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Growth Patterns of Ethnic Groups in Bexar County With Dynamic Leslie Models

Judith Arriaza, Zhanbo Yang & Flor de María García-Wukovits

Department of Mathematical Sciences
University of the Incarnate Word
4301 Broadway, San Antonio, TX 78259
yang@uiwtx.edu

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Abstract

The purpose of this study is to modify the Leslie model with a dynamic matrix for better population projections in Bexar County, where UIW is located and the authors reside. A dynamic matrix was used to improve the static Leslie model used in the previous study since human population growth is dynamic and complex. The matrix was constructed with functions that modeled the birth rates and survival rates. This allowed the rates to change from year to year. The population projections using the dynamic matrix were compared to the real population data and the static matrix. The researcher concluded that the dynamic matrix produced good population projections for the ethnic groups in Bexar County when compared with actual census data. Some preliminary projections were also made for the election cycles in the immediate and mid-range future.

Keywords: Dynamic Leslie Model; population model; curve fitting

MSC 2010 No.: 92D25, 92D40, 93A30

1. Introduction

The Leslie Model (Leslie (1945)) introduced a matrix model to produce population data vector streams. It divides the population into m (many) age groups, hence allows the researcher to study the age structures within the population at stage n (population vector X_n). The Leslie model is written as:

$$\mathbf{X_{n+1}} = \mathbf{L}\,\mathbf{X_n}$$

where X_n is the population vector at time n. X_0 represents the initial population vector.
The matrix L represents the so called "Leslie matrix"

$$\mathrm{L} = \begin{pmatrix} b_1 & b_2 & \dots & b_m \\ s_1 & 0 & \dots & 0 \\ 0 & s_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s_{m-1} \end{pmatrix},$$

where $b_i's$ are the birth rates for each age group and $s_i's$ are the survival rates from one age class to the next age class.

The birth rates and survival rates are often assumed to be constants and gathered from observation if the study is for a relatively short period of time. Since the birth rates and survival rates for human populations do change rather significantly over a longer period of time, using constants rates seems to be deficient.

This study attempted to use functions to capture the trend of changes of those rates over longer periods of time and then construct a secondary model based on those rate functions. Improved results were obtained from this study when compared to our previous study using the same set of underlying data with a static Leslie model. Our earlier study was not submitted for publication. We will include the results from that previous study in this paper for comparison purposes.

2. Literature Review

Since the introduction of the original Leslie model in 1945, there have been a number of different attempts where the Leslie model is modified in order to adapt to different situations and problems. Sharov (1996) described some of those modifications, which included "Partitioning the life cycle into stages", "Distributed delays" and "Variable matrix elements".

This study is a specific application of the idea of "Variable Matrix Elements". There are times when survival and reproduction rate of organisms "may depend on a variety of factors such as temperature, habitat characteristics, natural enemies, food, etc." [For details, see Sharov (1996)]. In order to "represent these dependencies, the elements of the Leslie model can be replaced by equations that specify survival and reproduction rates as functions of various factors" [Sharov (1996)]. This means that instead of having elements that are constants, these elements can be replaced with functions which allow the rates to change depending on some specified factors.

$$L = \begin{pmatrix} f_1(t) & f_2(t) & \dots & f_{m-1}(t) & f_m(t) \\ s_1(t) & 0 & \dots & 0 & 0 \\ 0 & s_2(t) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & s_{m-1}(t) & 0 \end{pmatrix}.$$

This method was used in a study of maple-birch trees in forests. In this study the constant Leslie model was modified by “making survivor growth and mortality of trees functions of stand basal area” [Lin and Buongiorno (1996)]. As a result, the variable-parameter model “had predictions close to the observations, except for the two largest size classes” [Lin and Buongiorno (1996)]. This study showed that variable-parameter models may be necessary in order to predict population dynamics.

For a human population model, since the birth rates and survival rates of the population are changing from year to year, “Variable Matrix Elements” is the most appropriate for the purpose of this study over the time frame covered in this paper.

Bexar County was chosen as the study subject for a number of reasons. (1), the county has a rich ethnic composition; (2) University of the Incarnate Word is located in this county and (3), all authors and co-authors are long time or life residents of this county so the results seemed to be very relevant to our lives.

3. Methodology

The main purpose of this quantitative study is to project the growth patterns of ethnic groups in Bexar County using a dynamic Leslie model.

To study population dynamics, we divided the population into 18 age groups, mostly over 5 year increments: 00-04, 05-09, 10-14, 15-19, 20-24, 25-29, 30-34, 35-39,40-44, 45-49, 50-54, 55-59, 60-64, 65-69, 70-74, 75-79, 80-84, 85+. This is mostly because the data we obtained were in 5 year increments. We also feel that the 5 year increment gives us sufficient detail in the age structure to distinguish different population characteristics, such as young child, primary and secondary school ages, college age, and working age, all the way to retirement age. Population of 85 year or older are grouped together because that is how the data came in.

The first step toward constructing the dynamic matrix was to find model functions for each of the eighteen age groups to fit the birth and survival rates. The birth rates and survival rates were gathered from the Texas Department of State Health Services. The birth rates were collected from 1990 to 2004 and the survival rates were collected from 1990 to 2010, which helped to determine the population trends for the four main ethnic groups in Bexar County throughout the past two decades. The same four ethnic categories: W (white), H (Hispanic), B (black) and O (other) were used as in the census data.

The population, birth rates and survival rates for the four ethnic groups were also divided into the same 18 age groups.

The birth rates data were from the vital statistics of the state of Texas. We interpreted the “survival” rate as the rate at which one age group transitions into the next age group, including immigrations and other migrations (which explained why some of rates are greater than 1. The fluctuations of the transition rates could be attributed to extended list of factors, such as the net survival rates from one age group into the next, the rate of migration and immigration, the level of industrialization, the availability of employment, social and natural environments, etc. We choose to aggravate all of those factors into one time factor at this time. Further study may focus on how to correlate the time factor with some of the other factors that are significant.

Upon observing the data trends for the birth rates and survival rates over the observed period, we found the following patterns:

(1) The data tends to be periodically oscillatory.

(2) The data is generally trending upward.

We also added a third assumption:

(3) The amplitude of oscillation in rates will gradually reduce over a long period of time.

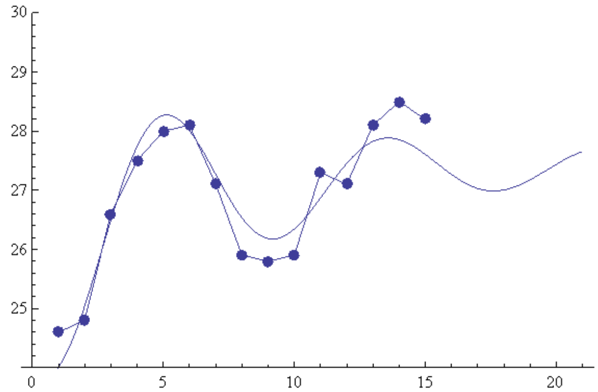
We made the assumption (3) because we think it is difficult and unreliable to assume that the oscillations far into future will be identical to the current pattern. So the further into the future years, the more we should think “in average” terms and avoid assuming large changes. Of course, it is almost not reasonable to assume that the changes in the future will be any less volatile than the current (it may be more volatile, if anything), having assumption (1) without (3) would imply that the current periodical pattern will be preserved indefinitely into the future. This made us equally uncomfortable. Since there is really no reliable way to validate this assumption one way or another, (other than waiting for 20 years and check against the census,) we choose to be on the “conservative” side by adding the assumption (3), which basically takes a “regression” approach for the future trends.

With those assumptions, we decided to use the following function template:

$$g(x) = a \cdot e^{-\frac{x}{10}} \cdot \sin(bx + c) + d \cdot \log\left(\frac{x}{10}\right) + e.$$

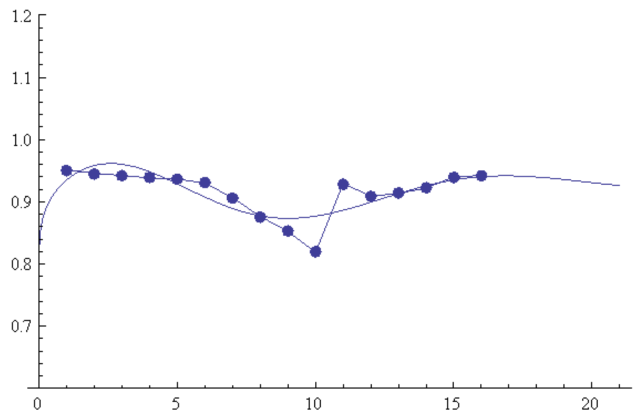
The sine function is used to capture the oscillatory trends with amplitude $ae^{-\frac{x}{10}}$ that converges to 0 over time. The parameters b and c are used to capture the period and the phrase shift. The log function is used to capture the increasing but concave downward trend, with the multiplier d to provide a vertical stretch. The last constant e is for the base line level.

We used curve fitting techniques to estimate the parameters $a, b, c, d\ and\ e$ for each age group for their birth rates and survival rates based on the actual census data. Figures 1 and 2 show two examples of the results for such curve fitting. The dots are the actual data points.



$$f(t)_{30-34} = 27.1431 + 0.429956\, Log[0.1\, t] + 2.37128 e^{(-0.1\, t)}\, Sin[3.98127 + 0.743266\, t]$$

Figure 1. Curve fitting example for White birth rates at age 30-34 using the combined function template



$$s(t)_{30-34} = 0.913034 + 0.0202266\, \mathrm{Log}[0.1\, \mathrm{t}] + 0.0982437\, \mathrm{e}^{-0.1\mathrm{t}}\, \mathrm{Sin}[6.72738 + 0.434488\, \mathrm{t}]$$

Figure 2. Curve fitting example for survival rates at age 30-34 for Whites using the combined function template

When all the parameters for each function were determined, we constructed a dynamic Leslie matrix for each of the ethnic group. Figure 3 is an example matrix for the white group.

In order to better calibrate our model against the real census data, we used Monte Carlo techniques to construct a set of weight factors for each of the survival functions with the objective of minimizing the standard deviation between our model and the real census data. Those weight factors were placed into a diagonal matrix as well. The final Leslie matrix was produced as the product of the weight factor matrix and the dynamic Leslie Matrix.

To see how well we were doing, we produced a 3D plot using the census data and the population data we produced using our model over the same time period as the available census data. Figures 4 and 5 are a visual inspection of how well the function fits the real census data.

```
Mw1 t_  :
 0, 0, 0.1  100,
  8.25  0.43  Log 1.23 t    100,
  22.71  0.81  Log t   2.34  Exp  t  10   Sin 2.28567  0.6439  t    100,
  33.3  1.79  Log t   2.83 Exp  t  10   Sin 1.02  0.78 t    100,
  26.12363  0.0868645  t  1.360806  Sin 4.0041770  0.7428896  t    100,
  4.8178358  0.2527723  t  0.0526785  t  Sin 7.020914  0.477742  t    100,
  2.408917  0.126386  t  0.026339  t  Sin 7.020914  0.4777424  t    100,
  1.605945  0.0842574  t  0.01755  t  Sin 7.0209145  0.4777  t    100,
  1.2044  0.0631  t  0.01316  t  Sin 7.0209  0.47774  t    100,
 0, 0, 0, 0, 0, 0, 0 ,
 0.93  0.02  Log 0.1 t   0.12  Exp  0.1 t   Sin 2.7  0.55  t , 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ,
 0, 0.93  0.02  Log 0.1 t   0.12  Exp  0.1 t   Sin 3.55  0.56  t , 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ,
 0, 0, 0.99  0.037  Log 0.1 t   0.08  Exp  0.1 t   Sin 2.796  0.53  t , 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ,
 0, 0, 0, 0.98  0.021  Log 0.1 t   0.16 Exp  0.1 t   Sin 3.54  0.499  t , 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ,
 0, 0, 0, 0, 1.05  0.004 Log t  10.0   0.196  Exp  0.1 t   Sin 3.194  0.479  t , 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ,
 0, 0, 0, 0, 0, 0.95  0.0098  Log t  10.0   0.17 Exp  0.1 t   Sin 2.6  0.61  t , 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ,
 0, 0, 0, 0, 0, 0, 0.91  0.02  Log 0.1 t   0.098  Exp  0.1 t   Sin 6.73  0.4345  t , 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ,
 0, 0, 0, 0, 0, 0, 0, 0.96  0.013  Log 0.1 t   0.02  Exp  0.1 t   Sin 3.09  0.80  t , 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ,
 0, 0, 0, 0, 0, 0, 0, 0, 0.96  0.018 Log 0.1 t   0.018 Exp  0.1 t  Sin 2.76  0.93 t , 0, 0, 0, 0, 0, 0, 0, 0, 0 ,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.99  0.01  Log 0.1 t   0.04 Exp  0.1 t   Sin 1.81  0.75  t , 0, 0, 0, 0, 0, 0, 0, 0 ,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.93  0.0057 Log 0.1 t   0.012 Exp  0.1 t  Sin 0.26  0.997 t , 0, 0, 0, 0, 0, 0, 0 ,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.899  0.035 Log 0.1 t   0.018 Exp  0.1 t  Sin 0.22  0.99 t , 0, 0, 0, 0, 0, 0 ,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.9  0.025 Log 0.1 t   0.038 Exp  0.1 t  Sin 3.87  0.56 t , 0, 0, 0, 0, 0 ,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.92  0.0166224 Log 0.1 t   0.026 Exp  0.1 t  Sin 2.83  0.83 t , 0, 0, 0, 0 ,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.87  0.046 Log 0.1 t   0.13 Exp  0.1 t  Sin 2.399  0.63 t , 0, 0, 0 ,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.81  0.047 Log 0.1 t   0.062 Exp  0.1 t  Sin 4.9  0.69 t , 0, 0 ,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.995  0.049 Log 0.1 t   0.064 Exp  0.1 t  Sin 3  0.88 t , 0
```

Figure 3. Dynamic Leslie matrix for White group

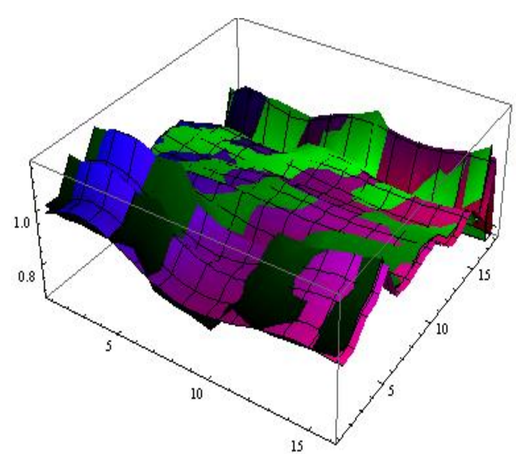


Figure 4. Comparison of the survival rates (the surface with grid) with census data

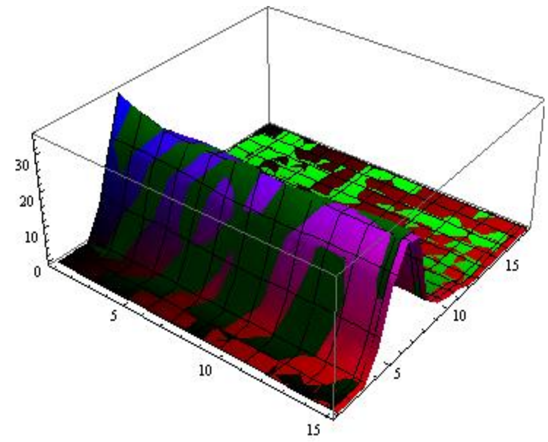


Figure 5. Comparison of the birth rates (the surface with grid) with census data

We also compared our results with 2000 and 2010 census data. Figures 6 and 7 show a comparison of the ethnic composition of Bexar County between the actual census and our projection for 2000 and 2010.

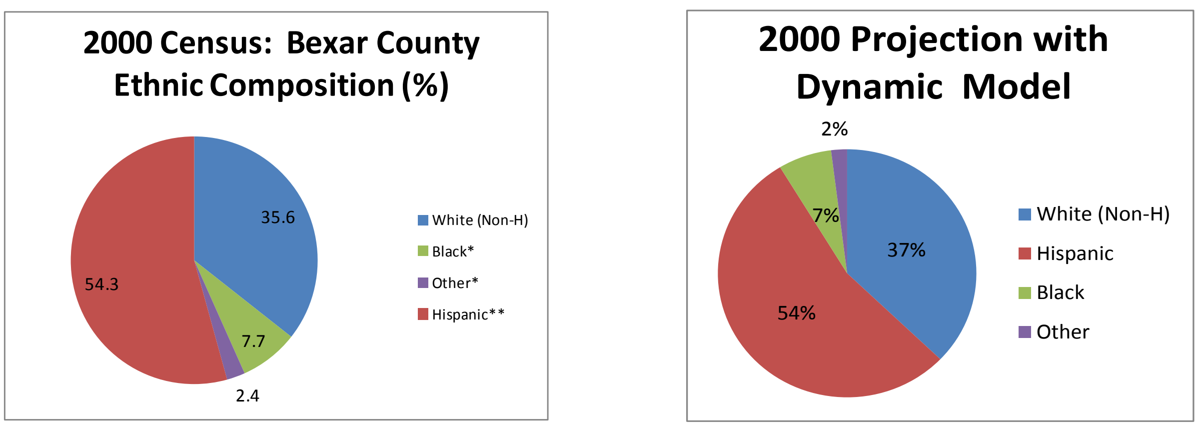


Figure 6. Ethnic composition is displayed by percentages.

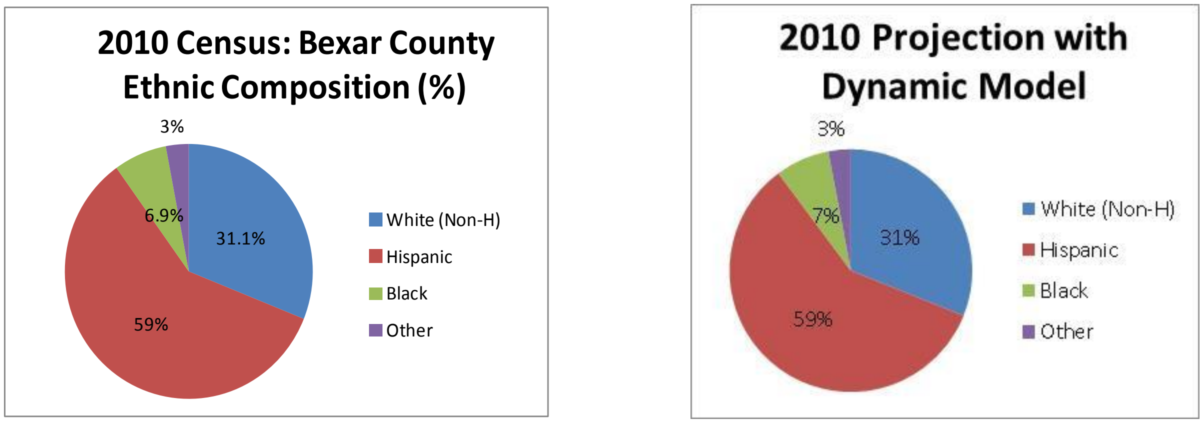


Figure 7. Ethnic composition is displayed by percentages

As it is demonstrated in those figures, the population projections were reasonably close to the 2000 and 2010 census data using the dynamic Leslie matrix.

At this point, the fitness of the model is reasonably satisfactory; with the population from the 1990 census and 1991-1994 census estimation as the initial population vector, the dynamic Leslie matrix was then multiplied with the population vector at each year to generate the population projection for the following year. A sequence of population vectors was produced. Once we went beyond the current time frame into the future years, it became the population projections of Bexar County.

4. Data Results and Conclusions

The population projections in this study were made from 1995 to 2020. The reason for this stopping point was because 2020 is the next future U.S. Census and also a year of the presidential election.

Figure 8 is an example of the yearly population projections for the white group. First, 3D plotting was used to compare the real population data with the projected data and to visualize the trends.

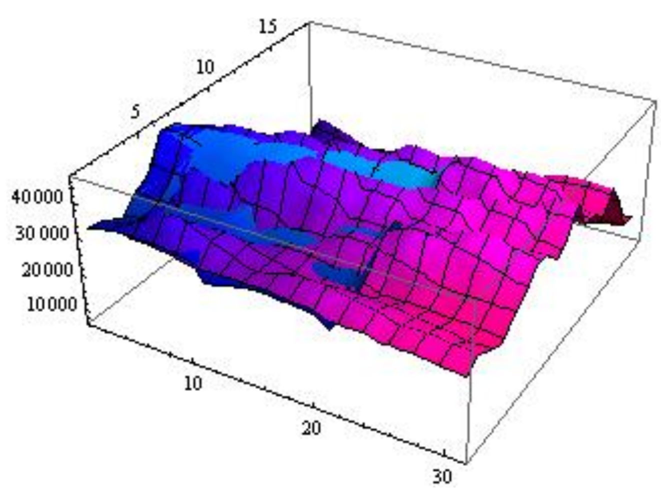


Figure 8. The surface with grid lines is the projected data for the white population using 3D plotting

Table 1 shows some segments of the population projections for the white group:

Table 1. Results and Projections from 1990-2020

Variable Matrix	1990	1991	1992		1999	2000	2001		2018	2019	2020
00-04	32579	33380	33744		31211	31160	31600		27056	26878	26583
05-09	31647	32056	32619		35931	34225	33180		28786	28948	28719
10-14	28943	29958	30468		34977	34478	33695		25772	26034	25988
15-19	32969	32247	32215		29671	29735	29120		24744	24459	24104
20-24	35711	35166	35268		32150	31445	31417		27491	27474	27325
25-29	45094	45223	43892		44783	45782	45616		30303	29664	29765
30-34	44617	45147	45396		39545	39253	39448		36846	35667	34265
35-39	41307	41836	42114		37605	37467	38509		32431	33503	33828
40-44	38295	39910	39918		44607	43937	42771		29822	29813	29673
45-49	29430	30847	33038		37132	37802	38645		32202	32370	31697
50-54	23764	24416	25243		36949	37576	37816		37986	37962	37997
55-59	22828	22504	22465		29055	30809	32043		30736	29754	29725
60-64	21823	21917	21902		22553	22711	23661		30433	29344	29220
65-69	23018	22098	21214		16801	16707	16724		25550	25909	25662
70-74	17813	19050	19805		20477	20659	21092		31280	30883	30823
75-79	12519	12909	13164		20006	20319	20144		21674	23481	24034
80-84	8217	8532	8764		7251	7647	8229		12875	13357	14173
85+	6500	6809	7117		8582	8877	9252		13564	13570	13652
Total Projectio	497074	504005	508346	......	529286	530592	532961	.....	499553	499071	497234

Table 2 and Figure 9 show the overall population projection when all four ethnic groups are combined, using the dynamic Leslie model.

Table 2. Population Projections for All Ethnic Groups from 2010-2020

Ethnicity	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
W	526806	528957	529840	531559	521228	520664	519336	519360	510409	510604	509566
H	998860	1039040	1062795	1093516	1133125	1177963	1205748	1242044	1291744	1343556	1375019
B	121877	131235	135236	138983	140572	152508	157401	161357	163106	177092	183553
O	50829	57964	61194	66376	68058	77488	81963	89224	91737	104439	110101
Total	1698371	1757195	1789065	1830433	1862982	1928622	1964448	2011985	2056996	2135690	2178239

As the result, according the projections in table 2 and Figure 7, the population in Bexar County will grow to over 1.9 million in 2016 and will continue to grow to over 2 million (about 2.1 million) in 2020. Even though the population is growing, the majority of the growth will be contributed by the Hispanic, Black, and Other ethnic groups, while the White ethnic group is projected to decline slightly.

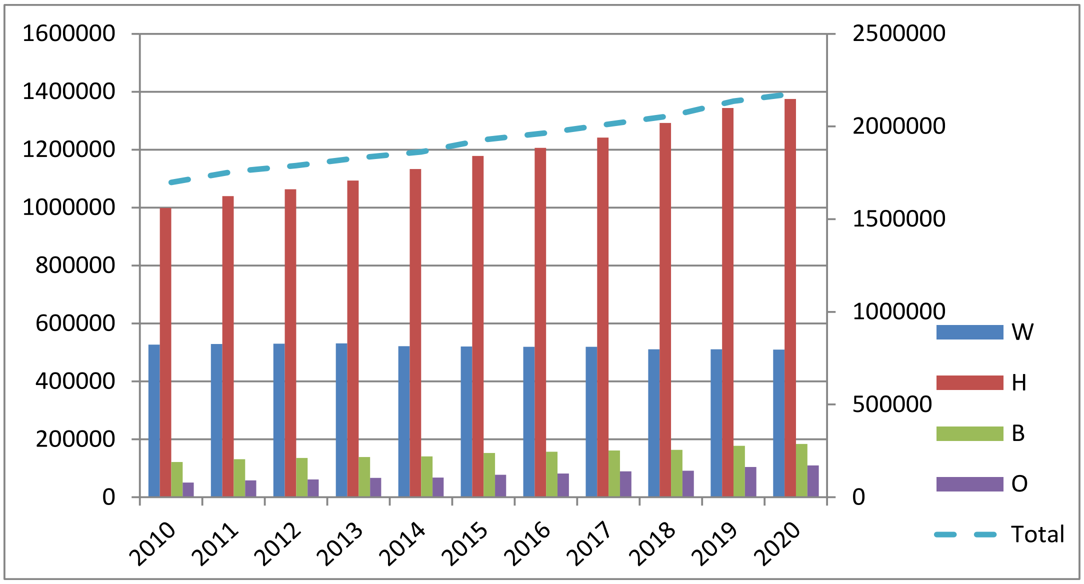


Figure 9. Population projections for ethnic group in Bexar County using a variable matrix

Figure 10 shows the population projections of ethnic groups in Bexar County over the next three presidential election cycles using the dynamic matrix.

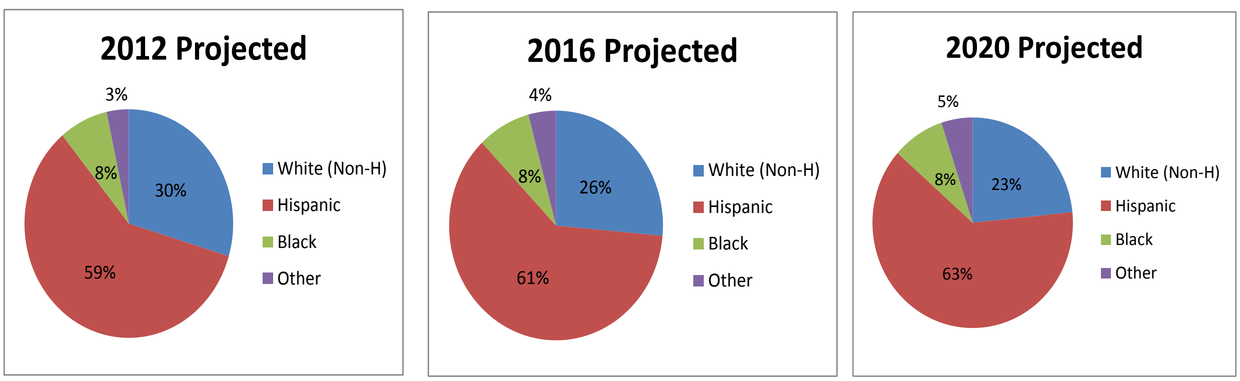


Figure 10. Ethnic composition displayed by percentages for the next three presidential elections

Figure 11 shows the 2010 census data of age groups compared with the projections using a dynamic matrix. One can observe that the results were reasonably close.

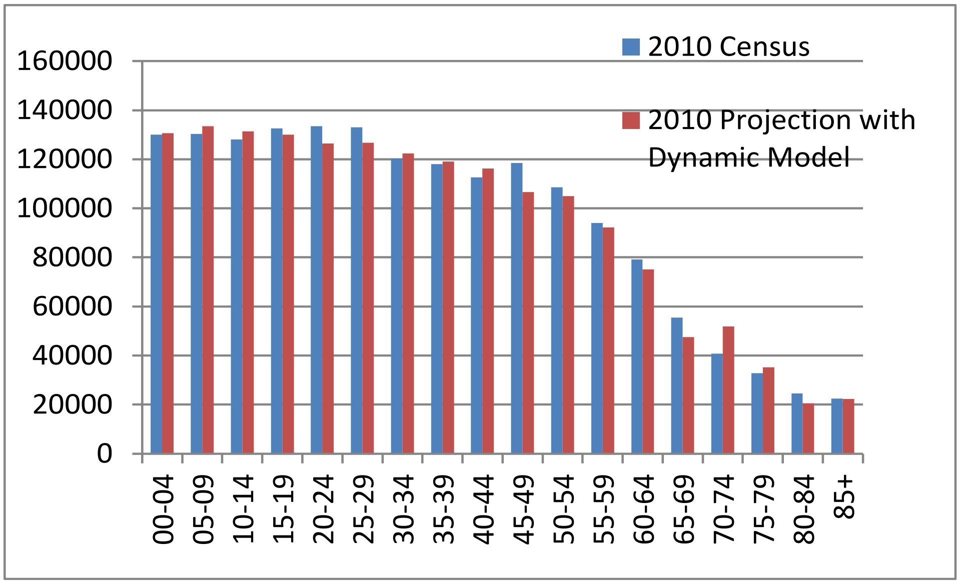


Figure 11. Comparison of total population by age groups

The projections for three presidential election cycles were also calculated with respect to the age group compositions in Bexar County and shown in Figure 12.

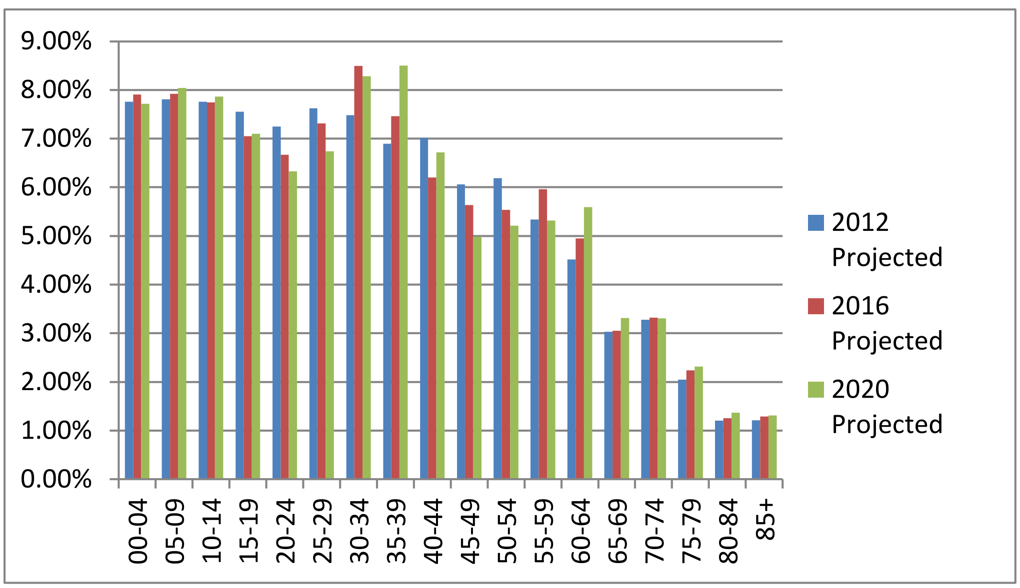


Figure 12. Total population by age groups in percentages for the next three presidential elections

We also tried to compare the dynamic model with a static model in our previous study and to see if any improvement was made. The following graphs (Figure 13) are the dynamic Leslie model projections compared to the real population data from 1990 to 2010 and the static Leslie model from the previous study.

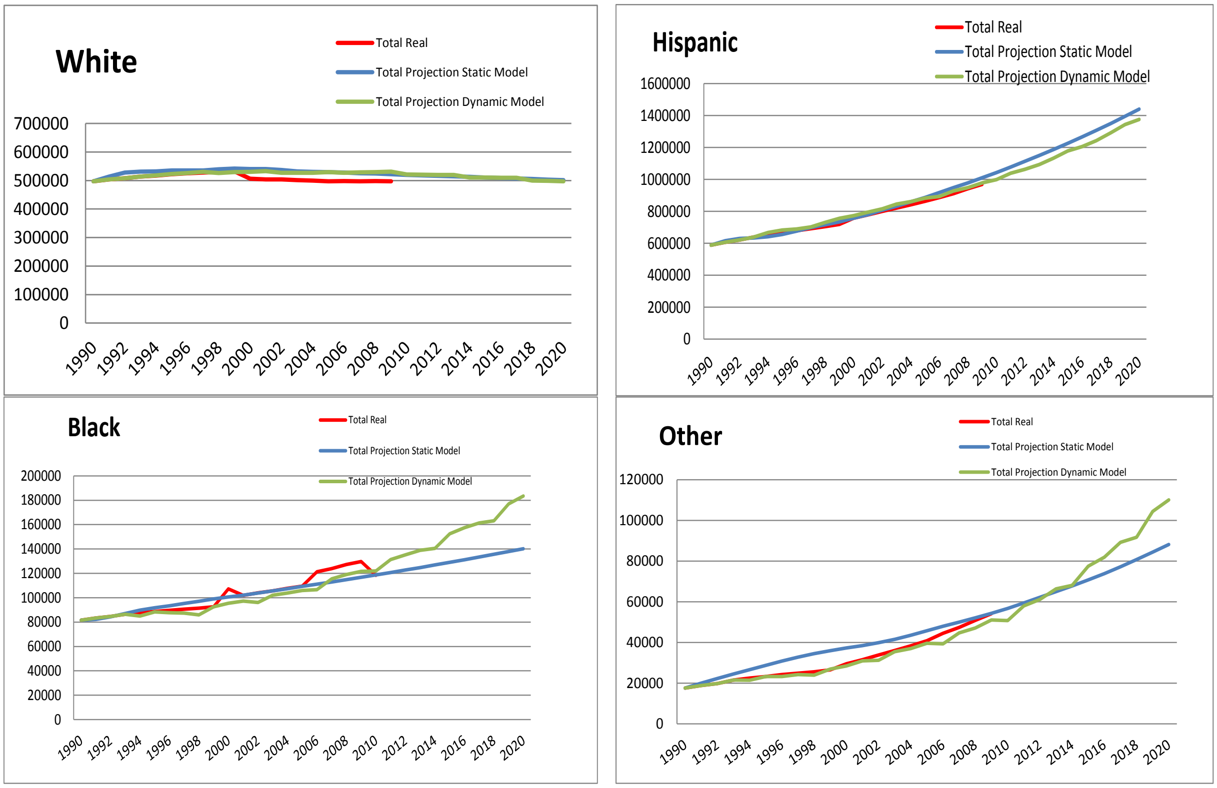


Figure 13. Real population data vs. static Leslie matrix and dynamic Leslie matrix

The green lines are the projections of the dynamic Leslie matrix and the blue ones are the projections of the static Leslie matrix. The dynamic Leslie matrix seems to give the best fit model for the Hispanic ethnic group. The Hispanic model under dynamic matrix is almost identical to the real population data (when that data is available). The second best fit is the other model. The dynamic matrix gives a good fit for the real data. In the model for the Black group, at first the dynamic matrix seems to closely model the real data but then begins to underestimate. The projections for the Black model seem to overestimate after the year 2010. The model for the White group using the dynamic matrix seems to be reasonably close, but then it seems to show a downward trend that is not entirely justifiable by the real data line, which may lead to under estimation by the model to some extent. The downward movement around 2000 in the real data is due to the fact that the definition and method of collection of data was changed by both Census Bureau and the State in 2000 (as indicted by the drop in the white population around the year 2000). The static matrix model seemed to fit the White model reasonably well. However, all four models seem to capture the general trends. Overall, the dynamic Leslie model seems to capture the trends better and produce better projections for the total population growth of Bexar County.

In Figure 14, the real populations of 2010 of the ethnic groups were also compared to the dynamic and static Leslie matrices by age groups.

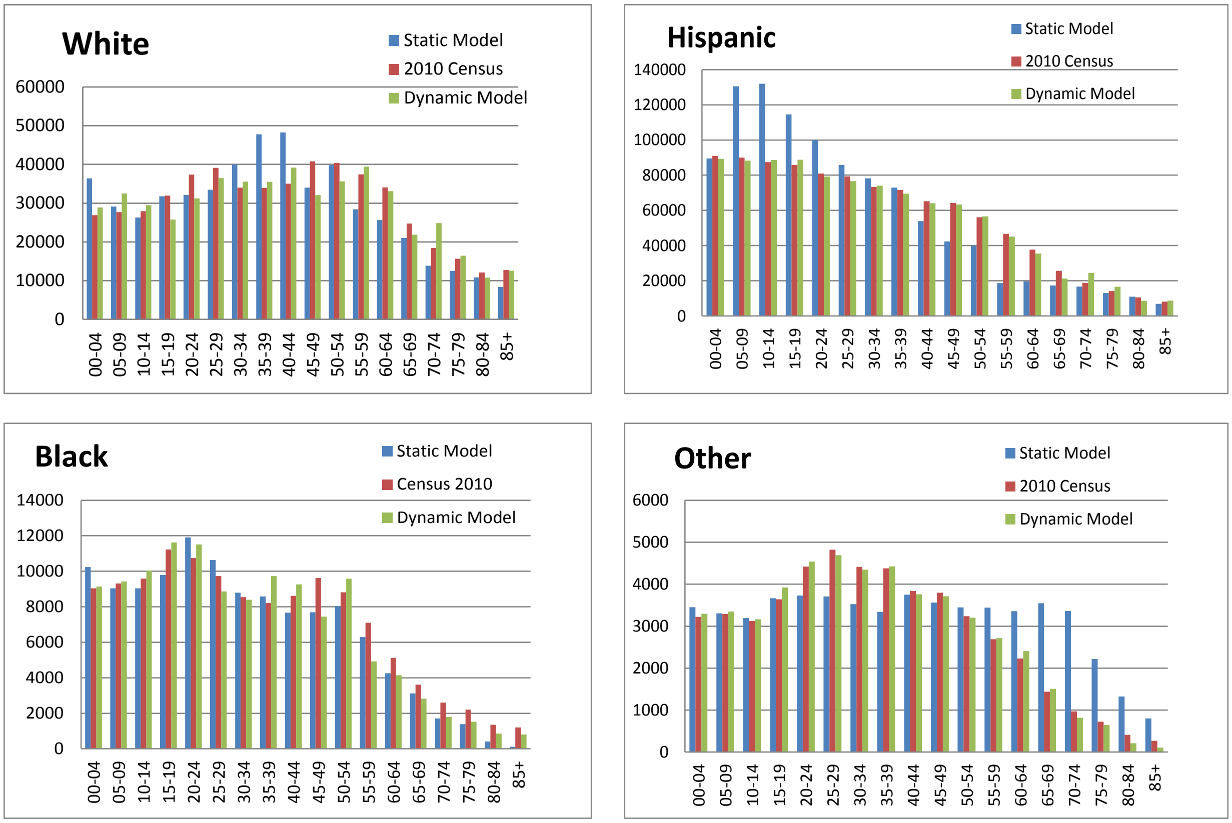


Figure 14. 2010 real population age groups vs. static Leslie matrix and dynamic Leslie matrix

When observing the line graphs, one can see that the dynamic Leslie matrix modeled each of the ethnic groups' age groups better than the static Leslie matrix. Not only did it model the age groups better but the results were more accurate. When the static and dynamic matrices projections were compared, the dynamic Leslie matrix had a smaller standard deviation than the static matrix except for the Black ethnic group.

5. Conclusions

In conclusion, we used curve fitting techniques to produce functions that capture the trend for the survival functions and birth rate functions. Those functions were then used to construct a dynamic Leslie matrix for each of the ethnic groups. With some moderation using weight factors, the model generally captured the general trend for population group and provided a detailed projection of each ethnic group and different age groups within the ethnic group. Those projects are important for civil and political leaders to plan for the city services and grow over the next 20 years.

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