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Testing Equality of Locally Stationary Covariances with Application to Mortality Rate Modeling

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Abstract

The strong assumption of the stationarity in the analysis of time series may cause some lack of fit problems because it is open to abuse. Therefore, nonstationary models have been developed when one or two items of stationarity are not true. One of the most important subclasses of nonstationary time series is locally stationary or time varying stationary family. In this family, the second order properties oscillate smoothly over the time; while they seem constant within a suitable window of time. When the finite-dimensional distributions are Gaussian, almost all properties of the process are characterized through the local covariance function. Thus, two locally stationary time series are clustered in a same group when the corresponding covariance functions are equal. We propose two testing methods for the equality of covariance functions of two independent locally stationary time series. Due to the local behavior of the locally stationary covariance function, we look at the local discrete Fourier transform of the sample covariance function to capture the frequency domain features of the observations. The test statistics are based on the limiting distributions of bootstrap estimator of centered spectral mean. The empirical power and type I error of testing procedures are examined through a simulation study. The motivating problem is the use of tests in model diagnostics of a well-known mortality rate modeling. Using the France population mortality experiences, we deduce that the time-varying ARMA models are more appropriate in comparison to the prevalent ARIMA models or some other nonlinear or conditionally heteroscedastic time series which have been widely used in the literature.

Keywords: Bootstrap; kernel smoothing; local periodogram; local spectral density function;

locally stationary time series; mortality rate; time varying ARMA process

MSC 2010 No.: 62M10; 62M15

1. Introduction

In a constant mean Gaussian process $\{X_t\}$, the covariance function $\gamma_X(t,s) = \mathrm{cov}(X_t, X_s)$ specifies all the behaviors of sample path. For instance, continuity, stationarity, symmetry and self-similarity are the most important features of the time series determined by the covariance function. In some problems canonical facts induce special covariance functions. However, laboratory situations or canonical facts exist circumstantially instead of certainly in real life data. So, testing the structure of the covariance function plays a major role in the time series analysis and the statistical aspects of this problem have come under explicit study. For stationary time series, several methods were provided to test the equality of the covariance functions of two independent time series in (Coates and Diggle, 1986) and (Lund et al., 2009). The same problem was investigated for nonstationary time series by (Choi et al., 2008) and (Bengtsson and Cavanaugh, 2008). Some of the testing procedures deal with the classification of covariance functions and determine whether a covariance function belongs or does not belong to a specific family or not. Among them we refer to (Priestley and Rao, 1969) and (Fuentes, 2005) for stationarity, (Guan et al., 2004) for isotropy, (Mitchell et al., 2005; Mitchell et al., 2006) for separability and (Li et al., 2008) for symmetry. The main idea of our test is similar to (Lund et al., 2009). Based on the asymptotic distribution of the estimate of spectral density function of a stationary time series, a test statistic is constructed using the ratio of peridograms. Since the asymptotic distribution of stationary periodogram belongs to the scale family, the distribution of the ratio does not depend on the parameter under the equality of covariance function.

Although stationarity plays an important role in time series analysis, in many problems the assumption does not hold true. For instance, stationarity is widely used in estimation of mortality rate models. Let $m_{\mathcal{X},t}$ be the central mortality rate at age $\mathcal{X} \in \{\mathcal{X}_1, \ldots, \mathcal{X}_K\}$, $K \in \mathbb{N}$ and period $t \in \{t_1, \ldots, t_T\}$ and define

$$\eta_{\mathcal{X}t} = E\left[\frac{d}{dt} \log m_{\mathcal{X}t}\right],$$

called the expected value of the mortality improvement rate. Then, the data is modeled through the generalized linear model $\eta_{\mathcal{X}t} = \beta_{\mathcal{X}}\kappa_t + \iota_{t-\mathcal{X}}$, where $\beta_{\mathcal{X}}$ is the effect of age, κ_t is the derivative of period effect and $\iota_{t-\mathcal{X}}$ is the derivative of cohort effect. Using the Newton-Raphson algorithm, the maximum likelihood estimates of parameters, $\widehat{\kappa}_{t_j}$ and $\widehat{\beta}_{\mathcal{X}_i}$, $j = 1, \ldots, T$ and $i = 2, \ldots, K$ are obtained under the normality assumption. Therefore, one may predict the period effect at t_{T+1}, $\widehat{\widehat{\kappa}}_{t_{T+1}}$ say, by employing the time series observations $\widehat{\kappa}_{t_1}, \ldots, \widehat{\kappa}_{t_T}$. To this end, the ARMA model is used by (Haberman and Renshaw, 2012). However, our observations for the rectangular age-period data of France shows a nonstaionarity behavior for this time series. According to Figure 5, the estimated period effects have a constant mean along the time varying second order properties, e.g. the variances.

A locally stationary (L-S) process is nonstationary but shows stationary behavior in small windows of time and so is appropriate for modeling $\{\widehat{\kappa}_{t_j}\}_j$. We are hopeful to find a Gaussian time series in the family of L-S processes which determines the geometric behavior of the observations. By Gaussian assumption we restrict the problem to find a suitable covariance function and for a candidate covariance function $h(\cdot,\cdot)$ the motivating problem is testing the hypothesis H_0 : $\gamma_X(\cdot,\cdot) = h(\cdot,\cdot)$ versus $H_1 : \gamma_X(t,s) \neq h(t,s)$ for some t, s. If $\{Y_t\}_t$ is generated with respect to the covariance function $h(\cdot,\cdot)$, then the mentioned hypotheses are justified through the testing problem of $H_0 : \gamma_X = \gamma_Y$ versus $H_1 : \gamma_X \neq \gamma_Y$.

To employ the spectral analysis methods, we first require a frequency domain representation for L-S processes. In spite of stationary time series, in nonstationary time series the spectral representation includes a stochastic integral with respect to a complex value independent increment stochastic process (Bonami and Estrade, 2003). Fortunately, the representation for L-S time series has a more appropriate form and then the spectral density function appears in a simple Reiman integration form (Dahlhaus, 1997). This form is useful in the construction of almost all estimators of the spectral density function. Among these estimators we prefer the one introduced by (Sergides and Paparoditis, 2007). They developed a bootstrap method to accelerate the convergence rate of CLT type of the sample distribution of the estimator. An admissible convergence rate is a critical property to control the type I error of our tests.

The covariance function of a L-S time series is stationary in small windows of time. We determine the size of each window as a proportion of the sample size T which means that the time series is almost stationary within windows of width uT where $u \in [0,1]$. Using the spectral representation, the covariance function of X at lag τ is defined by

$$c_X(u,\tau) := \text{cov}(X_{t,T}, X_{t+\tau,T}) = \int_{-\pi}^{\pi} f_X(u,\lambda) e^{i\lambda\tau} d\lambda, \tag{1}$$

for all $t = 1,\ldots,T$ where $f_X(u,\lambda)$ denotes the local spectral density function of $\{X_{t,T}\}_t$. Let $c_X(u,\tau)$ and $c_Y(u,\tau)$ be the covariance functions of $\{X_{t,T}\}_t$ and $\{Y_{t,T}\}_t$, respectively. Our purpose is testing

$$\begin{cases} H_0 : c_X(u,\tau) = c_Y(u,\tau), & \text{for all } \tau, \\ H_1 : c_X(u,\tau) \neq c_Y(u,\tau), & \text{for at least one } \tau. \end{cases} \tag{2}$$

Since the covariance function is studied only for $\tau \in \mathbb{N}$, the test is semi-parametric. For the same problem and under the stationary assumption, (Lund et al., 2009) employed the periodogram as an estimator of the spectral density, that is

$$I_X(\lambda_j) = \frac{1}{2\pi T} \left| \sum_{t=1}^{T} X_t e^{-it\lambda_j} \right|,$$

where $\lambda_j = 2\pi j/T$ denote the Fourier frequencies. Thus, the restriction of the test statistic

$$R_j = \frac{I_X(\lambda_j)/f_X(\lambda_j)}{I_Y(\lambda_j)/f_Y(\lambda_j)},$$

to the null hypothesis is

$$R_j\Big|_{H_0} = \frac{I_X(\lambda_j)}{I_Y(\lambda_j)},$$

where f_X and f_Y are the stationary unknown spectral density functions. The distribution of $R_j\Big|_{H_0}$ does not depend on unknown parameters and it is used as a pivotal quantity for test. Using an appropriate estimate of $f_X(u, \lambda)$, we hope to extend this idea to L-S time series. According to the lack of closed form for the distribution of local peridogram, it is not advisable to simple use of R_j for L-S time series. Thus, we present a test statistic by use of the limiting distribution of bootstrap estimator of the spectral density function introduced by (Sergides and Paparoditis, 2007).

The outline of the paper is structured as follows. Some notations and definitions for L-S time series and the bootstrap estimator are reviewed in Section . Using the limit distribution of the bootstrapped periodogram in Section , we present two test statistics. The performances of the purposed tests are evaluated using simulation studies in Section . The mortality rate modeling is also studied as a practical problem in this section.

2. Preliminaries

The family of L-S time series has been introduced by (Dahlhaus, 1997). A time series $\{X_{t,T}\}_t$ belongs to this family if for any $t = 1, \ldots, T$ there exists the sequence $\{\alpha_{t,T}(j)\}_j$ in such a way that $X_{t,T}$ satisfies the representation

$$X_{t,T} = \sum_{j=-\infty}^{\infty} \alpha_{t,T}(j)\varepsilon_{t-j}, \tag{3}$$

where $\{\varepsilon_t\}_t \sim IID(0,1)$. Moreover, for any $j \in \mathbb{Z}$ there exists a positive constant k and a sequence $\{l(j)\}_{j\in\mathbb{Z}}$ where

$$\sup_t |\alpha_{t,T}(j)| \leq \frac{k}{l(j)},$$

such that

$$\sum_{j=-\infty}^{\infty} \frac{|j|}{l(j)} < \infty,$$

and there exists a function $\alpha(\cdot, j) : (0,1] \longrightarrow \mathbb{R}$ satisfying

$$\begin{aligned}
&\sup_u |\alpha(u,j)| \leq \frac{k}{l(j)}, \\
&\sup_t \left|\alpha_{t,T}(j) - \alpha(\frac{t}{T}, j)\right| \leq \frac{k}{Tl(j)}, \\
&|\alpha(u,j) - \alpha(v,j)| \leq \frac{k\,|u-v|}{l(j)}.
\end{aligned}$$

This is the simple causal representation of time series which links the observed values to the pure sequence of a white noise. For stationary causal time series, the sequence of functions $\{\alpha_{t,T}(j)\}_j$ is replaced by an absolute summable sequence of real numbers not depending on t.

For clearance, let us to discuss about a very typical example. When the observed path behaves like an ARMA(p, q) stationary time series in all small windows of width uT, then we can model the observation as a L-S version of ARMA processes, called time varying autoregressive moving average (abbreviated by tvARMA) models with bounded variation coefficient functions (Dahlhaus and Polonik, 2009). Formally, $\{X_{t,T}\}_t$ is a tvARMA(p, q) process if it has the representation

$$X_{t,T} - \sum_{j=1}^{p} \phi_{j,p}(\frac{t}{T}) X_{t-j,T} = \sum_{k=0}^{q} \theta_{k,q}(\frac{t}{T}) \sigma(\frac{t-k}{T}) \varepsilon_{t-k}, \tag{4}$$

where $\sigma(\cdot)$, $\theta_{k,q}(\cdot)$ and $\phi_{j,p}(\cdot)$ are smooth functions of t. Note that the function σ controls the oscillation of variance of $X_{t,T}$. Also, $\{\varepsilon_t\}$ is an iid sequence of zero mean random variables and $\theta_{0,q}(s) \equiv \phi_{0,p}(s) \equiv 1$, $\theta_{j,q}(s) \equiv \theta_{j,q}(0)$ and $\phi_{j,p}(s) \equiv \phi_{j,p}(0)$ for $s < 0$. Consequently, the tvAR(p) and tvMA(q) are obtained from tvARMA(p, q) process by setting $q = 0$ and $p = 0$, respectively.

Using the representation (3) and similar to the stationary case, the local spectral density function of $\{X_{t,T}\}_t$ at frequency λ and time scaling parameter u is given by

$$f_X(u, \lambda) = \frac{1}{2\pi} |A(u, \lambda)|^2,$$

where

$$A(u, \lambda) = \sum_{j=-\infty}^{\infty} \alpha(u, j) e^{-i\lambda j},$$

and then (1) holds true for $\tau = 1, \ldots, [T/2]$. For instance, the local spectral density of a tvAR(p) is

$$f_{tv\mathrm{AR}}(u, \lambda) = \frac{\sigma^2}{2\pi} \left| 1 - \sum_{j=1}^{p} \phi_{j,p}(u) e^{-i\lambda j} \right|^{-2}. \tag{5}$$

This function was used in the bootstrap estimator of $f_X(u, \cdot)$. The discrete Fourier transform and consequently the periodogram is an intrinsic estimator of local spectral density function which is inconsistent (Mikosch and Norvaiša, 1997) but asymptotically unbiased (Brockwell and Davis, 1991, Chapter 11). Thus, it seems that the estimator proposed by (Sergides and Paparoditis, 2007) is a more appropriate starting point. Let $X_{1,T}, X_{2,T}, \ldots, X_{T,T}$ be observations from a L-S time series. The local periodogram is the periodogram of a segment of length N, as a sub window of $[0, T]$ around the time $\lfloor uT \rfloor$, at frequency λ which is defined by

$$I_N(u, \lambda) = \frac{1}{2\pi N} \left| \sum_{s=0}^{N-1} X_{\lfloor uT \rfloor - N/2 + s + 1, T} e^{-i\lambda s} \right|^2.$$

However, because of the closed form (5) for tvAR models, the estimator of local spectral density of this model is given by

$$\widehat{f}_{tv\mathrm{AR}}(u, \lambda) = \frac{\widehat{\sigma}_p^2}{2\pi} \left| 1 - \sum_{r=1}^{p} \widehat{\phi}_r(u) e^{-i\lambda r} \right|^2.$$

The coefficients vector $\widehat{\boldsymbol{\phi}}_u(p) = \left(\widehat{\phi}_1(u), \ldots, \widehat{\phi}_p(u)\right)'$ is the solution of the estimating equation

$$C(u,p)\widehat{\boldsymbol{\phi}}_u(p) = \mathbf{c}(u,p), \tag{6}$$

where $C(u,p) = [c(u, i-j)]_{i,j=0}^{p-1}$ and $\mathbf{c}(u,p) = (c(u,1), \ldots, c(u,p))'$. The unknown covariances are estimated using the simple moment estimators

$$\widehat{C}(u,p) = \frac{1}{N-p}\sum_{j=p}^{N} \mathbf{X}_j(u,p)\mathbf{X}_j'(u,p),$$

and

$$\widehat{\mathbf{c}}(u,p) = \frac{1}{N-p}\sum_{j=p}^{N} \mathbf{X}_j(u,p) X_{\lfloor uT \rfloor - N/2+j,T},$$

where

$$\mathbf{X}_j(u,p) = (X_{\lfloor uT \rfloor - N/2+j-1,T}, \ldots, X_{\lfloor uT \rfloor - N/2+j-p,T})'.$$

The moment estimate of $\sigma_p^2(u)$ is also considered as

$$\widehat{\sigma}_p^2(u) = \frac{1}{N-p}\sum_{j=p}^{N} X^2_{\lfloor uT \rfloor - N/2+j,T} - (\widehat{\boldsymbol{\phi}}_u(p))'\widehat{\mathbf{c}}(u,p).$$

Two estimators of the local spectral density function introduced by (Sergides and Paparoditis, 2007) are reviewed here: (i)

(1) $\tilde{f}(u,\lambda)$: Fit a tvAR(p) to the observation. The coefficients of this model are obtained from the Yule-Walker type equations (6). The order parameter, p, is estimated using the minimum AIC; however the estimator is robust against the changes of p. Estimate the parameter $q(u,\lambda) = f_X(u,\lambda)/f_{tv\mathrm{AR}}(u,\lambda)$ using the kernel smoothed estimator

$$\widehat{q}(u,\lambda) = \frac{1}{N}\sum_{j=-N/2}^{N/2} K\left(\frac{\lambda - \lambda_j}{h}\right)\frac{I_N(u,\lambda_j)}{\widehat{f}_{tv\mathrm{AR}}(u,\lambda_j)},$$

where K is a kernel function and h is the bandwidth of smoothing. Then, the smoothed estimator of the local spectral density function is

$$\tilde{f}(u,\lambda) = \widehat{q}(u,\lambda)\widehat{f}_{tv\mathrm{AR}}(u,\lambda).$$

(2) $I_N^*(u,\lambda)$: This is the bootsrapped local periodogram of the L-S time series. Fit again an tvAR(p) to the data and compute $\widehat{f}_{tv\mathrm{AR}}(u,\lambda)$. Generate the Gaussian pseudo observations $X^+_{1,T}, \ldots, X^+_{T,T}$ according to $\widehat{f}_{tv\mathrm{AR}}(u,\cdot)$ and then compute the local periodogram of these observations for all possible frequencies λ and call it $I^+_{N,tv\mathrm{AR}(p)}(u,\lambda)$. The bootsrapped local periodogram as the estimator of the local spectral density function is

$$I_N^*(u,\lambda) = \widehat{q}(u,\lambda) I^+_{N,tv\mathrm{AR}(p)}(u,\lambda).$$

We use the asymptotic distribution of the estimators to create the test statistic.

3. Test Statistic

Two independent sample paths $X_{1,T}, \ldots, X_{T,T}$ and $Y_{1,T} \ldots, Y_{T,T}$, are observed from two L-S time series for testing the hypotheses (2) at level α. Under the null hypothesis $f_X(u,\lambda) = f_Y(u,\lambda)$, for all λ, we expect the same feature for the estimates of f_X and f_Y. We need to define a distance for the estimates of spectral densities and create a rejection area based on large enough values of the distance function. Let

$$D_{j,X} = \sqrt{N}\left(\int_{-\pi}^{\pi} e^{i\lambda j} I^{*}_{N,X}(u,\lambda)d\lambda - \int_{-\pi}^{\pi} e^{i\lambda j}\tilde{f}_X(u,\lambda)d\lambda\right),$$

be the distance of the estimated local covariance functions with respect to $\tilde{f}(u,\cdot)$ and $I^{*}_{N,X}(u,\cdot)$ at lags $j = 1, \ldots, \lfloor T/2 \rfloor$ and define $\mathbf{D}_{T,X} = \left(D_{1,X}, \ldots, D_{\lfloor T/2 \rfloor,X}\right)'$ as the vector of distances. The normality of $\mathbf{D}_{T,X}$ needs two more assumptions: ASSUMPTION 1.

(1) The window width N satisfies $N \to \infty$ such that $N^{3/2}/T \to 0$ as $N \to \infty$.
(2) The smoothing bandwidth h is defined as a function of N satisfies $h \to 0$ such that $Nh \to \infty$ as $N \to \infty$.

Under the assumptions 1 and 2 (Sergides and Paparoditis, 2007)

$$\mathbf{D}_{T,X} \xrightarrow{P} N(\mathbf{0}, W_X), \tag{7}$$

as $T \to \infty$, where $W_X = [W_{\eta\zeta,X}]_{\eta,\zeta}$, $\eta, \zeta = 1, \ldots, \lfloor T/2 \rfloor$ is a covariance matrix with components

$$\begin{aligned} W_{\eta\zeta,X} &= \operatorname{cov}(D_{\eta,X}, D_{\zeta,X}) \\ &= 2\pi\left\{\int_{-\pi}^{\pi} e^{i\lambda\eta}\left\{e^{i\lambda\zeta} + e^{-i\lambda\zeta}\right\} f_X^2(u,\lambda)d\lambda\right. \\ &+ \left.\kappa_4(p)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i\lambda\eta}e^{-i\mu\zeta} f_X(u,\lambda) f_X(u,\mu)d\lambda d\mu\right\}, \end{aligned}$$

and

$$\kappa_4(p) = \frac{1}{\sigma_p^4(u)}\int_0^1 E\left[\tilde{X}_p(u) - \sum_{j=1}^{p}\hat{\phi}_j(u)\tilde{X}_{p-j}(u)\right]^4 - 3,$$

where

$$\tilde{X}_t(u) = \sum_{j=-\infty}^{\infty} \alpha(u,j)\varepsilon_{t-j}.$$

With appropriate notation for the time series $\{Y_{t,T}\}_t$, we have

$$\begin{aligned} W_X^{-1/2}\mathbf{D}_{T,X} &\xrightarrow{P} N(\mathbf{0}, I_{\lfloor T/2 \rfloor}), \\ W_Y^{-1/2}\mathbf{D}_{T,Y} &\xrightarrow{P} N(\mathbf{0}, I_{\lfloor T/2 \rfloor}), \end{aligned}$$

independently as $T \to \infty$ where I_k is the identity matrix of order k. Thus, define

$$R_\tau = \left(\frac{D_{\tau,X}/W_{\tau\tau,X}}{D_{\tau,Y}/W_{\tau\tau,Y}}\right)^2,$$

for $\tau = 1, \ldots \lfloor T/2 \rfloor$ where the restriction of R_τ to the parameter space induced by the null hypothesis is

$$R_\tau\big|_{H_0} = \left(\frac{D_{\tau,X}}{D_{\tau,Y}}\right)^2. \tag{8}$$

$R_\tau\big|_{H_0}$ is asymptotically distributed as $F_{1,1}$ and many goodness of fit approaches are opened to solve the testing problem. One may simply employ the nonparametric tests such as Kolmogorov-Smirnov or Pearson tests. Notice that $\{R_\tau\big|_{H_0}\}_\tau$ is the sequence of dependent random variables and using the mentioned methods requires replication in observations. Therefore, these methods have the lower powers in comparison to the methods introduced here. Our rejection methods for simultaneous hypotheses testing (2) are obtained by using the marginal quantiles of $R_\tau\big|_{H_0}$:

Test based on the pivotal quantity (8): We reject the null hypothesis for the large deviations of (8) from 1. Using the Bonferroni's method, reject the null hypothesis at level α when $R_\tau\big|_{H_0}$ exceeds $(1-\alpha/(2\lfloor T/2 \rfloor))$'th quantile or is smaller from the $\alpha/(2\lfloor T/2 \rfloor)$'th quantile of the $F_{1,1}$ distribution.

Test based on the Fisher's Z: The expected value of (8) does not exist and has a heavier tail in comparison with

$$\frac{1}{2}V_\tau = \frac{1}{2}\log R_\tau\big|_{H_0} \sim Fisher'sz(1,1).$$

Thus, the type I error of the other rejection area

$$|V_\tau| > F_{1-\alpha/(2\lfloor T/2 \rfloor);z(1,1)}, \quad \text{at least for one } \tau = 1, \ldots, \lfloor T/2 \rfloor,$$

does not exceed α, where $F_{p;z(m,n)}$ is the p'th quantile of the Fisher's Z distribution with m and n degrees of freedom (Aroian, 1941). The quantiles are generated using the Monte Carlo method.

4. Numerical results

Theoretic conclusions are examined in two numeric studies. The first one is a simulation study and the second is a mortality rate modeling problem.

A. Simulation study

Consider a tvAR(1) model with equation

$$X_{t,T} = 0.9\cos\left(1.5 - \cos(4\pi t/T)\right) X_{t-1,T} + \varepsilon_t, \tag{9}$$

where $\{\varepsilon_t\}_t$ is a standard Gaussian white noise. A realization of this time series is given in Figure 1. In the first step, we consider two independent sample paths $\{X_{t,T}\}_t$ and $\{Y_{t,T}\}_t$ of length $T = 512$ from this model and then examine the empirical type I error of the tests. To compute $\widehat{q}$ we use the Bartlett-Priestley's kernel given by

$$K(x) = \begin{cases} 3(4\pi)^{-1}\left(1 - (x/\pi)^2\right), & |x| \le \pi \\ 0, & \text{otherwise.} \end{cases}$$

The bandwidth h is set to 0.2, p is equal to one and the window length N is 40. This setting of simulation is replicated 100 times. The true decision is not to reject the null hypothesis in this case. All of the resulted $R_\tau\big|_{H_0}$'s belong to the interval $(F_{0.05/512;1,1}, F_{1-0.05/512;1,1})$ an thus the empirical type I error is equal to zero. The same results take place for testing based on V_τ and both tests are at level 0.05. In comparison to (Sergides and Paparoditis, 2009) which implemented the same simulation study, the type I error is controlled at level $.05$ in both methods. The distance of

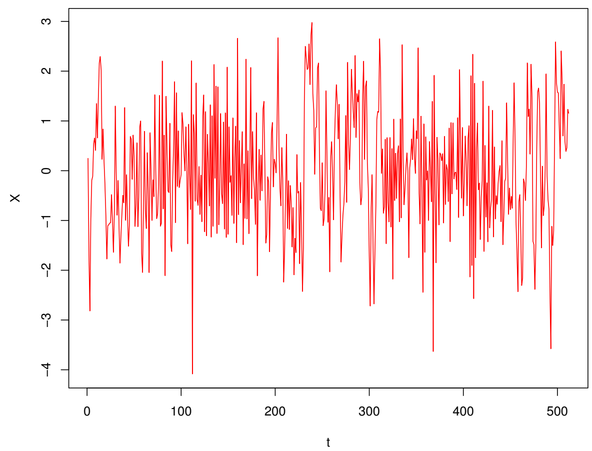

Fig. 1: A realization from (9) with $T = 512$ observations.

the empirical type I error and $\alpha = 0.05$ is due to the use of Bonferroni method in the simultaneous hypotheses testing. At this step, the procedure is also repeated for two independent observations from a tvMA(1) model with corresponding difference equation

$$X_{t,T} = 1.1\cos\left(1.5 - \cos(4\pi t/T)\right)\varepsilon_{t-1} + \varepsilon_t, \tag{10}$$

and a realization of this model is shown in Figure 2. In this case the empirical type I error based on $R_\tau\big|_{H_0}$ exceeds the level 0.05 and is equal to 0.06; while the empirical type I error of the second approach is 0.04.

In the second step we examine the empirical powers of tests along the sensitivity of tests using the sample paths of two different L-S time series. Let $\{X_{t,T}\}_t$ and $\{Y_{t,T}\}_t$ be two independent tvAR(1) and tvMA(1) time series respectively. According to Figures 1 and 2, these two time series are structurally different and we expect the tests to simply detect the differences. The tests based on $R_\tau\big|_{H_0}$ and V_τ achieve the empirical powers $.93$ and 0.94, respectively. Now suppose that the tvAR model is accepted in the background and we want to check the sensitivity of tests to the order of tvAR model. Again we simulate a realization under the model (9) and let $\{Y_{t,T}\}_t$, is a sample path from tvAR(2) satisfying

$$\begin{aligned} Y_{t,T} &= 0.9\cos(1.5 - \cos(4\pi t/T))Y_{t-1,T} \\ &+ 0.3\cos(1 - \cos(4\pi t/T))Y_{t-2,T} + \varepsilon_t. \end{aligned} \tag{11}$$

A realization of $\{Y_{t,T}\}$ is shown in Figure 3. In this case, the empirical powers based on $R_\tau\big|_{H_0}$

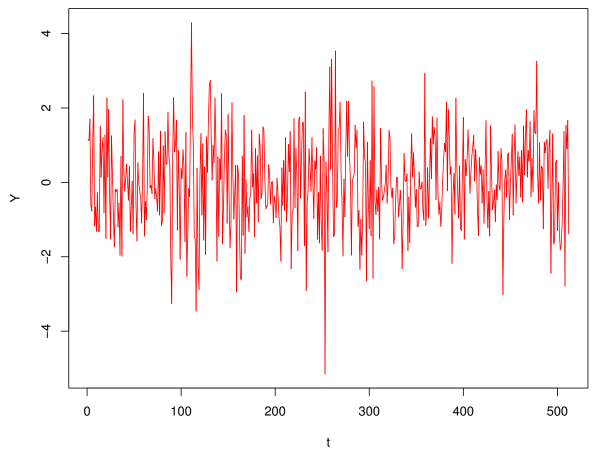

Fig. 2: A realization from (10) with $T = 512$ observations.

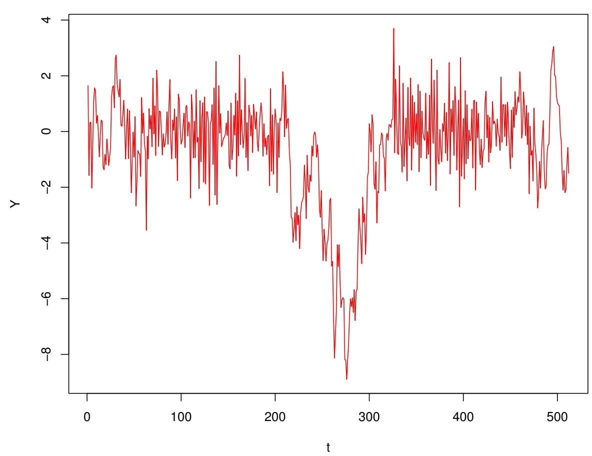

Fig. 3: A realization from (11) with $T = 512$ observations.

and V_τ are improved to the values 0.95 and 0.96, respectively. Returning to Figures 1 and 3, two models (9) and (11) are different in the local oscillation and the bootsrapped periodogram $I^*_{N,X}(u, \lambda)$ is the size of induced oscillation with respect to this particular frequency and this fact causes the improvements in the powers. Since the visual differences in $tv\text{AR}(1)$ and $tv\text{AR}(2)$ is the size of their local oscillations, then $I^*_{N,X}$'s are quite different which verifies the results of tests. (Sergides and Paparoditis, 2009) examined a $tv\text{AR}(1)$ versus $tv\text{AR}(2)$ with different parameters to compute the empirical power and type I error of the test; while we examine both differences in parameters and differences in the orders of models. In other words, in the first setting we examine two $tv\text{AR}(1)$ with different parameters and in the second setting we test the order of a $tv\text{AR}(p)$ model with known parameters. They achieved the power equal to $.96$ in testing a $tv\text{AR}(1)$ versus another $tv\text{AR}(1)$ with different parameter.

Regarding Figure 3, one may suggest the use of SETAR models to explain the piecewise behavior of the mean function. Visually, the hypothesis $H_0 : True\ model\ is$ AR is rejected versus $H_1 : True\ model\ is$ SETAR or theoretically use (Tsay, 1989) to this end which tends to the p-value equal to 0.030. However, constructing a theoretic testing procedure to compare the efficiency of SETAR with tvAR is not as easy as the previous schematic decision and we only compare the squared sum of residuals (SSE) of the fitted models. The SSE of fitted 2 and 3-regimes SETAR models are 1091.66 and 1032.19, respectively while the SSE of tvAR(2) model is 648.49 which shows the efficiency of tvAR models against SETAR in analyzing the data in Figure 3.

We also provide a step 3 in simulation study which tests the mentioned L-S time series versus ARIMA models. Using minimum AIC, ARIMA$(2, 2, 1)$ and ARIMA$(0, 1, 2)$ are fitted to the sample paths plotted in Figures 2 and 3, respectively. Using the test provided by (Lund et al., 2009) we examine the performance of ARIMA models in explanation of these two models. Thus, let $X_{1,T+d}, \dots, X_{T+d,T+d}$ be the generated sample path from models (9) or (10) and we want to test the hypotheses H_0 : ARIMA(p, d, q) versus $H_1 : not\ H_0$ with the known estimated parameters. We generate the observation $Y_1, \dots, Y_{T+d}$ under H_0 and define $W_t := (1-B)^d X_{t,T+d}$ and $Z_t := (1-B)^d Y_t$ where B is the backward operator. Thus,under the null hypothesis the series $\{W_t\}$ and $\{Z_t\}$ are two independent ARMA(p, q) time series with the same parameters and we can test the equality of covariance functions of these two stationary time series using the methods introduced by (Lund et al., 2009). We replicate this procedure for 100 sample paths from (9) and (10) and the null is rejected 98 and 97 times for these models, respectively.

B. A real data

A rectangular mortality data array is constructed by unit squares of size one year by triplets $(d_{\mathcal{X}t}, e_{\mathcal{X}t}, \omega_{\mathcal{X}t})$, for ages $\mathcal{X} = \mathcal{X}_1, \dots, \mathcal{X}_k$ and periods $t = t_0 + 1, \dots, t_0 + T$ where t_0 is the base year, $d_{\mathcal{X}t}$ is reported number of deaths, , $e_{\mathcal{X}t}$ is the exposure to the risk of death and $\omega_{\mathcal{X}t}$ is used to indicate the empty data cells. A rectangular data array for a generation is depicted in Figure 4. The observations gathered in a rectangular data array are the essentials of life table which is the basis to compute many quantities in life insurances such as optimum premiums, life expectancy and many other demographic indexes. Employing the `demography` package in `R`, we have the life table and hence the rectangular data array of France within 1950–2006. We hope to analyze the stochastic process $\{e_{\mathcal{X}t}\}_t$ as a sample path of a time series. Under the normality assumption, we need to testify the covariance function of this time series to estimate the predicted number of the exposure to risk. It is prevalent to look at the reported number of deaths or exposure to the risk via the central mortality rate which is define by $m_{\mathcal{X},t} = d_{\mathcal{X}t}/e_{\mathcal{X}t}$. To predict $m_{\mathcal{X}t}$ and so $e_{\mathcal{X}t}$ in actuarial problems, m is participated in a generalized linear model such as (Lee and Carter, 1992)

$$\zeta_{\mathcal{X}t} = \beta_{\mathcal{X}} \kappa_t,$$

where $\zeta_{\mathcal{X}t} = E[m_{\mathcal{X},t}]$, $\beta_{\mathcal{X}}$ is the age effect and κ_t is the period effect. The estimated period effects $\{\widehat{\kappa}_t\}_t$ construct a time series realization and $\widehat{\widehat{\kappa}}_{t_0+T+1}$ is the predicted value of period effect in the next year. This prediction is a critical value in forecasting the mortality rate of the next year.

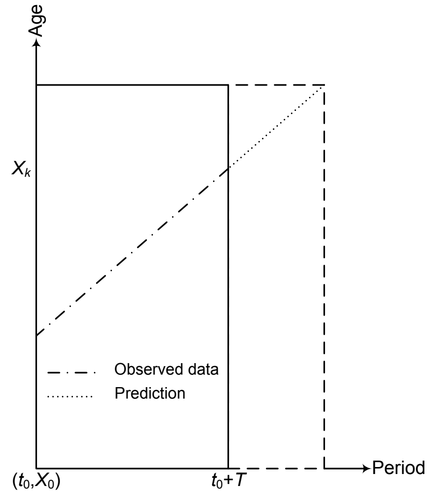

Fig. 4: A schematic diagram of a rectangular age-period data array. The rectangle with solid line determines the territory of the observed values on the life table. The observations for each generation contain the triplets $(d_{\mathcal{X}t}, e_{\mathcal{X}t}, \omega_{\mathcal{X}t})$ which are located on an oblique line. The rectangle with dashed edges represents the prediction space.

Typically, $\{\widehat{\kappa}_t\}_t$ is modeled as a realization of an ARIMA$(p, 1, q)$ time series. (Haberman and Renshaw, 2012) introduced the improvement mortality rate

$$Z_{\mathcal{X}} = 2\frac{1 - m_{\mathcal{X}t}/m_{\mathcal{X},t-1}}{1 + m_{\mathcal{X}t}/m_{\mathcal{X},t-1}},$$

and employed the generalized linear model

$$\eta_{\mathcal{X}t} = \beta_{\mathcal{X}}\kappa_t^*, \tag{12}$$

where $\eta_{\mathcal{X}t} = E[Z_{\mathcal{X},t}]$ and κ_t^* is the derivative of κ_t and hence we expect an ARMA(p, q) model for $\{\widehat{\kappa}_t^*\}_t$. We compute $\widehat{\kappa}_t^*$ for the mortality data of France within 1950–2006. The computed values are very sensitive to the initial values of the Newton-Raphson algorithm and according to the computation restrictions we did not control the convergence rules stringently. Figure 5 shows the values of $\widehat{\kappa}_t^*$ for this data. Time varying properties are obvious in this time plot; however, using the both testing methods in Section ,the hypothesis H_0 : tvAR(1) model versus H_1 : not H_0 is not rejected. Therefore, comparing the SSE of the tvAR and ARMA models which are respectively equal to 62.684 and 67.508, we prefer the use of tvAR models instead of other stationary models.

5. Conclusion

There are many methods to test the equality of covariance functions under the stationary assumption. Among them, we focus on the methods introduced by (Coates and Diggle, 1986) and developed by (Lund et al., 2009). We first choose an appropriate estimator of the local spectral density function. Then, under the normality assumption we use the fact of the same local spectral density then the same local covariance function. Based on this idea, two pivotal

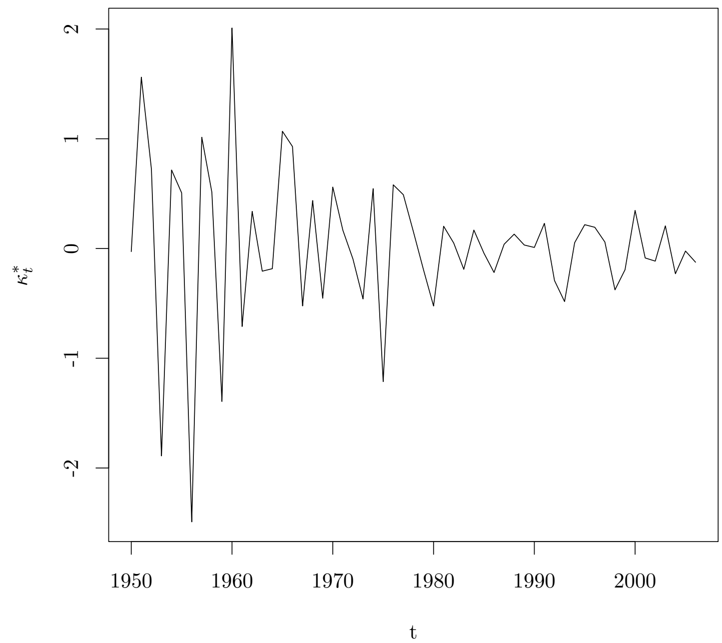

Fig. 5: Estimated values of κ_t^* for the mortality rate data of France under the model (12).

quantities R and $V$are introduced which under the null hypothesis $H_0 : c_X(u, \tau) = c_Y(u, \tau)$, for all τ, their densities do not depend on the unknown parameters. The simulation study confirms the domination of the first test function by the second one in view of the empirical power. We also study the empirical type I error of both tests to remove the danger of ever-rejection in high power tests.

This inference is used in a classic actuarial problem of the prediction of mortality rate. Typically, the observed time effects of the Lee-Carter type models for the improved mortality rate are considered as stationary and usually an ARMA time series. The observed time effects in literatures (including our results in this paper) usually have a time varying structures particularly in the variance. According to the presented testing methods, we show the preference of the L-S time series in comparison to the ARMA models.

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