



## Numerical Solution of Linear Fredholm Integro-Differential Equations by Non-standard Finite Difference Method

**Pramod Kumar Pandey**

Department of Mathematics  
Dyal Singh College (University of Delhi)  
Lodhi road, New Delhi 110003 India

Department of Natural Resource Engineering Management  
University of Kurdistan Hewler  
Erbil, Kurdistan Region, Iraq  
[pramod\\_10p@hotmail.com](mailto:pramod_10p@hotmail.com)

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### Abstract:

In this article we consider a non-standard finite difference method for numerical solution of linear Fredholm integro-differential equations. The non-standard finite difference method and the repeated / composite trapezoidal quadrature method are used to transform the Fredholm integro-differential equation into a system of non-linear algebraic equations. The numerical experiments on some linear model problems show the simplicity and efficiency of the proposed method. It is observed from the numerical experiments that our method is convergent and second order accurate.

**Keywords:** Composite Trapezoidal Method, Fredholm Integro-differential equations, Non-Linear Equation, Non-Standard Finite difference, Quadrature formulas

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### 1. Introduction

Mathematical modeling of real-life problems usually results in some form of functional equations, e.g. algebraic equations, differential equations, integral equations and others. The

occurrence of differential equations and integral equations is common in many areas of the sciences and engineering. However, research work in this field results in a new specific topic, where both differential and integral operators appear together in the same equation. These new type of equations are known as integro-differential equations. Many mathematical formulations in natural science, i.e., study of fluid, biology and chemical kinetics, contain integro-differential equations. In particular, the conversion of boundary value problems in differential equations to integro-differential equations, with limits of integration, considered as constant, is termed Fredholm integro-differential equations. This class of problems has gained importance in the literature with a variety of applications. In most cases it is not possible to obtain exact solutions of these problems using theoretical / analytical methods. In these cases we need an approximate solution of these problems and with development in the computational facilities in the past few decades, we have seen substantial progress in the development of approximate solutions of these problems. In the literature, there are different approaches and varieties of numerical and analytical methods that are used to solve Fredholm integro-differential equations namely compact finite difference method Zhao (2006), an extrapolation method Chang (1982), Taylor series Yalcinbas (2002), method of regularization Phillips (1962), Tikhonov (1963). In the recent past, there are many new methods reported in literature, such as the variational method He (2009), Adomian decomposition method Wazwaz (1999)}, variational iterations method Saadati et al. (2008), and references therein.

In this article we consider a method for the numerical solution of linear Fredholm integro-differential equations of the form

$$y'(x) = f(x, y) + \int_a^b K(x, t)y(t)dt, \quad a \leq x \leq b. \quad (1)$$

subject to the initial condition

$$y(x) = \alpha,$$

where  $\alpha$  is real constant. The functions  $f(x, y)$  and the kernel  $K(x, t)$  are known. The solution  $y(x)$  is to be determined.

The emphasis in this article is on the development of efficient numerical methods as opposed to proving theoretical concepts of convergence and existence. Thus, existence and uniqueness of the solution to problem (1) is assumed. We further assume that problem (1) is well posed. We will not consider the specific assumption on source function to ensure existence and uniqueness of the solution to problem (1). It is important to prove theorems on uniqueness, existence, and convergence and that can be found in the literature Hu et al. (1987), Hairer et al. (1993).

Over the last few decades, a variety of specialized method Van (1988), Ramos (2007) for the numerical solution of initial value problems in ODEs has been reported in the literature. These methods generated impressive and accurate numerical results for the model problems considered in experiment. Hence, the purpose of this article is to propose a non-standard finite difference method similar to Ramos (2007) for numerical solution of problem (1).

To the best of our knowledge, no method similar to the proposed method for the numerical solution of problem (1) has been discussed in the literature to date. We hope that others may find the proposed method appealing, and an improvement to those existing finite difference methods for the numerical solution of integro-differential equations.

We present our work in this article as follows. In Section 2, we derive a non-standard finite difference method. In Section 3, we discuss local truncation error in the proposed method. We present the application of the proposed method for solving problem (1) and explanatory numerical results to show the efficiency of the new method in Section 4. Discussion and conclusion on the performance of the new method are presented in Section 5.

## 2. The Non-Standard Finite Difference Method

We define  $N$  finite nodal points of the domain  $[a, b]$ , in which the solution of problem (1) is desired, as  $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$  using uniform step length  $h$  such that  $x_i = a + i.h, i = 0(1)N$ . Suppose that we wish to determine the numerical approximation of the theoretical solution  $y(x)$  of problem (1) at the nodal points  $x_i, i = 1, 2, \dots, N$ . We denote the numerical approximation of  $y(x)$  at node  $x_i$  as  $y_i$ . Let us denote  $f_i$  as the approximation of the theoretical value of the source function  $f(x, y(x))$  at node  $x_i, i = 0, 1, 2, \dots, N$ . Further, we have assumed that  $f(x, t)$  is a separable kernel otherwise by using Taylor series expansion for the kernel to reduce it to a separable kernel. Thus the integro-differential Equation (1) at node  $x_i$  may be written as

$$y'(x_i) = f(x_i, y_i) + \int_a^b K(x_i, t)y(t)dt. \quad (2)$$

We approximate the integral that appears in Equation (2) by the repeated / composite trapezoidal quadrature method Jain et al. (1987) which yields the following

$$\int_a^b K(x_i, t)y(t)dt = \sum_{j=0}^N \{K(x_i, t)\lambda_j y(t_j) + E_{tj}\}, \quad (3)$$

where  $a = t_0 < t_1 < t_2 < \dots < t_{N-1} < t_N = b, j = 0, 1, 2, \dots, N$  using uniform step length  $h$  such that  $t_j = a + j.h, j = 0, 1, 2, \dots, N, E_{tj}$  is the truncation error in  $j^{th}$  interval and quadrature nodes  $\lambda_j, j = 0, 1, 2, \dots, N$  are numerical coefficients such that

$$\lambda_j = \begin{cases} \frac{1}{2}h, & \text{if } j = 0, N, \\ h, & \text{otherwise } j = 1, 2, \dots, N - 1, \end{cases}$$

do not depend on the function  $y(t)$ . The term  $E_{tj}$  in (3) depends on  $N$  and large  $N$  reduces  $E_{tj}$  considerably. Thus, substituting the value of the integral term in (2) from (3) we have after neglecting the error terms,

$$y'_i = f_i + \sum_{j=0}^N K_{i,j} \lambda_j y_j. \quad (4)$$

Let us define a new source function  $G(x, y)$  as,

$$G(x, y) = f(x, y) + \sum_{j=0}^N K(x, t) \lambda_j y(t_j),$$

and at each node  $x = x_i$  we can write  $G(x, y)$  as,

$$G(x_i, y_i) = f_i + \sum_{j=0}^N K_{i,j} \lambda_j y_j \quad (5)$$

and substitute this so defined source function in (4), we have

$$y'_i = G(x_i, y_i). \quad (6)$$

Let us assume a local assumption as in Lambert (1991) that no previous truncation errors have been made i.e.  $y(x_i) = y_i$  and following the ideas in Van (1988), Ramos (2007), Pandey (2013), we propose a non-standard finite difference method for the approximation of the analytical solution  $y(x_{i+1})$  of the problem (1) at node  $x = x_{i+1}$  as,

$$y_{i+1} = y_i + \frac{2hG_i^2}{2G_i - hG'_i}, \quad i = 0, 1, 2, \dots, N-1, \quad (7)$$

where  $G'_i = \frac{\partial G_i}{\partial x}$ . Thus, we obtain a system of nonlinear equations at each nodal point  $x_{i+1}$ ,  $i = 0, 1, 2, \dots, N-1$ .

For computational purposes in Section 4, we use the following finite difference approximation in place of  $hG'_i$  in (7).

$$hG'_i = G_{i+1} - G_i. \quad (8)$$

Thus, from (8) we can write (7) as,

$$y_{i+1} = y_i + \frac{2hG_i^2}{G_{i+1} - 3G_i}, \quad i = 0, 1, 2, \dots, N-1, \quad (9)$$

which is an implicit nonlinear system of equations. Some complexity exists in the system and so computation is difficult. Since we need to solve a nonlinear system with a large number of equations, we have to rely on some iterative type method. We apply Newton-Raphson iterative method to solve the above system of nonlinear equations.

### 3. Local Truncation Error

The local truncation error at the node  $x = x_{i+1}$  using exact arithmetic, is

$$T_{i+1} = y(x_i + h) - y_{i+1}.$$

At the nodal point  $x = x_{i+1}$ ,  $i = 0, 1, 2, \dots, N - 1$ , the truncation error  $T_{i+1}$  in method (7) may be written as Jain (1987),

$$T_{i+1} = y_{i+1} - y_i - \frac{2hG_i^2}{2G_i - hG_i'}.$$

$$T_{i+1} = y_{i+1} - y_i - hG_i \left(1 - \frac{hG_i'}{2G_i}\right)^{-1}.$$

Writing the Taylor series expansion for  $y$  at nodal point  $x = x_i$  and using the binomial expansion along with  $y_i'' = G_i'$  and  $y_i' = G_i$ , we have

$$T_{i+1} = \frac{h^3}{12} \left( 2y_i^{(3)} - 3 \frac{(y_i'')^2}{y_i'} \right). \quad (10)$$

Thus, we obtain a truncation error at each node of  $O(h^3)$ .

### 4. Numerical Experiments

To illustrate our method and demonstrate its computational efficiency, we consider two model problems. In each model problem, we took uniform step size  $h$ . In Tables 1 and 2, we show the maximum absolute error (*MAY*), computed for different values of  $N$  defined as

$$MAY = \max_{1 \leq i \leq N} |y(x_i) - y_i|.$$

The order of convergence  $O_N$  of the method (9) is estimated by the formula

$$O_N = \log_m \left( \frac{MAY_N}{MAY_{mN}} \right),$$

where  $m$  can be estimated by considering the ratio  $N$ 's.

We use Newton-Raphson iteration method to solve the system of nonlinear equations arising from equation (9). All computations are performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The solutions are computed on  $N$  nodes and iteration is continued until either the maximum difference between two successive iterates is less than  $10^{-10}$  or the number of iterations reaches  $10^3$ .

**Problem 1.**

The model linear problem Darania (2007) is given by

$$y'(x) = 1 - \frac{1}{3}y(x) + \int_0^1 xty(t)dt, \quad y(0) = 0, \quad 0 \leq x \leq 1.$$

The analytical solution is  $y(x) = x$ . The *MAY* computed by method (9) for different values of  $N$  and number of iterations, *Iter*, are presented in Table 1.

**Table 1.** Maximum absolute error  $|y(x_i) - y_i|$  in problem 1

	$N$				
	128	256	512	1024	2048
<i>MAY</i>	.35524368(-4)	.76293945(-5)	.19073486(-5)	.47683716(-6)	.11920929(-6)
<i>Iter.</i>	3	2	2	2	2

**Problem 2.**

The model linear problem Darania (2007) is given by

$$y'(x) = y(x) + e^x - x + \int_0^1 y(t)dt, \quad y(0) = 0, \quad 0 \leq x \leq 1.$$

The analytical solution is  $y(x) = xe^x$ . The *MAY* computed by method (9) for different values of  $N$  and number of iterations, *Iter*, are presented in Table 2.

**Table 2.** Maximum absolute error  $|y(x_i) - y_i|$  in problem 2

	$N$				
	128	256	512	1024	2048
<i>MAY</i>	.40245056(-3)	.10228157(-3)	.26464462(-4)	.66757202(-5)	.26226044(-5)
<i>Iter.</i>	11	10	7	6	4

## 5. Conclusion

A uniform step size method to determine the numerical solution of Fredholm integro-differential equation problems has been developed. This method has been used for transforming Fredholm integro-differential equation to a system of nonlinear algebraic equations, i.e. at each nodal point  $x = x_i, i = 1, 2, 3, \dots, N$ , we obtain a system of algebraic equations given by (7) which is not easy to solve analytically. The proposed method produced a good approximate numerical solution for two model problems with uniform step size. The numerical results of the model problems showed that the proposed method is computationally efficient. The rate of convergence of the present method is quadratic. The idea presented in this article leads to the possibility to develop nonstandard uniform step size difference methods to solve higher order integro-differential equations. Work in this direction is in progress.

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