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On factorization of a special type of vandermonde rhotrix

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Abstract

Vandermonde matrices have important role in many branches of applied mathematics such as combinatorics, coding theory and cryptography. Some authors discuss the Vandermonde rhotrices in the literature for its mathematical enrichment. Here, we introduce a special type of Vandermonde rhotrix and obtain its LR factorization namely left and right triangular factorization, which is further used to obtain the inverse of the rhotrix.

Keywords: Vandermonde Matrix; Vandermonde Rhotrix; Special Vandermonde Rhotrix; Left and Right Triangular Rhotrix

AMS 2010 Classification: 15A33, 15A09, 15A23

1. Introduction

Vandermonde matrix V is an $l \times m$ matrix with terms of a geometric progression in each row that is

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & . & \alpha_1^{m-1} \\ 1 & \alpha_2 & \alpha_2^2 & . & \alpha_2^{m-1} \\ 1 & \alpha_3 & \alpha_3^2 & . & \alpha_3^{m-1} \\ . & . & . & . & . \\ 1 & \alpha_l & \alpha_l^2 & . & \alpha_l^{m-1} \end{bmatrix}. \tag{1.1}$$

Due to a wide range of applications of the Vandermonde matrices in different areas of mathematical sciences as well as other sciences, they have attained much importance, see Lacan and Fimes (2004), Lin and Costello (2004) and Sharma and Rehan (2014). The solutions of the linear system of equations $Vx = b$ have been studied by Björck and Pereyra (1970) and Tang and Golub (1981). The solutions of the equation $x = V^{-1}b$ leads to the factorizations of V^{-1}, such as Lower and Upper factorizations and 1-banded factorizations. Oruc and Phillips (2000) obtained the formula for the LU factorization of V and expressed the matrices L and U as a product of 1-banded matrices. Yang (2004, 2005) modified the results of Oruc and obtained a simpler formula. In the recent literature, special generalized Vandermonde matrices have attracted a great amount of attention. Demmel and Koev (2005) studied totally positive generalized Vandermonde matrices and gave a formula for the entries of the bidiagonal factorization and the LDU factorization. Yang and Holtti (2004) discussed various types of the generalized Vandermonde matrices. Li and Tan (2008) discussed the LU factorization of the special class of the generalised Vandermonde matrices which was introduced by Liu (1968). This matrix arrises while solving the equation

$$a_m = c_1 a_{m-1} + c_2 a_{m-2} + ... + c_p a_{m-p}, \quad m \geq p, (p \text{ fixed}) \tag{1.2}$$

where $c_1, c_2, ..., c_p$ are constants and $c_p \neq 0$. If the equation (1.2) has distinct real roots $v_1, v_2, ..., v_q$ with multiplicities $u_1, u_2, ..., u_q$ respectively and $\sum_{i=1}^{q} u_i = m,$ then the corresponding generalised Vandermonde matrix has the following form:

$$V'_{\{q;u_1,u_2,..,u_q\}} = \begin{bmatrix} 1 & 0 & ... & 0 & ... & 1 & 0 & ... & 0 \\ v_1 & v_1 & ... & v_1 & ... & v_q & v_q & ... & v_q \\ : & : & : & : & : & : & : & : & : \\ v_1^{m-1} & (m-1)v_1^{n-1} & ... & (m-1)^{u_1-1}v_1^{m-1} & ... & v_q^{m-1} & (m-1)v_q^{m-1} & ... & (m-1)^{u_q-1}v_q^{m-1} \end{bmatrix}. \tag{1.3}$$

A special class of generalized Vandermonde matrices $V'_{R\{2;1,m-1\}}$ is defined by Li and Tan (2008) as follows: For $u_1 = 1, u_2 = m-1, q = 2, V'_{R\{2;1,m-1\}}$ is the transpose of $V'_{\{q;u_1,u_2,..,u_q\}}$ and given as

$$V'_{R\{2;1,m-1\}} = \begin{bmatrix} 1 & v_1 & v_1^2 & . & v_1^{m-1} \\ 1 & v_2 & v_2^2 & . & v_2^{m-1} \\ 0 & v_2 & 2v_2^2 & . & (m-1)v_2^{m-1} \\ . & . & . & . & . \\ 0 & v_2 & 2^{m-2}v_2^2 & . & (m-1)^{m-2}v_2^{m-1} \end{bmatrix}. \tag{1.4}$$

The study of rhotrices is introduced in the literature of mathematics by Ajibade (2003). Rhotrix is a mathematical object, which is in some way between 2×2 dimensional and 3×3 dimensional matrices. The dimension of a rhotrix is the number of entries in the horizontal or vertical diagonal of the rhotrix and is always an odd number. A rhotrix of dimension 3 is defined as

$$R(3)=\left\langle \begin{array}{ccc} & a_{11} & \\ a_{31} & a_{21} & a_{12} \\ & a_{32} & \end{array} \right\rangle, \tag{1.5}$$

where $a_{11}, a_{12}, a_{21}, a_{31}, a_{32}$ are real numbers. Sani (2007) extended the dimension of a rhotrix to any odd number $n \geq 3$ and gave the row-column multiplication & inverse of a rhotrix as follows:

Let

$$Q(3)=\left\langle \begin{array}{ccc} & b_{11} & \\ b_{31} & b_{21} & b_{12} \\ & b_{32} & \end{array} \right\rangle,$$

Then

$$R(3)oQ(3)=\left\langle \begin{array}{ccc} & a_{11}b_{11}+a_{12}b_{31} & \\ a_{31}b_{11}+a_{32}b_{31} & a_{21}b_{21} & a_{11}b_{12}+a_{12}b_{32} \\ & a_{31}b_{12}+a_{32}b_{32} & \end{array} \right\rangle.$$

Also,

$$(R(3))^{-1}=\frac{1}{a_{11}a_{32}-a_{31}a_{12}}\left\langle \begin{array}{ccc} & a_{32} & \\ -a_{31} & \dfrac{a_{11}a_{32}-a_{31}a_{12}}{a_{21}} & -a_{12} \\ & a_{11} & \end{array} \right\rangle,$$

provided $a_{21}(a_{11}a_{32}-a_{31}a_{12})\neq 0$. The algebra and analysis of rhotrices are discussed in the literature by Ajibade (2003), Sani (2004), Sani (2007), Aminu (2010), Tudunkaya and Makanjuola (2010), Absalom et al. (2011), Sharma and Kanwar (2011), Sharma and Kanwar (2012a, 2012b, 2012c), Kanwar (2013), Sharma and Kanwar (2013), Sharma and Kumar (2013), Sharma et al. (2013a, 2013b), Sharma and Kumar (2014a, 2014b, 2014c) and Sharma et al. (2014). Sharma et al. (2013b) have introduced Vandermonde rhotrix which is defined as

$$V_n = \left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & a_0 & & & \\ & & 1 & 1 & a_2 & a_1 & a_0^2 & & \\ & . & . & . & . & . & . & . & \\ 1 & . & & a_{\frac{n+1}{2}} & a_{\frac{n-1}{2}} & a_{\frac{n-3}{2}}^{\frac{n-3}{2}} & & . & a_0^{\frac{n-1}{2}} \\ & . & & . & . & . & & . & \\ & & & . & . & . & . & & \\ & & & a_{n-1}^{\frac{n-3}{2}} & a_{n-2}^{\frac{n-3}{2}} & a_{n-3}^{\frac{n-1}{2}} & & & \\ & & & & a_{n-1}^{\frac{n-1}{2}} & & & & \end{array} \right\rangle. \tag{1.6}$$

In the present paper, we have introduce special type of generalised Vandermonde rhotrices and factored it into L_5 and R_5 rhotrices namely left and right triangular rhotrices. Further, we factor the rhotrices L_5 and R_5 as product of left and right triangular rhotrices. As an application, we obtain the inverse of the rhotrix considered with the help of the said factorization.

2. Notations and Symbols

The following notations are used in the present paper:

m :	a positive integer
n :	an odd number
$V'_{\{q;u_1,u_2,..,u_q\}}$:	generalized Vandermonde matrix
$V'_{R\{2;1,m-1\}}$:	special Vandermonde matrix
$R(3), Q(3)$:	3-dimensional rhotrices
V_n :	n-dimensional Vandermonde rhotrix
$V_{R\{2;1,n-1\}}$:	n-dimensional special Vandermonde rhotrix
$L_n, L_n^{(1)}, L_n^{(2)}$:	n -dimensional left triangular rhotrix
$R_n, R_n^{(1)}, R_n^{(2)}$:	n -dimensional right triangular rhotrix
a_{ij} :	entries in the rhotrix

3. Left and right triangular factorization

In this section, we introduce a special type of Vandermonde rhotrix. We factor $V_{R\{2;1,4\}}$ and $V_{R\{2;1,2\}}$

in terms of left and right triangular rhotrices in Theorem 3.1 and Theorem 3.2. Further factorization of left and right triangular rhotrices is given in Theorem 3.3.

Definition:

Let v_1, v_2 are two numbers such that v_2 has multiplicity $(n-1)$, then the rhotrix

$$V_{R\{2;1,n-1\}} = \left\langle \begin{array}{c}
1 \\
1 \quad 1 \quad v_1 \\
0 \quad 0 \quad v_2 \quad v_2 \quad v_1^2 \\
. \quad 0 \quad v_2 \quad v_2 \quad 2v_2^2 \quad v_2^2 \quad . \\
. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
0 \quad 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad v_1^{\frac{n-1}{2}} \\
v_2 \quad v_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
2^{n-2}v_2^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
. \quad . \quad . \quad . \quad . \quad . \quad . \\
. \quad . \quad . \quad . \quad . \\
. \quad . \quad . \\
(\frac{n-1}{2})^{n-2} v_2^{\frac{n-1}{2}}
\end{array} \right\rangle . \tag{3.1}$$

is defined as special generalised Vandermonde rhotrix.

Theorem 3.1.

Let $V_{R\{2;1,4\}}$ be a 5-dimensional special Vandermonde rhotrix, then $V_{R\{2;1,4\}}$ can be factored as $V_{R\{2;1,4\}} = L_5 R_5$, where L_5 is a left triangular rhotrix and R_5 is a right triangular rhotrix and the entries of L_5 are

$$\begin{aligned}
& a_{ij} = 0, i \le j, i, j \ne 1; \\
& a_{ij} = 0, i > j, j = 1, i \ne 2, 3; \\
& a_{ij} = 1, i > j, i = 2, 3, 4;\ j = 1, 2; \\
& a_{41} = 0; \\
& a_{ij} = \frac{v_2}{v_2 - v_1}, j = 2, i = j + 3; \\
& a_{ij} = -1, i = 2 + j, j = 3,
\end{aligned}$$

and of R_5 are

$$\begin{aligned}
& a_{ij} = 0, i > j, i = 3, 4, 5;\ j = 1, 2; \\
& a_{ij} = 0, j = 2, i = j + 3; \\
& a_{ij} = v_2, i = 2, 4;\ j = 2; \\
& a_{ij} = v_1, i = 1, j = i + 1;
\end{aligned}$$

$$a_{ij} = {v_1}^2, i = 1, j = 2 + i;$$
$$a_{ij} = v_2 - v_1, j = 2, i = j + 1;$$
$$a_{ij} = 2{v_2}^2 - {v_1}^2, i = j, i = 3;$$
$$a_{ij} = \frac{v_2(2{v_2}^2 - {v_1}^2)}{v_2 - v_1} - 8{v_2}^2, i = 2 + j, j = 3.$$

Proof:

Let $V_{R\{2;1,4\}}$ be 5-dimensional special Vandermonde rhotrix defined as

$$V_{R\{2;1,4\}} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & v_1 & \\ 0 & 0 & v_2 & v_2 & {v_1}^2 \\ & v_2 & v_2 & 2{v_2}^2 & \\ & & 8{v_2}^2 & & \end{array} \right\rangle, \tag{3.2}$$

then (3.2) can be factored in the product of two rhotrices as

$$V_{R\{2;1,4\}} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & \frac{v_2}{v_2 - v_1} & 1 & 0 & \\ & & -1 & & \end{array} \right\rangle$$

$$\cdot \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & v_1 & \\ 0 & 0 & v_2 - v_1 & v_2 & {v_1}^2 \\ & 0 & v_2 & 2{v_2}^2 - {v_1}^2 & \\ & & \frac{v_2(2{v_2}^2 - {v_1}^2)}{v_2 - v_1} - 8{v_2}^2 & & \end{array} \right\rangle$$

$$= L_5 R_5, \tag{3.3}$$

where the rhotrix L_5 is a left triangular rhotrix and R_5 is a right triangular rhotrix.

∎

Theorem 3.2.

Let $V_{R\{2;1,2\}}$ be a 3-dimensional special Vandermonde rhotrix, then $V_{R\{2;1,2\}}$ can be factored as $V_{R\{2;1,2\}} = L_3 R_3,$ where L_3 is a left triangular rhotrix and R_3 is a right triangular rhotrix. The entries of L_3 are

$$a_{ij} = 1, i = 1,2,3, j = 1;$$
$$a_{12} = 0;$$
$$a_{ij} = v_2 - v_1, j = 2, i = j+1,$$

and of R_3 are

$$a_{ij} = 1, i = 1,2,3, j = 1,2;$$
$$a_{ij} = v_1, i = 1, j = i+1;$$
$$a_{ij} = 0, i = j+2, j = 1.$$

Proof:

Let $V_{R\{2;1,2\}}$ be a 3-dimensional special Vandermonde rhotrix defined as

$$V_{R\{2;1,2\}} = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & v_1 \\ & v_2 & \end{array} \right\rangle, \tag{3.4}$$

then

$$V_{R\{2;1,2\}} = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & v_2 - v_1 & \end{array} \right\rangle . \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & v_1 \\ & 1 & \end{array} \right\rangle . \tag{3.5}$$

Using (3.4) in (3.5), we get

$$\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & v_1 \\ & v_2 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & v_2 - v_1 & \end{array} \right\rangle . \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & v_1 \\ & 1 & \end{array} \right\rangle . \tag{3.6}$$

From row-column multiplication of rhotrices, we get

$$\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & v_1 \\ & v_2 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & v_1 \\ & v_2 & \end{array} \right\rangle,$$

which verifies the result.

∎

Theorem 3.3.

Let $V_{R\{2;1,4\}}$ be a 5-dimensional special Vandermonde rhotrix, then $V_{R\{2;1,4\}}$ can be factored as $V_{R\{2;1,4\}} = L_5^{(1)} R_5^{(1)} L_5^{(2)} R_5^{(2)}$, where the entries of $L_5^{(1)}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 0, i > j, j = 1, i \ne 2, 3;$$
$$a_{ij} = 1, i > j, i = 2, 3, 4, j = 1, 2;$$
$$a_{ij} = \frac{v_2 - v_1}{v_2}, i = j + 1, j = 2;$$
$$a_{ij} = 0, j = 1, i = j + 3;$$
$$a_{ij} = -1, i = 2 + j, j = 3,$$

the entries of $R_5^{(1)}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 0, i > j, j = 1, i = 3, 4, 5;$$
$$a_{ij} = 1, i > j, i = 2, 4, 5; j = 1, 2, 3;$$
$$a_{ij} = \frac{v_2}{v_2 - v_1}, i = j + 1, j = 2;$$
$$a_{52} = 0,$$

the entries of $L_5^{(2)}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 0, i > 1, i = 3, 4, 5; j = 1;$$
$$a_{ij} = 0, i > 1, i = 3, j = i;$$
$$a_{ij} = 1, i > j, i = 2, 3, 4; j = 1, 2;$$
$$a_{22} = a_{52} = 0;$$
$$a_{42} = 1$$
$$a_{ij} = \frac{v_2(2v_2{}^2 - v_1{}^2)}{v_2 - v_1} - 8v_2{}^2, i = 2 + j, j = 3,$$

and the entries of $R_5^{(2)}$ are

$$a_{ij} = 1, i = j, i = 1;$$
$$a_{ij} = v_1, i < j;$$
$$a_{ij} = v_1{}^2, i < j;$$
$$a_{ij} = 0, i > j, j = 1, i = 3, 4, 5;$$
$$a_{42} = a_{22} = v_2;$$
$$a_{53} = 1, j = 3, i = j + 2;$$
$$a_{ij} = 0, i > 1, i = 3, 4, 5; j = 1, 2;$$
$$a_{ij} = (2v_2{}^2 - v_1{}^2), i = j, j = 3.$$

Proof:

Let $V_{R\{2;1,4\}}$ be a special type of Vandermonde rhotrix as defined in (3.2). From (3.3), we have

$$V_{R\{2;1,4\}} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & \dfrac{v_2}{v_2 - v_1} & 1 & 0 & \\ & & -1 & & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & v_1 & \\ 0 & 0 & v_2 - v_1 & v_2 & v_1{}^2 \\ & 0 & v_2 & 2v_2{}^2 - v_1{}^2 & \\ & & \dfrac{v_2(2v_2{}^2 - v_1{}^2)}{v_2 - v_1} - 8v_2{}^2 & & \end{array} \right\rangle$$

$$= L_5 R_5 .$$

Now, we further factor L_5 and R_5

$$L_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & \dfrac{v_2}{v_2 - v_1} & 1 & 0 & \\ & & -1 & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 0 & \\ 0 & 0 & \dfrac{v_2 - v_1}{v_2} & 0 & 0 \\ & 1 & 1 & 0 & \\ & & -1 & & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & \dfrac{v_2}{v_2 - v_1} & 0 & 0 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle , \tag{3.7}$$

which are left and right triangular rhotrices. Therefore,

$$L_5 = L_5^{(1)} R_5^{(1)} .$$

Similarly,

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & v_1 & \\ 0 & 0 & v_2 - v_1 & v_2 & v_1{}^2 \\ & 0 & v_2 & 2v_2{}^2 - v_1{}^2 & \\ & & \dfrac{v_2(2v_2{}^2 - v_1{}^2)}{v_2 - v_1} - 8v_2{}^2 & & \end{array} \right\rangle .$$

$$= \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 & \\ & & \frac{v_2(2v_2^{\,2}-v_1^{\,2})}{v_2-v_1}-8v_2^{\,2} & & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & v_1 & \\ 0 & 0 & v_2-v_1 & v_2 & v_1^{\,2} \\ & 0 & v_2 & 2v_2^{\,2}-v_1^{\,2} & \\ & & 1 & & \end{array} \right\rangle, \tag{3.8}$$

which are left and right triangular rhotrices. Therefore,

$$R_5 = L_5^{(2)} R_5^{(2)}.$$

Hence,

$$V_{R\{2;1,4\}} = L_5^{(1)} R_5^{(1)} L_5^{(2)} R_5^{(2)}.$$

∎

4. Application of factorization of special vandermonde rhotrix

In this section, we apply the factorization of special, the Vandermonde rhotrix to find the inverse of the rhotrix. The inverse of $V_{R\{2;1,4\}}$ in terms of the inverses of L_5, R_5 is given in Theorem 4.1. We also obtain the inverse of $V_{R\{2;1,2\}}$ in Theorem 4.2. We find the inverse of $V_{R\{2;1,4\}}$ in terms of $L_5^{(1)}, R_5^{(1)}, L_5^{(2)}, R_5^{(2)}$ in Theorem 4.3.

Theorem 4.1.

Let $V_{R\{2;1,4\}}$ be a 5-dimensional special Vandermonde rhotrix, then $V^{-1}{}_{R\{2;1,4\}} = R_5^{-1} L_5^{-1}$, where the entries of R_5^{-1} are

$$a_{ij} = 1, i \ge 1, i = 1,2;\ j = 1;$$
$$a_{ij} = 0, i > 1, i = 3,4,5;\ j = 1,2;$$
$$a_{ij} = -1, i = j,\ j = 2;$$
$$a_{ij} = \frac{v_1}{v_1 - v_2}, i = 1;\ j = 2;$$
$$a_{ij} = -\frac{v_1^{\,2} - 2v_1 v_2}{v_1^{\,2} - 8v_1 v_2 + 6v_2^{\,2}}, i = 1,\ j = i+2;$$
$$a_{ij} = -\frac{1}{v_1 - v_2}, i = j+1,\ j = 2;$$
$$a_{ij} = \frac{1}{v_2}, j = 2, i = 2 + j;$$
$$a_{ij} = \frac{v_1^{\,2} - 2v_2^{\,2}}{v_1^{\,2} v_2 - 8v_1 v_2^{\,2} + 6v_2^{\,3}}, j = 3, i = j;$$
$$a_{ij} = \frac{v_1 - v_2}{v_1^{\,2} - 8v_1 v_2^{\,2} + 6v_2^{\,3}}, j = 3, i = j+2;$$

and L_5^{-1} has entries

$$
\begin{aligned}
&a_{ij} = 1, i \le j, i, j \ne 1;\\
&a_{ij} = 0, i < j, j = 2,3, i = 1,2;\\
&a_{ij} = 1, i \ge j, i = j+1, j > 2;\\
&a_{41} = 0;\\
&a_{ij} = \frac{v_2}{v_2 - v_1}, j = 1, i = j+4;\\
&a_{ij} = -\frac{v_2}{v_2 - v_1}, j = 2, i = j+3;\\
&a_{ij} = -1, i = 2+j, j = 3.
\end{aligned}
$$

Proof:

Let $V_{R\{2;1,4\}}$ be a 5-dimensional special Vandermonde rhotrix as defined in (3.2), then the inverse of $V_{R\{2;1,4\}}$ is

$$
V^{-1}{}_{R\{2;1,4\}} = \left\langle \begin{array}{ccccc}
 & & \frac{6v_2^2}{v_1^2 - 8v_1v_2 + 6v_2^2} & & \\
 & -8\frac{v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} & 1 & \frac{v_1^2 - 8v_1v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} & \\
\frac{1}{v_1^2 - 8v_1v_2 + 6v_2^2} & 0 & \frac{8v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} & -1 & -\frac{v_1^2 - 2v_1v_2}{v_1^2v_2 - 8v_1v_2 + 6v_2^3} \\
 & \frac{-1}{v_1^2 - 8v_1v_2 + 6v_2^2} & \frac{1}{v_2} & \frac{1}{v_2}\frac{v_1^2 - 2v_2^2}{v_1^2 - 8v_1v_2 + 6v_2^3} & \\
 & & -\frac{1}{v_2}\frac{(v_1 - v_2)}{v_1^2 - 8v_1v_2 + 6v_2^2} & &
\end{array} \right\rangle. \tag{4.1}
$$

Now,

$$
L_5^{-1} = \left\langle \begin{array}{ccccc}
 & & 1 & & \\
 & -1 & 1 & 0 & \\
\frac{v_2}{v_1 - v_2} & 0 & 1 & 0 & 0 \\
 & -\frac{v_2}{v_1 - v_2} & 1 & 0 & \\
 & & -1 & &
\end{array} \right\rangle \tag{4.2}
$$

and

$$R_5^{-1} = \left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad \dfrac{v_1}{v_1 - v_2} \\ 0 \quad 0 \quad -\dfrac{1}{v_1 - v_2} \quad -1 \quad -\dfrac{v_1^2 - 2v_1v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} \\ 0 \quad \dfrac{1}{v_2} \quad \dfrac{v_1^2 - 2v_2^2}{v_1^2v_2 - 8v_1v_2^2 + 6v_2^3} \\ \dfrac{v_1 - v_2}{v_1^2v_2 - 8v_1v_2^2 + 6v_2^3} \end{array} \right\rangle. \tag{4.3}$$

On multiplying (4.2) and (4.3), we get

$$R_5^{-1}L_5^{-1} = \left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad \dfrac{v_1}{v_1 - v_2} \\ 0 \quad 0 \quad -\dfrac{1}{v_1 - v_2} \quad -1 \quad -\dfrac{v_1^2 - 2v_1v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} \\ 0 \quad \dfrac{1}{v_2} \quad -\dfrac{v_1^2 - 2v_2^2}{v_1^2v_2 - 8v_1v_2^2 + 6v_2^3} \\ \dfrac{v_1 - v_2}{v_1^2v_2 - 8v_1v_2^2 + 6v_2^3} \end{array} \right\rangle \cdot \left\langle \begin{array}{c} 1 \\ -1 \quad 1 \quad 0 \\ \dfrac{v_2}{v_1 - v_2} \quad 0 \quad 1 \quad 0 \quad 0 \\ -\dfrac{v_2}{v_1 - v_2} \quad 1 \quad 0 \\ -1 \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{c} \dfrac{6v_2^2}{v_1^2 - 8v_1v_2 + 6v_2^2} \\ -8\dfrac{v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} \quad 1 \quad \dfrac{v_1^2 - 8v_1v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} \\ \dfrac{1}{v_1^2 - 8v_1v_2 + 6v_2^2} \quad 0 \quad \dfrac{8v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} \quad -1 \quad -\dfrac{v_1^2 - 2v_1v_2}{v_1^2v_2 - 8v_1v_2 + 6v_2^2} \\ \dfrac{-1}{v_1^2 - 8v_1v_2 + 6v_2^2} \quad \dfrac{1}{v_2} \quad \dfrac{1}{v_2}\dfrac{v_1^2 - 2v_2^2}{v_1^2 - 8v_1v_2 + 6v_2^3} \\ -\dfrac{1}{v_2}\dfrac{(v_1 - v_2)}{v_1^2 - 8v_1v_2 + 6v_2^2} \end{array} \right\rangle.$$

$$= V^{-1}{}_{R\{2;1,4\}}.$$

Therefore,

$$V^{-1}{}_{R\{2;1,4\}} = R_5^{-1}L_5^{-1}.$$

∎

Theorem 4.2.

Let $V_{R\{2;1,2\}}$ be a 3-dimensional special Vandermonde rhotrix, then $V_{R\{2;1,2\}}^{-1} = R_3^{-1}L_3^{-1}$, where the entries of R_3^{-1} are

$$a_{ij} = 1, i = 1, j = 1;$$
$$a_{ij} = -v_1, i = 1, j = i + 1;$$
$$a_{ij} = 1; j = 1, i = j + 1;$$
$$a_{ij} = 0, i = j + 2, j = 1;$$
$$a_{ij} = 1, j = 2, i = j + 1,$$

and entries of L_3^{-1} are

$$a_{ij} = 1, i = 1, j = 1;$$
$$a_{ij} = 0, i = 1, j = i + 1; \cdot$$
$$a_{ij} = 1, i = j + 1, j = 1;$$
$$a_{ij} = \frac{1}{v_1 - v_2}, i = j + 2, j = 1;$$
$$a_{ij} = \frac{-1}{v_1 - v_2}, i = j + 1, j = 2.$$

Proof:

Let $V^{-1}{}_{R\{2;1,2\}}$ be a 3-dimensional special Vandermonde rhotrix as defined in (3.5), then

$$V^{-1}{}_{R\{2;1,2\}} = \left\langle \begin{array}{ccc} & \frac{-v_2}{v_1 - v_2} & \\ \frac{1}{v_1 - v_2} & 1 & \frac{v_1}{v_1 - v_2} \\ & \frac{-1}{v_1 - v_2} & \end{array} \right\rangle . \tag{4.4}$$

Now,

$$L_3^{-1} = \left\langle \begin{array}{ccc} & 1 & \\ \frac{1}{v_1 - v_2} & 1 & 0 \\ & \frac{-1}{v_1 - v_2} & \end{array} \right\rangle \tag{4.5}$$

and

$$R_3^{-1} = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & -v_1 \\ & 1 & \end{array} \right\rangle . \tag{4.6}$$

On multiplying (4.5) and (4.6), we get

$$R_3^{-1}L_3^{-1} = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & -v_1 \\ & 1 & \end{array} \right\rangle \left\langle \begin{array}{ccc} & 1 & \\ \frac{1}{v_1 - v_2} & 1 & 0 \\ & \frac{-1}{v_1 - v_2} & \end{array} \right\rangle,$$

$$= \left\langle \begin{array}{ccc} & \frac{-v_2}{v_1 - v_2} & \\ \frac{1}{v_1 - v_2} & 1 & \frac{v_1}{v_1 - v_2} \\ & \frac{-1}{v_1 - v_2} & \end{array} \right\rangle$$

$$= V^{-1}{}_{R\{2;1,2\}}.$$

Therefore,

$$V_{R\{2;1,2\}} = R_3^{-1}L_3^{-1}.$$

■

Theorem 4.3.

Let $V_{R\{2;1,4\}}$ be a 5-dimensional special Vandermonde rhotrix, then

$$V_{R\{2;1,4\}}{}^{-1} = \left(R_5^{(2)}\right)^{-1}\left(L_5^{(2)}\right)^{-1}\left(R_5^{(1)}\right)^{-1}\left(L_5^{(1)}\right)^{-1},$$

where entries of $\left(R_5^{(1)}\right)^{-1}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 0, i \ge j, j = 1, 3, i = 3, 4, 5;$$
$$a_{ij} = 1, i \ge j, j = 1, 2, 3;$$
$$a_{ij} = \frac{v_2}{v_2 - v_1}, i = j, j = 2,$$

entries of $\left(L_5^{(1)}\right)^{-1}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 1, i \ge j, j = 1, 2, 3, 4;$$
$$a_{ij} = \frac{v_2}{v_1 - v_2}, i > j, i = 3, 5, j = 1;$$

$$a_{ij} = -\frac{v_2}{v_1 - v_2}, i \geq j, j = 2, i = 3, 5;$$
$$a_{ij} = -1, j = 3, i = j + 2,$$

entries of $\left(R_5^{(2)}\right)^{-1}$ are

$$a_{ij} = 1, i = j, i, j = 1;$$
$$a_{ij} = 0, i > j, j = 1, i = 3, 4, 5;$$
$$a_{ij} = 1, j = 1, i = j + 1;$$
$$a_{ij} = 0; j = 2, i = j + 3;$$
$$a_{ij} = \frac{v_1}{v_1 - v_2}, i = 1, j = i + 1;$$
$$a_{ij} = \frac{v_2 v_1{}^2 - 2 v_1 v_2{}^2}{v_1 - v_2}, i = 1, j = i + 2;$$
$$a_{ij} = -1; j = i, i = 2;$$
$$a_{ij} = \frac{-1}{v_1 - v_2}, j = 2, i = j + 1;$$
$$a_{ij} = \frac{1}{v_2}, i = 2 + j, j = 2;$$
$$a_{ij} = 1, j = 3, i = j + 2,$$

and entries of $\left(L_5^{(2)}\right)^{-1}$ are

$$a_{ij} = 0, i \leq j, i, j \neq 1;$$
$$a_{ij} = 0, i > j, j = 1, i = 3, 4, 5;$$
$$a_{ij} = 1, i \geq j, j = 1, 2;$$
$$a_{ij} = 1, i \leq j, j = 2, 3;$$
$$a_{ij} = 0, j = 2, i = j + 3;$$
$$a_{ij} = \frac{v_1 - v_2}{v_1{}^2 v_2 - 8 v_1 v_2{}^2 + 6 v_2{}^3}, j = 3, i = j + 2;$$

Proof:

Let $V_{R\{2;1,4\}}$ be a special type of Vandermonde rhotrix as defined in (3.2) and its inverse is given in (4.1). Now,

$$\left(R_5^{(1)}\right)^{-1} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & \frac{v_2 - v_1}{v_2} & 0 & 0 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle \tag{4.7}$$

$$\left(L_5^{(1)}\right)^{-1}=\left\langle \begin{array}{c} 1 \\ \frac{v_2}{v_1-v_2} \quad 1 \quad 0 \\ \frac{v_2}{v_1-v_2} \quad 0 \quad \frac{-v_2}{v_1-v_2} \quad 0 \quad 0 \\ \frac{-v_2}{v_1-v_2} \quad 1 \quad 0 \\ -1 \end{array} \right\rangle \tag{4.8}$$

$$\left(R_5^{(2)}\right)^{-1}=\left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad \frac{v_1}{v_1-v_2} \\ 0 \quad 0 \quad \frac{-1}{v_1-v_2} \quad -1 \quad \frac{v_1^2v_2-2v_1v_2^2}{v_1-v_2} \\ 0 \quad \frac{1}{v_2} \quad \frac{-\left(v_1^2-2v_2^2\right)}{v_1-v_2} \\ 1 \end{array} \right\rangle \tag{4.9}$$

And

$$\left(L_5^{(2)}\right)^{-1}=\left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 1 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ \frac{v_1-v_2}{v_1^2v_2-8v_1v_2^2+6v_2^3} \end{array} \right\rangle. \tag{4.10}$$

On multiplying (4.7)-(4.10), we get

$$\left(R_5^{(2)}\right)^{-1}\left(L_5^{(2)}\right)^{-1}\left(R_5^{(1)}\right)^{-1}\left(L_5^{(1)}\right)^{-1}=$$

$$=\left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad \frac{v_1}{v_1-v_2} \\ 0 \quad 0 \quad \frac{-1}{v_1-v_2} \quad -1 \quad \frac{v_1^2v_2-2v_1v_2^2}{v_1-v_2} \\ 0 \quad \frac{1}{v_2} \quad \frac{-\left(v_1^2-2v_2^2\right)}{v_1-v_2} \\ 1 \end{array} \right\rangle \cdot \left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 1 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ \frac{v_1-v_2}{v_1^2v_2-8v_1v_2^2+6v_2^3} \end{array} \right\rangle$$

$$
\cdot\left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & \dfrac{v_2 - v_1}{v_2} & 0 & 0 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccccc} & & 1 & & \\ & \dfrac{v_2}{v_1 - v_2} & 1 & 0 & \\ \dfrac{v_2}{v_1 - v_2} & 0 & \dfrac{-v_2}{v_1 - v_2} & 0 & 0 \\ & \dfrac{-v_2}{v_1 - v_2} & 1 & 0 & \\ & & -1 & & \end{array} \right\rangle .
$$

$$
= \left\langle \begin{array}{ccccc} & & \dfrac{6{v_2}^2}{{v_1}^2 - 8v_1v_2 + 6{v_2}^2} & & \\ & -8\dfrac{v_2}{{v_1}^2 - 8v_1v_2 + 6{v_2}^2} & 1 & \dfrac{{v_1}^2 - 8v_1v_2}{{v_1}^2 - 8v_1v_2 + 6{v_2}^2} & \\ \dfrac{1}{{v_1}^2 - 8v_1v_2 + 6{v_2}^2} & 0 & \dfrac{8v_2}{{v_1}^2 - 8v_1v_2 + 6{v_2}^2} & -1 & -\dfrac{{v_1}^2 - 2v_1v_2}{{v_1}^2 - 8v_1v_2 + 6{v_2}^2} \\ & \dfrac{-1}{{v_1}^2 - 8v_1v_2 + 6{v_2}^2} & \dfrac{1}{v_2} & \dfrac{1}{v_2}\dfrac{{v_1}^2 - 2{v_2}^2}{{v_1}^2 - 8v_1v_2 + 6{v_2}^2} & \\ & & -\dfrac{1}{v_2}\dfrac{v_1 - v_2}{{v_1}^2 - 8v_1v_2 + 6{v_2}^2} & & \end{array} \right\rangle
$$

$$
= V_{R\{2;1,4\}}^{-1}.
$$

Hence,

$$
V_{R\{2;1,4\}}^{-1} = \left(R_5^{(2)}\right)^{-1}\left(L_5^{(2)}\right)^{-1}\left(R_5^{(1)}\right)^{-1}\left(L_5^{(1)}\right)^{-1}.
$$

■

5. Conclusion

We have discussed a special type of generalized Vandermonde rhotrix and given its left and right triangular factorization. We further factored the rhotrices L_5 and R_5 as the product of the left and right triangular rhotrices. Also, we have applied the above said factorization to find the inverse of the rhotrix under consideration.

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