Exponentially Fitted Variants of the Two-Step Adams-Bashforth Method for the Numerical Integration of Initial Problems

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Abstract

In this paper, we propose new variants of the two-step Adams-Bashforth and the one-step Adams-Moulton methods for the numerical integration of ordinary differential equations (ODEs). The methods are constructed geometrically from an exponentially fitted osculating parabola. The accuracy and stability of the proposed variants is discussed and their applicability to some initial value problems is also considered. Numerical experiments demonstrate that the exponentially fitted variants of the two-step Adams-Bashforth and the one-step Adams-Moulton methods outperform the existing classical two-step Adams-Bashforth and one-step Adams-Moulton methods respectively.

1. Introduction

For the numerical integration of the initial value problem (IVP)

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0, \quad x_0 \le x \le x_n, \tag{1.1}$$

the two-step Adams-Bashforth and the one-step Adams-Moulton methods are given by

$$y_{n+1} = y_n + \frac{h}{2}\left[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})\right], \tag{1.2}$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right], \tag{1.3}$$

respectively.

We set $x_n = x_0 + nh$, $n = 0,1,2,3,...,h =$ step-size, $f_n = f(x_n, y_n)$ and y_n is assumed to be a close approximation to the value $y(x_n)$, where $y(x)$ is the unique solution of (1.1). The explicit method (1.2) has a second order convergence and its stability interval is $(-1,0)$. The implicit method (1.3) has also a second order convergence and is A-stable. Many different ways like discretization, the weight function approach, numerical integration etc. may be used to construct the single-step and multi-step methods. The theory of interpolation and collocation has been extensively used to construct numerical methods for ordinary differential equations. Omolehin *et al.* (2003) proposed a new class of Adams-Bashforth schemes for the numerical solution of (1.1) by using collocation approach. Arevalo *et al.* (2002) have given a collocation formulation of multi-step methods for variable step-size extensions. Odejide and Adeniran (2012) proposed a five-step ninth order hybrid linear multi-step method for the solution of first order initial value problems via the interpolation and collocation procedure.

In this paper, we propose geometrically constructed new classes of the two-step Adams-Bashforth and the one-step Adams-Moulton methods by using the concept of osculating curves, from which the classical two-step Adams-Bashforth and one-step Adams-Moulton methods can be obtained as special cases.

This paper is organized as follows: In Section 2, we recall basic definitions. The constructions of new variants of the two-step Adams-Bashforth and the one-step Adams-Moulton methods are carried out in Section 3. Error analysis of the variants is carried out in Section 4. The stability analysis of the variants is presented in Section 5, and finally the numerical results are discussed in Section 6.

2. Basic Definitions

Definition 2.1. (Consistency)

A linear multi-step method is said to be consistent if it has an order of convergence, say $p \geq 1$.

Definition 2.2. (Root Condition)

A linear multi-step method is said to satisfy the root condition if the roots of the corresponding characteristic equation, $p(\xi) = 0$, lie inside the unit circle in the complex plane, and are simple if they lie on the unit circle.

Definition 2.3. (A -stability)

A one-step method is A -stable if it may be applied to the test equation

$$y' = \lambda y,$$

where λ is a complex number with $\mathrm{Re}(\lambda) < 0$, it yields $y_{n+1} = Q(z) y_n$, where $z = \lambda h$ such that $Q(z) < 1$.

Definition 2.4. (Osculating curves)

Two curves $f(x)$ and $g(x)$ are said to be osculating at a point $x = x_0$ to degree n if

$$f(x_0) = g(x_0), \quad f'(x_0) = g'(x_0), ..., f^{(n)}(x_0) = g^{(n)}(x_0).$$

3. Construction of New Families of Methods

Consider the initial value problem (1.1). We are interested in finding the approximate solution of (1.1) at $x_{n+1} = x_n + h$.

Let

$$y = \phi(x), \tag{3.1}$$

be the actual solution curve of (1.1) in the xy -plane.

Now, consider an exponentially fitted parabola in the following form

$$y = e^{p(x-x_n)} \left[a(x - x_n)^2 + b(x - x_n) + c \right]. \tag{3.2}$$

where a, b and c are arbitrary constants to be determined by using some tangency conditions at $x = x_n$. Here, the quantity p will play a very important role. If we consider (3.2) as an osculating parabola to the solution curve $y = \phi(x)$ at $x = x_n$, then we have

$$y(x_n) = \phi(x_n), \quad y'(x_n) = \phi'(x_n), \quad y''(x_n) = \phi''(x_n). \tag{3.3}$$

Using these conditions, we obtain

$$a = \frac{f'(x_n, y_n) + p^2 y_n - 2pf(x_n, y_n)}{2}, \quad b = f(x_n, y_n) - py_n, \quad c = y_n. \tag{3.4}$$

Therefore, at $x = x_{n+1}$ Equation (3.2) gives

$$y_{n+1} = e^{ph}\left[\left(\frac{f'(x_n, y_n) + p^2 y_n - 2pf(x_n, y_n)}{2}\right)h^2 + (f(x_n, y_n) - py_n)h + y_n\right]. \tag{3.5}$$

Now, we shall discuss the role of the quantity p and derive various methods.

3.1. Taylor Series Method

Consider

$$y_{n+1} = e^{hp}\left[\left(\frac{f'(x_n, y_n) + p^2 y_n - 2pf(x_n, y_n)}{2}\right)h^2 + (f(x_n, y_n) - py_n)h + y_n\right]. \tag{3.6}$$

If we let $p = D = \dfrac{d}{dx}$ be a differential operator and assume that all higher order derivatives of y_n exist, then from Equation (3.6), we obtain, on expanding e^{hp} in Taylor expansion,

$$y_{n+1} = \left[1 + hD + \frac{h^2 D^2}{2!} + \ldots\right]y_n, \tag{3.7}$$

which is the well-known Taylor series method.

3.2. Family of The Two-Step Adams-Bashforth Method

Let $p \in \Re$ such that $|ph| < 1$. Using Taylor expansion for e^{ph}, we obtain the following expression

$$e^{ph} = 1 + ph + \frac{p^2 h^2}{2} + O(p^3 h^3).$$

Using this expression in (3.5), yields on simplification,

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2}f'(x_n, y_n) + p\frac{h^3}{2}f'(x_n, y_n)$$
$$- \frac{p^2 h^3}{2}f(x_n, y_n) + O(h^4). \tag{3.8}$$

In order to avoid computation of the second order derivative of y_n in (3.8), we approximate it with a backward difference at $x = x_n$,

$$f'(x_n, y_n) \approx \frac{f(x_n, y_n) - f(x_{n-1}, y_{n-1})}{h}.$$

Using this approximate value of $f'(x_n, y_n)$ in (3.8),

$$y_{n+1} = y_n + \frac{h}{2}[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})] + p\frac{h^2}{2}[f(x_n, y_n) - f(x_{n-1}, y_{n-1})]$$
$$- \frac{p^2 h^3}{2} f(x_n, y_n) + O(h^4). \tag{3.9}$$

Ignoring the third and higher order terms of h in (3.9), the following simplified expression for y_{n+1} is

$$y_{n+1} = y_n + \frac{h}{2}[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})] + p\frac{h^2}{2}[f(x_n, y_n) - f(x_{n-1}, y_{n-1})]. \tag{3.10}$$

These are the new variants of the two-step Adams-Bashforth method, which will be proved to have the same order of convergence as the classical two-step Adams-Bashforth method. We also call formula (3.10) a one-parameter family of the two-step Adams-Bashforth method. For $p = 0$, one can obtain the well-known two-step Adams-Bashforth method (1.2) for the numerical solution of (1.1).

A method for selection of the parameter p :

Rewrite (3.10) in the following form

$$y_{n+1} = y_n + F_1 + pF_2, \tag{3.11}$$

where

$$F_1 = \frac{h}{2}[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})] \text{ and } F_2 = \frac{h^2}{2}[f(x_n, y_n) - f(x_{n-1}, y_{n-1})].$$

From this iteration formula, we can see that at every iteration step, the value of the parameter p is such that

$$p = \begin{cases} +ve, & if \quad F_1 F_2 \geq 0, \\ -ve, & if \quad F_1 F_2 \leq 0. \end{cases} \tag{3.12}$$

3.3. Family of the One-Step Adams-Moulton Method

In order to avoid the computation of second order derivative of y_n in the formula (3.8), we approximate it with a forward difference given by

$$f'(x_n, y_n) \approx \frac{f(x_{n+1}, y_{n+1}) - f(x_n, y_n)}{h}.$$

Substituting this approximate value of $f'(x_n, y_n)$ into the formula (3.8), we have

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})] + p\frac{h^2}{2}[f(x_{n+1}, y_{n+1}) - f(x_n, y_n)]$$
$$- \frac{p^2 h^3}{2} f(x_n, y_n) + O(h^4). \tag{3.13}$$

Ignoring the third and higher order terms of h in (3.13), the simplified expression for y_{n+1} is given by

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})] + p\frac{h^2}{2}[f(x_{n+1}, y_{n+1}) - f(x_n, y_n)]. \tag{3.14}$$

These are the new variants of the one-step Adams-Moulton method, which will be proved to have the same order of convergence as the one-step Adams-Moulton method. We also call formula (3.14) a one-parameter family of the one-step Adams-Moulton method. For $p = 0$, one can obtain the well-known one-step Adams-Moulton method for the numerical solution of (1.1).

A method for selection of the parameter p :

Rewrite (3.14) in the following form

$$y_{n+1} = y_n + G_1 + pG_2, \tag{3.15}$$

where

$$G_1 = \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})] \text{ and } G_2 = \frac{h^2}{2}[f(x_{n+1}, y_{n+1}) - f(x_n, y_n)].$$

From this iteration formula, we can see that at every iteration step, the parameter p is such that

$$p = \begin{cases} +ve, & if \quad G_1 G_2 \geq 0, \\ -ve, & if \quad G_1 G_2 \leq 0. \end{cases} \tag{3.16}$$

4. Error Analysis

The notion of an error is a very important criterion in the study of any numerical method. In this section we shall discuss the error encountered in the use of the proposed families of methods.

4.1. Accuracy of the Family of the Adams-Bashforth Method

Consider the family (3.10)

$$y_{n+1} = y_n + \frac{h}{2}[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})] + p\frac{h^2}{2}[f(x_n, y_n) - f(x_{n-1}, y_{n-1})] \qquad (4.1)$$

Since

$$f(x_n, y_n) = y_n' \quad \text{and} \quad f(x_{n-1}, y_{n-1}) = y_{n-1}' = y_n' - hy_n'' + \frac{h^2}{2} y_n''' + O(h^3),$$

we obtain the following expression

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2} y_n'' + h^3 \left(\frac{-y_n'''}{4} + \frac{py_n''}{2} \right) + O(h^4). \qquad (4.2)$$

Assuming the existence of higher order derivatives of $y(x)$, we can expand $y(x_{n+1})$ about $x = x_n$ in the following form by Taylor expansion

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{h^3}{6} y'''(x_n) + O(h^4). \qquad (4.3)$$

After subtracting (4.2) from (4.3), we obtain the following local truncation error

$$LTE = \left(\frac{5}{12} y_n''' - \frac{p}{2} y_n'' \right) h^3 + O(h^4).$$

Hence, the proposed family of the two-step Adams-Bashforth method has a second order convergence. We have shown that the proposed family has order of convergence >1. Therefore, the proposed family (3.10) is consistent.

Remark:

By putting $\frac{5}{12} y_n''' - \frac{p}{2} y_n'' = 0$, we obtain the value of $p = \frac{5y_n'''}{6y_n''}$. It is worth noting that if we substitute this value of p into (3.10) and discretize second and third order derivatives of y_n by backward differences, then we obtain the following three-step Adams-Bashforth method

$$y_{n+1} = y_n + \frac{h}{12}\left[23f(x_n, y_n) - 16f(x_{n-1}, y_{n-1}) + 5f(x_{n-2}, y_{n-2})\right]. \tag{4.4}$$

4.2. Accuracy of the Family of the Adams-Moulton Method

Consider the family (3.14)

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right] + p\frac{h^2}{2}\left[f(x_{n+1}, y_{n+1}) - f(x_n, y_n)\right]. \tag{4.5}$$

Since

$$f(x_n, y_n) = y_n' \text{ and } f(x_{n+1}, y_{n+1}) = y_{n+1}' = y_n' + hy_n'' + \frac{h^2}{2}y_n''' + O(h^3).$$

By substituting these values in (4.5), we obtain the following expression

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + h^3\left(\frac{y_n'''}{4} + \frac{py_n''}{2}\right) + O(h^4). \tag{4.6}$$

Assuming the existence of higher order derivatives of $y(x)$, we can expand $y(x_{n+1})$ about $x = x_n$ in the following form by Taylor expansion

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y'''(x_n) + O(h^4). \tag{4.7}$$

After subtracting (4.6) from (4.7), we obtain the following local truncation error

$$LTE = \left(\frac{-1}{12}y_n''' - \frac{p}{2}y_n''\right)h^3 + O(h^4).$$

Hence, the proposed family of the one-step Adams-Moulton method (3.14) has second order convergence. We have shown that the proposed family has order of convergence >1. Therefore, the proposed family (3.14) is consistent.

5. Stability Analysis

It is well-known that a linear multi-step method is convergent if and only if the method is consistent and satisfies the root condition. It is not always possible to test the root condition, when multi-step methods involve some parameters. To test the root condition of two-step methods when the corresponding characteristic equation, say $a_0\xi^2 + a_1\xi + a_2 = 0$, has

coefficients which are functions of some parameters, we use the transformation $\xi = \dfrac{1+z}{1-z}$ which maps the interior of the unit circle $|\xi| = 1$ onto the left half plane $\operatorname{Re}(z) < 0$, and the point $\xi = 1$ onto $z = 0$. Therefore, we can obtain the transformed characteristic equation as

$$p(z) = b_0 z^2 + b_1 z + b_2 = 0.$$

Then the *Routh − Hurwitz criterion* (which gives the necessary and sufficient conditions for the roots of characteristic equation to have negative real parts) is satisfied if and only if

$$b_0 > 0, \quad b_1 > 0 \quad \text{and } b_2 > 0. \tag{5.1}$$

The conditions (5.1) ensure that the roots of the original polynomial

$$a_0 \xi^2 + a_1 \xi + a_2 = 0,$$

from which $p(z)$ is derived satisfy $|\xi_i| < 1$. In the following theorem, we shall use the *Routh − Hurwitz criterion* to check the stability of the family of the two-step Adams-Bashforth method.

Theorem 5.1.

The family of the two-step Adams-Bashforth method (3.10) has stability interval $(-2, 0)$.

Proof:

Since the proposed family (3.10) has a parameter p, therefore we will use the *Routh − Hurwitz criterion* to check its stability.

Consider the test equation

$$y' = \lambda y \text{, where } \lambda < 0. \tag{5.2}$$

When the family of the two-step Adams-Bashforth method (3.10) is applied to the test equation (5.2), we obtain

$$y_{n+1} - \left(1 + \frac{3\overline{h}}{2} + \frac{ph\overline{h}}{2}\right) y_n + \left(\frac{\overline{h}}{2} + \frac{ph\overline{h}}{2}\right) y_{n-1} = 0 \text{, where } \overline{h} = \lambda h < 0. \tag{5.3}$$

Its characteristic equation given by

$$\xi^2 - \left(1 + \frac{3\bar{h}}{2} + \frac{ph\bar{h}}{2}\right)\xi + \left(\frac{\bar{h}}{2} + \frac{ph\bar{h}}{2}\right) = 0. \tag{5.4}$$

Let $\xi = \dfrac{1+z}{1-z}$, then from (5.4) we get the transformed characteristic equation given by

$$\left(2 + 2\bar{h} + ph\bar{h}\right)z^2 + \left(2 - \bar{h}(1 + ph)\right)z - \bar{h} = 0. \tag{5.5}$$

Let $k_1 = \left(2 + 2\bar{h} + ph\bar{h}\right)$, $k_2 = \left(2 - \bar{h}(1 + ph)\right)$ and $k_3 = -\bar{h}$,

Then, (5.5) can be written as

$$k_1 z^2 + k_2 z + k_3 = 0. \tag{5.6}$$

The *Routh − Hurwitz criterion* is satisfied if

$$k_1 > 0, \ k_2 > 0 \text{ and } k_3 > 0.$$

Since $k_2 > 0, \ k_3 > 0$, but k_1 will be positive if $\bar{h} > \dfrac{-2}{2 + ph}$.

Hence, for $|ph| < 1$, the stability interval of the proposed family (3.10) is $(-2, 0)$.

Theorem 5.2.

The family of the one-step Adams-Moulton method (3.14) is A-stable for $p \geq 0$.

Proof:

Consider the test equation

$$y' = \lambda y, \tag{5.7}$$

where λ is a complex number with $\text{Re}(\lambda) < 0$.

When the family of the one-step Adams-Moulton method (3.14) is applied to the test equation (5.7), we obtain

$$y_{n+1} = \left(1 + \frac{z}{2} - \frac{kz}{2}\right)y_n + \left(\frac{z}{2} + \frac{kz}{2}\right)y_{n+1}, \tag{5.8}$$

where

$$z = \lambda h \text{ and } k = ph \text{ with } |ph| < 1.$$

Equation (5.8) can be rewritten as

$$\frac{y_{n+1}}{y_n} = \frac{2 + (1 - k)z}{2 - (1 + k)z} = Q(z). \tag{5.9}$$

Now, consider

$$|Q(z)| = \left| \frac{y_{n+1}}{y_n} \right| = \left| \frac{2 + (1 - k)z}{2 - (1 + k)z} \right|. \tag{5.10}$$

Let $z = -x - iy$, where $x > 0$, then from Equation (5.10) we obtain

$$|Q(z)| = \frac{\sqrt{(2 - (1 - k)x)^2 + ((1 - k)y)^2}}{\sqrt{(2 + (1 + k)x)^2 + ((1 + k)y)^2}}. \tag{5.11}$$

Again, let $k_1 = (1 - k) > 0$ and $k_2 = (1 + k) > 0$, then Equation (5.11) can be written as

$$|Q(z)| = \frac{\sqrt{(2 - k_1 x)^2 + (k_1 y)^2}}{\sqrt{(2 + k_2 x)^2 + (k_2 y)^2}}, \tag{5.12}$$

$$\Rightarrow |Q(z)| = \frac{\sqrt{4 + k_1^2 x^2 - 4k_1 x + k_1^2 y^2}}{\sqrt{4 + k_2^2 x^2 + 4k_2 x + k_2^2 y^2}}, \tag{5.13}$$

$$\Rightarrow |Q(z)| < \frac{\sqrt{4 + k_1^2 x^2 + k_1^2 y^2}}{\sqrt{4 + k_2^2 x^2 + 4k_2 x + k_2^2 y^2}}, \tag{5.14}$$

since $\sqrt{4 + k_1^2 x^2 + k_1^2 y^2} < \sqrt{4 + k_2^2 x^2 + 4k_2 x + k_2^2 y^2}$, for $p \geq 0$.

Thus, for $p \geq 0$, we have $|Q(z)| < 1$ where $\text{Re}(z) < 0$. Hence, the family of the one-step Adams-Moulton method (3.14) is A-stable for $p \geq 0$.

6. Numerical Results

In this section, we check the usefulness and efficiencies of the newly proposed methods. In first two problems, we employ the classical two-step Adams-Bashforth method $(CAB)(1.2)$ and the two new proposed methods obtained from (3.10), namely $|p| = 2(NAB)$, $|p| = 1(NAB)$ respectively. We also discuss results obtained by using the classical one-step Adams-Moulton method $(CAM)(1.3)$ and a new proposed Adams-Moulton method obtained from (3.14), namely $|p| = \dfrac{1}{32}(NAM)$ in the third problem. A comparative study of absolute error (AE) and relative error (RE) among the classical methods and the new proposed methods shows that the proposed methods outperform the existing classical methods. The fourth problem has been taken from Lambert (1991) in which restriction of the stability interval is discussed for new proposed two-step Adams-Bashforth method, namely $|p| = \dfrac{1}{4}(NAB)$. All the computational work has been done using 'Matlab' version 7.9.0.529(R2009b).

Example 1.

Consider the IVP: $y' = -y^2$, $x \in [0, 1]$ with $y_0 = 1$ and, $y_1 = 0.8400\,(h = 0.2)$ and $y_1 = 0.9525\,(h = 0.05)$. The values of y_1 are obtained by using the second order Taylor series method. The exact solution of the problem is $y(x) = \dfrac{1}{1+x}$. The results are listed in Table 1.

Table 1. Numerical results of Problem 1 using (CAB) and (NAB) at $|p| = 2$

x	Exact solution	For $h = 0.2$				For $h = 0.05$			
		AE_{CAB}	AE_{NAB}	RE_{CAB}	RE_{NAB}	AE_{CAB}	AE_{NAB}	RE_{CAB}	RE_{NAB}
0.4	0.7143	0.0140	0.0022	0.0196	0.0031	0.0009	0.0001	0.0013	0.0001
0.6	0.6250	0.0147	0.0004	0.0235	0.0006	0.0009	0.0000	0.0014	0.0000
0.8	0.5556	0.0144	0.0011	0.0259	0.0020	0.0008	0.0001	0.0014	0.0001
1.0	0.5000	0.0135	0.0020	0.0270	0.0040	0.0008	0.0001	0.0016	0.0002

Example 2.

Consider the IVP: $y' = -50y + 51\cos(x) + 49\sin(x)$; $x \in [0, 0.5]$ with $y_0 = 1$ and, $y_1 = 1.0950\,(h = 0.1)$ and $y_1 = 1.0488\,(h = 0.05)$. The values of y_1 are obtained by using the second order Taylor series method. For the given problem, the values of the step-sizes are taken

outside the stability interval $(0, 0.04)$ of the proposed method. The exact solution of the problem is $y(x) = \sin(x) + \cos(x)$. The results are listed in Table 2.

Table 2. Numerical results of Problem 2 using (CAB) and (NAB) at $|p| = 1$

x	Exact solution	For $h = 0.1$				For $h = 0.05$			
		AE_{CAB}	AE_{NAB}	RE_{CAB}	RE_{NAB}	AE_{CAB}	AE_{NAB}	RE_{CAB}	RE_{NAB}
0.2	1.1787	0.0006	0.0001	0.0005	0.0000	0.0016	0.0010	0.0014	0.0008
0.3	1.2509	0.0051	0.0019	0.0041	0.0015	0.0160	0.0105	0.0128	0.0084
0.4	1.3105	0.0349	0.0117	0.0266	0.0089	0.1587	0.1092	0.1211	0.0833
0.5	1.3570	0.2397	0.0841	0.1766	0.0620	1.5715	1.1320	1.1581	0.8342

Example 3.

Consider the IVP: $y' = 2\tan^{-1}(x)$; $x \in [0,1]$ with $y(0) = 0$. The exact solution of the problem is $y(x) = 2x\tan^{-1}(x) - \log(1 + x^2)$. With different step-sizes $h = 0.2$ and $h = 0.1$, the results are listed in Table 3.

Table 3. Numerical results of Problem 3 using (CAM) and (NAM) at $|p| = \dfrac{1}{32}$

x	Exact solution	For $h = 0.2$				For $h = 0.1$			
		AE_{CAM}	AE_{NAM}	RE_{CAM}	RE_{NAM}	AE_{CAM}	AE_{NAM}	RE_{CAM}	RE_{NAM}
0.2	0.0397	0.0002	0.0000	0.0050	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.1560	0.0009	0.0005	0.0058	0.0032	0.0002	0.0001	0.0013	0.0006
0.6	0.3410	0.0018	0.0011	0.0053	0.0032	0.0004	0.0003	0.0012	0.0008
0.8	0.5849	0.0026	0.0018	0.0044	0.0031	0.0007	0.0005	0.0012	0.0008
1.0	0.8776	0.0033	0.0023	0.0038	0.0026	0.0008	0.0005	0.0009	0.0005

6.1. Significance of the Stability Interval

In this subsection, we shall discuss how the family of the two-step Adams-Bashforth method performs when restriction of the stability interval is relaxed.

Example 4.

Consider the IVP: $y' = f(x, y) = -20y;\ x \in [0,1]$ with $y_0 = 1$ and, $y_1 = 0.8200\ (h = 0.01)$ and $y_1 = 5\ (h = 0.2)$. The values of y_1 are obtained by using the second order Taylor series method. The exact solution of the problem is $y(x) = e^{-20x}$. The stability interval of the proposed variants of the two-step Adams-Bashforth method (3.10) is $(-2, 0)$, i.e., $\bar{h} \in (-2, 0)$. Therefore, we have $-2 < \bar{h} < 0,\ i.e.,\ -2 < \lambda h < 0$. In this problem, the value of $\lambda = \dfrac{\partial f}{\partial y} = -20$. For this value of λ, we have $0 < h < 0.1$. In Table 4, results are shown for $h = 0.01,\ h \in (0, 0.1)$ and for $h = 0.2 \notin (0, 0.1)$.

Table 4. Numerical results of Problem 4 using (CAB) at $|p| = \dfrac{1}{4}$

x	Exact solution	Approximate solution for $h = 0.01$	Approximate solution for $h = 0.2$
0.4	0.00033546262790	0.000386145450765	-23.39999999999999
0.6	0.00000614421235	0.000007594637663	129.84000000000000
0.8	0.00000011253517	0.000000149369936	-711.3240000000001
1.0	0.00000000206115	0.000000002937780	3900.416400000001

From the above Table 4, we observe that for $h = 0.01,\ h \in (0, 0.1)$, the approximate solution obtained by the new proposed two-step Adams-Bashforth method for $|p| = \dfrac{1}{4}$ is very accurate to the exact solution at nodal points. The approximate solution for $h = 0.2 \notin (0, 0.1)$ obtained by the same method is very absurd as compare to the exact solution at nodal points. Therefore, selection of the suitable step-size is very important to implement these methods for accurate results. Hence, the proposed family of the two-step Adams-Bashforth method is unstable for $\bar{h} \notin (-2, 0)$.

7. Conclusions

We have proposed new geometrically constructed variants of the two-step Adams-Bashforth and the one-step Adams-Moulton methods for the approximate solution of the initial value Example

(1.1). The methods were derived from an exponentially fitted parabola. The accuracy and stability of the proposed methods were considered and their applicability to some initial value problems was discussed. Numerical results presented in Section 6 overwhelmingly support that the exponentially fitted variants of the proposed methods outperform the existing classical methods. This work may be extended to variants of higher order multi-step methods of Adams-Bashforth and Adams-Moulton types by considering exponentially fitted osculating polynomials of any degree.

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