geq \lceil 2(n-1)/3 \rceil . \text{ Thus, } i(G_n) = \lceil 2(n-1)/3 \rceil .$$

Koh et al. (1980) defined a web graph as follows:

Definition 2.10.

A web graph is the graph obtained by joining the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle.

$W(t, n-1)$ is the generalized web with t cycles each of order $n-1$.

Theorem 2.11.

For the web graph $W(2, n-1)$ $(n \geq 4)$,

$$i(W(2, n-1)) = n .$$

Proof:

Denote the apex vertex of wheel W_n as c. For the web graph $W(2, n-1)$,

$$*V(W(2, n-1))* = 3n-2$$

and

$$|E(W(2,n-1))| = 5(n-1).$$

The web graph $W(2,n-1)$ contains W_n, the outer cycle C_{n-1} and $n-1$ pendant vertices.

In order to dominate $n-1$ pendant vertices of $W(2,n-1)$, at least $n-1$ vertices of $W(2,n-1)$ are required. Moreover, it is also possible to take $n-1$ pairwise non-adjacent vertices which can dominate the pendant vertices as well as all vertices of the outer cycle C_{n-1} of $W(2,n-1)$ and the vertex c dominates the remaining vertices of $W(2,n-1)$. Hence, at least $n-1$ (pendant vertices) $+1$ (apex vertex) $= n$ pairwise non-adjacent vertices are required to dominate all vertices of $W(2,n-1)$. Therefore, for any independent dominating set S of $W(2,n-1)$, $|S| \geq n$ implying that

$$i(W(2,n-1)) = n.$$

Theorem 2.12.

For the web graph $W(3,n-1)$ where $n \geq 4$,

$$i(W(3,n-1)) = \lceil (4n-2)/3 \rceil.$$

Proof:

Label the vertices of the web graph $W(3,n-1)$ as follows:

Denote the vertices of the innermost cycle of $W(3,n-1)$ successively as $v_{1,1}, v_{1,2}, v_{1,3}, \ldots, v_{1,n-1}$. Then denote the vertices adjacent to $v_{1,1}, v_{1,2}, v_{1,3}, \ldots, v_{1,n-1}$ on the second cycle (middle cycle) as $v_{2,1}, v_{2,2}, v_{2,3}, \ldots, v_{2,n-1}$ respectively, and the vertices adjacent to $v_{2,1}, v_{2,2}, v_{2,3}, \ldots, v_{2,n-1}$ on the third cycle (outermost cycle) as $v_{3,1}, v_{3,2}, v_{3,3}, \ldots, v_{3,n-1}$. Next denote the pendant vertices of $W(3,n-1)$ as $u_1, u_2, u_3, \ldots, u_{n-1}$ and the apex vertex of W_n as c. Then,

$$|V(W(3,n-1))| = 4n-3$$

and

$$|E(W(3,n-1))| = 7(n-1).$$

Now, to attain the minimum cardinality, every independent dominating set of $W(3,n-1)$ should contain the vertex c because

$$deg(c) = n-1 = \Delta(W(3,n-1))$$

and the vertex c dominates the innermost cycle of $W(3,n-1)$. Moreover, in order to dominate $n-1$ pendant vertices of $W(3,n-1)$, at least $n-1$ vertices are required and these $n-1$ pendant vertices dominate themselves as well as all vertices of outermost cycle of $W(3,n-1)$. Now, by Proposition 2.5, $i(C_n) = \lceil n/3 \rceil$. Therefore, at least $\lceil (n-1)/3 \rceil$ pairwise non-adjacent vertices are required to dominate all vertices of middle cycle of $W(3,n-1)$. Thus, for $n \equiv 0,1 (mod 3)$, at least

$$1+n-1+\lceil (n-1)/3 \rceil = \lceil (4n-2)/3 \rceil$$

pairwise non-adjacent vertices are required to dominate the web graph $W(3,n-1)$. But for $n \equiv 2(mod 3)$, to retain the minimum cardinality of S, it should contain a vertex $v_{3,n-1}$ because $v_{3,n-1}$ dominates a pendant vertex as well as a vertex of the middle cycle of $W(3,n-1)$. Hence, at least

$$n-2+(n-2)/3=(4n-8)/3$$

pairwise non-adjacent vertices are essential to dominate the remaining pendant vertices and the remaining vertices of the outermost as well as the middle cycles of $W(3,n-1)$. Therefore, at least

$$1+1+(4n-8)/3=(4n-2)/3$$

pairwise non-adjacent vertices are required to dominate all vertices of $W(3,n-1)$ for $n \equiv 2(mod 3)$. Thus, every independent dominating set of $W(3,n-1)$ must contain at least $\lceil (4n-2)/3 \rceil$ vertices of $W(3,n-1)$.

Hence, we construct a vertex set $S \subset V(W(3,n-1))$ as follows:

$$S = \begin{cases} \{c, v_{2,2}, v_{2,5}, \ldots, v_{2,3i+2}, u_1, u_2, \ldots, u_{n-1}\}, & \text{if } n \equiv 0 \text{ or } 1 (\text{mod } 3), \\ \{c, v_{2,2}, v_{2,5}, \ldots, v_{2,3i+2}, u_1, u_2, \ldots, u_{n-2}\} \bigcup \{v_{3,n-1}\}, & \text{otherwise,} \end{cases}$$

where $0 \le i \le \lfloor (n-3)/3 \rfloor$ with $*S* = \lceil (4n-2)/3 \rceil$ for $n \ge 4$.

Since each vertex in $W(3,n-1)$ is either in S or is adjacent to a vertex in S, it follows that the set S is a dominating set of $W(3,n-1)$. Moreover, the set S is also an independent set of $W(3,n-1)$ because no two vertices in S are adjacent. Therefore, the set S is an independent dominating set of $W(3,n-1)$. As

$$*S* = \lceil (4n-2)/3 \rceil,$$

the set S is of minimum cardinality.

Hence, the set S is an independent dominating set with minimum cardinality implying that

$$i(W(3,n-1)) = \lceil (4n-2)/3 \rceil .$$

Illustration 2.13.

In Figure 2, the web graph $W(3, 4$ is shown in which the set of solid vertices is its independent dominating set with minimum cardinality.

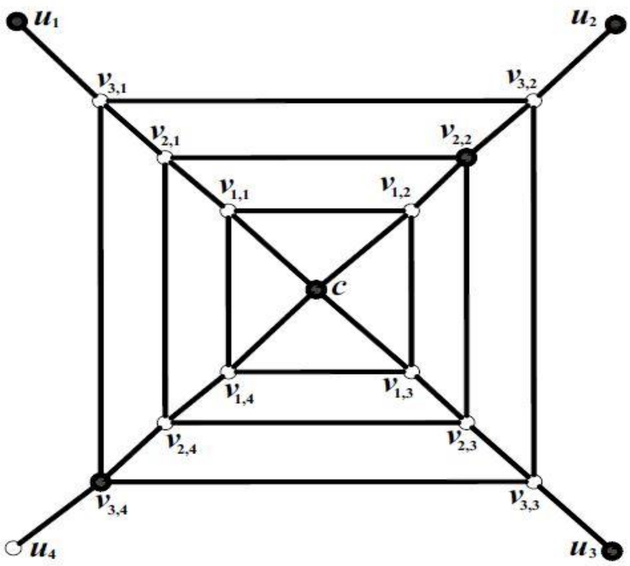

Figure 2. $W(3,4)$

Definition 2.14.

Duplication of an edge $e = uv$ by a new vertex v' in a graph G produces a new graph G' by adding a vertex v' such that $N(v') = \{u, v\}$.

Theorem 2.15.

If G is the graph obtained by duplication of each edge of W_n by a vertex then

$$i(G) = \left\lfloor \frac{n + 2\lfloor n/2 \rfloor}{2} \right\rfloor.$$

Proof:

Consider the wheel $W_n = C_{n-1} + K_1$ with $v_1, v_2, \ldots, v_{n-1}$ as its rim vertices and c as its apex vertex. Let $e_1, e_2, \ldots, e_{n-1}$ be the rim edges of W_n which are duplicated by new vertices $w_1, w_2, \ldots, w_{n-1}$ respectively and let $f_1, f_2, \ldots, f_{n-1}$ be the spoke edges of W_n which are duplicated by the vertices $u_1, u_2, \ldots, u_{n-1}$ respectively. Then, $*V(G)* = 3n - 2$ and $*E(G)* = 6(n-1)$.

Now, the vertices $v_2, v_4, \ldots, v_{2i}$ for $1 \le i \le \lfloor (n-1)/2 \rfloor$ will dominate all vertices of C_{n-1} as well as the vertices $u_1, u_2, \ldots, u_{n-1}$ of G for odd n and $u_1, u_2, \ldots, u_{n-2}$ of G for even n. Moreover, these vertices will also dominate the vertices $w_2, w_4, \ldots, w_{2i}$ and c of G.

Hence, we construct a vertex set $S \subset V(G)$ as follows:

$$S = \begin{cases} \{v_2, v_4, \ldots, v_{2i}, w_1, w_3, \ldots, w_{2j+1}\}, & \text{if } n \text{ is odd}, \\ \{v_2, v_4, \ldots, v_{2i}, w_1, w_3, \ldots, w_{2j+1}\} \bigcup u_{n-1}\}, & \text{if } n \text{ is even}, \end{cases}$$

where $1 \le i \le \lfloor (n-1)/2 \rfloor$ and $0 \le j \le \lfloor (n-2)/2 \rfloor$ with

$$*S* = \left\lfloor \frac{n+2\lfloor n/2 \rfloor}{2} \right\rfloor, \text{ for } n \ge 4.$$

Since every vertex $v \in V(G) - S$ is adjacent to at least one vertex in S, it follows that the set S is a dominating set of G. The set S is also an independent set of G because no two vertices in S are adjacent. Hence, the set S is an independent dominating set of G. .Moreover, since $deg(v_k) = 6$ and $deg(v) = 2$ for $v \in V(G) - \{c, v_k\}$ and from the adjacency nature of the vertices of G, one can observe that at least $\left\lfloor \frac{n+2\lfloor n/2 \rfloor}{2} \right\rfloor$ pairwise non-adjacent vertices of G are required to dominate all vertices of G. As

$$*S* = \left\lfloor \frac{n+2\lfloor n/2 \rfloor}{2} \right\rfloor,$$

the set S is of minimum cardinality. Hence, the set S is an independent dominating set of G with minimum cardinality implying that

$$i(G) = \left\lfloor \frac{n+2\lfloor n/2 \rfloor}{2} \right\rfloor .$$

Illustration 2.16.

In Figure 3, the graph obtained by duplication of each edge of W_9 by a vertex is shown in which the set of solid vertices is its independent dominating set S with $|S| = 8$.

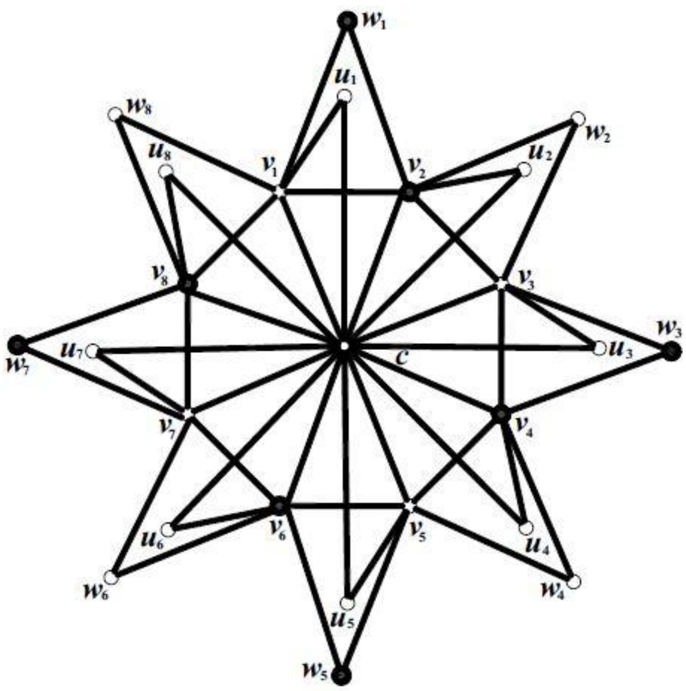

Figure 3. Duplication of each edge by a vertex in W_9

3. Conclusion

Some graphs with equal domination and independent domination numbers are studied by Vaidya and Pandit (2014) while here we investigated independent domination number of some wheel related graphs. We pose the following open problem:

Problem: To investigate the exact value of $i(G)$ for the generalized web graph.

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