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Several New Families of Jarratt's Method for Solving Systems of Nonlinear Equations

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Abstract

In this study, we suggest and analyze a new and wide general class of Jarratt's method for solving systems of nonlinear equations. These methods have fourth-order convergence and do not require the evaluation of any second or higher-order Fréchet derivatives. In terms of computational cost, all these methods require evaluations of one function and two first-order Fréchet derivatives. The performance of proposed methods is compared with their closest competitors in a series of numerical experiments. It is worth mentioning that all the methods considered here are found to be effective and comparable to the robust methods available in the literature.

Keywords: Numerical analysis; systems of nonlinear equations; iterative methods; order of convergence; Jarratt's method

AMS-MSC 2010 No: 41A25, 65H05

1. Introduction

This paper addresses the problem of finding real roots of nonlinear system of the form

$$F(x) = 0, \tag{1}$$

where

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T, \; x = (x_1, x_2, \ldots, x_n)^T$$

and $F : \mathrm{R}^n \to \mathrm{R}^n$ is a sufficiently differentiable vector function.

Many problems about finding a root of (1) have emerged in many sciences and engineering applications. The zeros of a nonlinear system can not in general be expressed in closed form, thus iterative methods for approximating solutions of systems of nonlinear equations are the most frequently used techniques. Therefore, finding a root of (1) has become one of the most important and challenging problems in computational mathematics. Many robust and efficient methods for solving (1) are already engaged. One of the most basic procedures for approximating solutions of the nonlinear system $F(x) = 0$, is the quadratically convergent Newton's method Traub (1964) and is given by

$$x^{(n+1)} = x^{(n)} - \{F'(x^{(n)})\}^{-1} F(x^{(n)}), n = 0,1,\ldots, \tag{2}$$

where $\{F'(x)\}^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ of the function $F(x)$.

In order to improve the local order of convergence of Newton's method, a number of methods have been proposed in the literature. For a system of k equations in k unknowns, the first-order Fréchet derivative is a matrix with k^2 evaluations while the second-order Fréchet derivative has $\frac{k^2(k+1)}{2}$ evaluations. This implies that a huge amount of computational work is required to evaluate every iteration Amat et al. (2003). Third order iterative methods like Halley's method Amat et al. (2003), Gutierrez and Hernandez (1997) and Chebyshev's method Amat et al. (2003), Gutierrez and Hernandez (1997) are close relatives of Newton's method. These methods require the evaluation of the second-order Fréchet derivative per iteration. Therefore, despite their cubic convergence, they are considered less practical from the computational point of view.

Multipoint iterative methods for solving nonlinear systems play a significant role in the field of iterative processes since they circumvent the drawbacks of one-point iterations, such as Newton's method. Such constructions occasionally possess a better order of convergence and efficiency index for solving the systems of nonlinear equations. In recent years, some new higher order iterative methods have been developed and analyzed to solve the nonlinear systems without using the second-order Fréchet derivative *cf.* Homeier (2004), Grau-Sa'nchez et al. (2011), Sharma et al. (2013), Cordero et al. (2009), Darvishi and Barati (2007) and Nedzhibov (2008).

In this paper, our main objective is to develop a wide general class of fourth-order Jarratt's method Jarratt (1966) for solving nonlinear systems without using the second or any higher-order Fréchet derivatives. For this purpose, we extend the scheme of Behl et al. (2013) to the k-dimensional case in a simple way. We also perform different numerical tests that confirm the theoretical results and allow us to compare the methods with some other recently published methods.

2. Description of New General Class of Jarratt's Method

More recently, Behl et al. (2011) have proposed a new optimal family of Jarratt's method for solving scalar nonlinear equations. This is given by

$$\begin{cases} y_n = & x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = & x_n - \dfrac{[(\alpha_1^2 - 22\alpha_1\alpha_2 - 27\alpha_2^2)f'(x_n) + 3(\alpha_1^2 + 10\alpha_1\alpha_2 + 5\alpha_2^2)f'(y_n)]f(x_n)}{2[\alpha_1 f'(x_n) + 3\alpha_2 f'(y_n)][3(\alpha_1+\alpha_2)f'(y_n) - (\alpha_1 + 5\alpha_2)f'(x_n)]}, \end{cases} \tag{3}$$

where $\alpha_1, \alpha_2 \in \mathrm{R}$ such that neither $\alpha_1 = \alpha_2$ nor $\alpha_1 = -3\alpha_2$ (otherwise these families of methods have a third -order of convergence).

In this section, we intend to develop an iterative scheme of higher order for solving systems of nonlinear equations without using second-order Fr e' chet derivative. For this purpose, we introduce the following modification over the family **Error! Reference source not found.** for multi-dimensional case:

$$\begin{cases} y^{(n)} = & x^{(n)} - \beta h(x^{(n)}), \\ \phi(x^{(n)}) = & x^{(n)} - \dfrac{1}{2}\{\eta(x^{(n)})\}^{-1}\nu(x^{(n)})h(x^{(n)}), \end{cases} \tag{2}$$

where

$$\nu(x^{(n)}) = [(\alpha_1^2 - 22\alpha_1\alpha_2 - 27\alpha_2^2)I + 3(\alpha_1^2 + 10\alpha_1\alpha_2 + 5\alpha_2^2)B(x^{(n)})], \tag{3}$$

$$\begin{aligned} \eta(x^{(n)}) &= [\alpha_1 I + 3\alpha_2 B(x^{(n)})][3(\alpha_1+\alpha_2)B(x^{(n)}) - (\alpha_1 + 5\alpha_2)I], \\ & B(x^{(n)}) = H(x^{(n)})F'(y^{(n)}),\ y^{(n)} = y(x) = x^{(n)} - \beta h(x^{(n)}), \\ & h(x^{(n)}) = H(x^{(n)})F(x^{(n)}) \text{ and } H(x^{(n)}) = \{F'(x^{(n)})\}^{-1}, \end{aligned} \tag{4}$$

and where I denotes the $k \times k$ identity matrix and $\alpha_1, \alpha_2 \in \mathrm{R}$ where $\alpha_1, \alpha_2 \in \mathrm{R}$ such that neither $\alpha_1 = \alpha_2$ nor $\alpha_1 = -3\alpha_2$.

3. Convergence Analysis

In order to explore the convergence properties of scheme (2), we recall the following results of Taylor's series expression on vector functions [see Örtega and Rheinboldt (1970)] and lemma proved by Nedzhibov [(see Nedzhibov (2008)].

Lemma 3.1.

Let $F: V \subset \mathrm{R}^n \to \mathrm{R}^n$ be a C^p function defined on $V = \{x^{(n)} : \| x - a \| < r\}$; then for any $\| v \| \le r$, the following expression holds:

$$F(a+v) = F(a) + F'(a)v + \frac{1}{2!}F''(a)vv + \frac{1}{3!}F'''(a)vvv + \ldots + \frac{1}{(p-1)!}F^{(p-1)}(a)(v, \ldots v) + \mathrm{R}_p, \tag{5}$$

where

$$\| \mathrm{R}_p \| \le \sup_{x \in V} \frac{\| v \|^p}{p!} \| F^{(p)}(x^{(n)}) - F^{(p)}(a) \|. \tag{6}$$

Lemma 3.2.

Let $F: V \subset \mathrm{R}^n \to \mathrm{R}^n$ is a C^4 function, and has a locally convergent unique root $\alpha \in V$. Further, suppose that the Jacobian $F'(x^{(n)})$ is invertible in a neighborhood of α, then the following expressions hold:

$$\begin{aligned}
h'(\alpha) &= I, \\
h''(\alpha) &= -H(\alpha)F''(\alpha), \\
h'''(\alpha) &= 2H''(\alpha)F'(\alpha) + H'(\alpha)F''(\alpha), \\
B(\alpha) &= I, \\
B'(\alpha) &= -\beta H(\alpha)F''(\alpha), \\
B''(\alpha) &= H''(\alpha)F'(\alpha) + 2(1-\beta)H'(\alpha)F''(\alpha) + (1-\beta)^2 H(\alpha)F'''(\alpha) + \beta(H(\alpha)F''(\alpha))^2.
\end{aligned} \tag{9}$$

Theorem 3.1.

Let $F: V \subset \mathrm{R}^n \to \mathrm{R}^n$ is a C^4 function in an open convex set $V \subset \mathrm{R}^n$. Assume that there exits an $\alpha \in V$ such that $F(\alpha) = 0$ and $F'(\alpha)^{-1}$ exits. Then there exits an $\varepsilon > 0$ such that for every initial guess $x^{(0)} \in U(\alpha, \varepsilon)$, the sequence of iterates generated by $x^{(n+1)} = \phi(x^{(n)})$ is well defined, converges to α, and has fourth-order convergence when $\beta = \frac{2}{3}$.

Proof:

From equation (2), we get

$$\phi(x^{(n)})-\alpha = x^{(n)}-\alpha-\frac{1}{2}\{\eta(x^{(n)})\}^{-1}\nu(x^{(n)})h(x^{(n)}). \tag{7}$$

We introduce the following notations

$$\phi(x^{(n)})-\alpha=\gamma, \tag{8}$$

$$x^{(n)}-\alpha=\varepsilon. \tag{9}$$

Now, equation (7) can be rewritten as

$$\gamma=\frac{1}{2}\{\eta(x^{(n)})\}^{-1}[2\eta(x^{(n)})e-\nu(x^{(n)})h(x^{(n)})]. \tag{10}$$

Using Lemma 3.1, we can represent $h(x^{(n)})$ by using the following Taylor's series expansion

$$h(x^{(n)})=h(\alpha)+h'(\alpha)e+\frac{1}{2}h''(\alpha)ee+\frac{1}{6}h'''(\alpha)eee+O(\|e\|^4). \tag{11}$$

Since α is a root of system (1.1), therefore, $F(\alpha)=0 \Rightarrow h(\alpha)=H(\alpha)F(\alpha)=0$ and $h'(\alpha)=I$ by Lemma 3.2. Therefore, equation (11) further gives

$$h(x^{(n)})=e+\frac{1}{2}h''(\alpha)ee+\frac{1}{6}h'''(\alpha)eee+O(\|e\|^4). \tag{12}$$

Similarly, we express

$$B(x^{(n)})=B(\alpha)+B'(\alpha)e+\frac{1}{2}B''(\alpha)ee+O(\|e\|^3). \tag{13}$$

Using Lemma 3.2 and equation (13) in (3), we obtain

$$\begin{aligned}\nu(x^{(n)})=4(\alpha_1^2+2\alpha_1\alpha_2-3\alpha_2^2)I+3\beta h''(\alpha)(\alpha_1^2+10\alpha_1\alpha_2+5\alpha_2^2)e\\ +\frac{3}{2}B''(\alpha)(\alpha_1^2+10\alpha_1\alpha_2+5\alpha_2^2)ee+O(\|e\|^3).\end{aligned} \tag{14}$$

Using (12) and (14), we have

$$\begin{aligned}\nu(x^{(n)})h(x^{(n)})=4(\alpha_1^2+2\alpha_1\alpha_2-3\alpha_2^2)e+h''(\alpha)[\alpha_1^2(2+3\beta)+3\alpha_2^2(-2+5\beta)+2\alpha_1\alpha_2(2+15\beta)]ee\\ +\frac{1}{6}[4h'''(\alpha)(\alpha_1^2+2\alpha_1\alpha_2-3\alpha_2^2)+9B''(\alpha)(\alpha_1^2+10\alpha_1\alpha_2+5\alpha_2^2)\\ +9h''(\alpha)^2\beta(\alpha_1^2+10\alpha_1\alpha_2+5\alpha_2^2)]eee+O(\|e\|^4).\end{aligned} \tag{18}$$

Using (13) in (4), we obtain

$$\eta(x^{(n)}) = 2(\alpha_1^2 + 2\alpha_1\alpha_2 - 3\alpha_2^2)I + 3\beta h''(\alpha)(\alpha_1^2 + 6\alpha_1\alpha_2 + \alpha_2^2)e + \frac{3}{2}[B''(\alpha)(\alpha_1^2 + 6\alpha_1\alpha_2 + \alpha_2^2) + 6h''(\alpha)^2\beta^2\alpha_2(\alpha_1 + \alpha_2)]ee + O(\|e\|^3). \tag{19}$$

Substituting **Error! Reference source not found.** and **Error! Reference source not found.** in (10), we have

$$\begin{aligned}\gamma = \frac{1}{2}\{\eta(x^{(n)})\}^{-1}\{&h''(\alpha)(\alpha_1^2 + 2\alpha_1\alpha_2 - 3\alpha_2^2)(-2+3\beta)ee \\ &+ \frac{1}{6}[9B''(\alpha)(\alpha_1^2 + 2\alpha_1\alpha_2 - 3\alpha_2^2) - 4h'''(\alpha)(\alpha_1^2 + 2\alpha_1\alpha_2 - 3\alpha_2^2) \\ &- 9h''(\alpha)^2\beta(\alpha_1^2 + \alpha_2^2(5-12\beta) - 2\alpha_1\alpha_2(-5+6\beta))]eee + O(\|e\|^4)\}.\end{aligned} \tag{15}$$

For $\beta = \dfrac{2}{3}$, equation (15) becomes

$$\gamma = \{\eta(x^{(n)})\}^{-1}\frac{1}{12}(\alpha_1^2 + 2\alpha_1\alpha_2 - 3\alpha_2^2)\{9B''(\alpha) - 6h''(\alpha)^2 - 4h'''(\alpha)\}eee + O(\|e\|^4). \tag{16}$$

According to Lemma 3.2 and substituting the expressions of $h'(\alpha), h''(\alpha)$ and $B''(\alpha)$ in (16), we have

$$\begin{aligned}\gamma = \{\eta(x^{(n)})\}^{-1}\frac{1}{12}(\alpha_1^2 + 2\alpha_1\alpha_2 - 3\alpha_2^2)\{&H''(\alpha)F'(\alpha) + 2H'(\alpha)F''(\alpha) \\ &+ H(\alpha)F'''(\alpha)\}eee + O(\|e\|^4).\end{aligned} \tag{22}$$

Let us differentiate twice the equation $H(x^{(n)})F'(x^{(n)}) = I$; we thereby obtain

$$H''(x^{(n)})F'(x^{(n)}) + 2H'(x^{(n)})F''(x^{(n)}) + H(x^{(n)})F'''(x^{(n)}) = 0. \tag{17}$$

Using equation (17) in **Error! Reference source not found.**, finally we get

$$\gamma = O(\|e\|^4). \tag{18}$$

This completes the proof of the theorem.

3.1. Special Cases

Finally, by using different specific values of α_1 and α_2, which are defined in Theorem 3.3, we get the various methods from formula (2) as follows:

(i) For $\alpha_1 = 0$, family (2) reads as

$$\begin{cases} y^{(n)} = & x^{(n)} - \frac{2}{3}\{F'(x^{(n)})\}^{-1}F(x^{(n)}), \\ x^{(n+1)} = & x^{(n)} - \left\{2(\alpha_2 F'(x^{(n)}) + 3\alpha_2 F'(y^{(n)}))(-(1+5\alpha_2)F'(x^{(n)}) + 3(\alpha_2+1)F'(y^{(n)}))\right\}^{-1} \\ & \times\left\{-(27\alpha_2^2 + 22\alpha_2 - 1)F'(x^{(n)}) + 3(5\alpha_2^2 + 10\alpha_2 + 1)F'(y^{(n)})\right\}F(x^{(n)}). \end{cases} \quad (25)$$

This is a new fourth-order family of methods for solving systems of nonlinear equations.

Sub special cases of family Error! Reference source not found.

(a) For $\alpha_2 = 0$, family **Error! Reference source not found.** reads as

$$\begin{cases} y^{(n)} = & x^{(n)} - \frac{2}{3}\{F'(x^{(n)})\}^{-1}F(x^{(n)}), \\ x^{(n+1)} = & x^{(n)} - \frac{1}{2}\left\{3F'(y^{(n)}) - F'(x^{(n)})\right\}^{-1}\left\{3F'(y^{(n)}) + F'(x^{(n)})\right\}\{F'(x^{(n)})\}^{-1}F(x^{(n)}). \end{cases} \quad (26)$$

This is the well-known Jarratt’s method [Nedzhibov (2008)] for solving systems of nonlinear equations.

(b) For $\alpha_2 = -2$, family **Error! Reference source not found.** reads as

$$\begin{cases} y^{(n)} = & x^{(n)} - \frac{2}{3}\{F'(x^{(n)})\}^{-1}F(x^{(n)}), \\ x^{(n+1)} = & x^{(n)} - \frac{1}{2}\left\{\left[F'(x^{(n)}) - 6F'(y^{(n)})\right]\left[3F'(x^{(n)}) - F'(y^{(n)})\right]\right\}^{-1}\left\{F'(y^{(n)}) - 21F'(x^{(n)})\right\}F(x^{(n)}). \end{cases} \quad (27)$$

This is a new fourth-order method for solving systems of nonlinear equations.

(ii) For $\alpha_2 = 1$, family (2) reads as

$$\begin{cases} y^{(n)} = & x^{(n)} - \frac{2}{3}\{F'(x^{(n)})\}^{-1}F(x^{(n)}), \\ x^{(n+1)} = & x^{(n)} - \left\{2\left[\alpha_1 F'(x^{(n)}) + 3F'(y^{(n)})\right]\left[-(\alpha_1+5)F'(x^{(n)}) + 3(\alpha_1+1)F'(y^{(n)})\right]\right\}^{-1} \\ & \times\left\{(\alpha_1^2 - 22\alpha_1 - 27)F'(x^{(n)}) + 3(\alpha_1^2 + 10\alpha_1 + 5)F'(y^{(n)})\right\}F(x^{(n)}). \end{cases} \quad (19)$$

This is another new fourth-order family of methods for solving systems of nonlinear equations.

Sub special cases of family (19)

(a) For $\alpha_1 = 0$, family (19) reads as

$$\begin{cases} y^{(n)} = & x^{(n)} - \dfrac{2}{3}\{F'(x^{(n)})\}^{-1}F(x^{(n)}), \\ x^{(n+1)} = & x^{(n)} - \dfrac{1}{2}\left\{3F'(y^{(n)}) - 5F'(x^{(n)})\right\}^{-1}\left\{5F'(y^{(n)}) - 9F'(x^{(n)})\right\}\left\{F'(y^{(n)})\right\}^{-1}F(x^{(n)}). \end{cases} \tag{20}$$

This is a modification over the well-known Jarratt's method Jarratt (1966) for solving systems of nonlinear equations.

(b) For $\alpha_1 = 5$, family (19) reads as

$$\begin{cases} y^{(n)} = & x^{(n)} - \dfrac{2}{3}\{F'(x^{(n)})\}^{-1}F(x^{(n)}), \\ x^{(n+1)} = & x^{(n)} - \left\{\left[5F'(x^{(n)}) + 3F'(y^{(n)})\right]\left[5F'(x^{(n)}) - 9F'(y^{(n)})\right]\right\}^{-1}\left\{28F'(x^{(n)}) - 60F'(y^{(n)})\right\}F(x^{(n)}). \end{cases} \tag{30}$$

This is a new fourth-order method for solving systems of nonlinear equations.

(iii) For $\alpha_1 = 50$ and $\alpha_2 = \dfrac{1}{10}$, family (2) reads as

$$\begin{cases} y^{(n)} = & x^{(n)} - \dfrac{2}{3}\{F'(x^{(n)})\}^{-1}F(x^{(n)}), \\ x^{(n+1)} = & x^{(n)} - \left\{2\left[500F'(x^{(n)}) + 3F'(y^{(n)})\right]\left[1503F'(y^{(n)}) - 505F'(x^{(n)})\right]\right\}^{-1} \\ & \times\left\{765015F'(y^{(n)}) + 238973F'(x^{(n)})\right\}F(x^{(n)}). \end{cases} \tag{21}$$

This is again a new fourth-order method for solving systems of nonlinear equations.

Note that family (2) can produce several new multipoint families of Jarratt's method without using second-order Fréchet derivative for simple roots of nonlinear system by fixing one of the disposable parameters namely, α_1 or α_2.

5. Computational Efficiency

The traditional way to obtain an assessment of the efficiency index Ostrowski (1973) of iterative methods is given by $E = \rho^{\frac{1}{C}}$, where ρ is the order of convergence and C is the computational cost per iteration. For the system of k non-linear equations in k unknowns, the computational cost per iteration is given by [see Grau-Sánchez et al. (2011)]

$$C(u_0, u_1, k) = u_0 a0k + u_1 a1k^2 + P(k), \tag{22}$$

where a_0 and a_1 represent the number of evaluations of $F(x)$ and $F'(x)$ respectively, $P(k)$ is the number of products per iteration and u_0 and u_1 are the ratios between products and valuations required to express the value of $C(u_0, u_1, k)$ in terms of product.

Now, let us compare the efficiency index of the proposed methods namely **Error! Reference source not found.** (MJM_1) (ϕ_1) and **Error! Reference source not found.** (MJM_2) (ϕ_2) with that of Newton's method (ϕ_3) (NM), third order method by Homeier (HM) (ϕ_4) [Homeier (2004)] and harmonic mean Newton's method $(HMNM)$ (ϕ_5) [Grau-Sa'nchez et al. (2011)], fourth-order methods by Sharma et al. (WNM) (ϕ_6) [Sharma et al. (2013)], Cordero et al. (CM) (ϕ_7)[Cordero et al. (2009)] and Darvishi (DM) (ϕ_8) [Darvishi and Barati (2007)].

The HM is given by

$$\left.\begin{aligned} y_1^{(n)} &= x^{(n)} - \frac{1}{2}\{F'(x^{(n)})\}^{-1} F(x^{(n)}), \\ x^{(n+1)} &= \varphi_4(x^{(n)}, y_1^{(n)}) = x^{(n)} - \{F'(y_1^{(n)})\}^{-1} F(x^{(n)}). \end{aligned}\right\} \tag{23}$$

The $HMNM$ is given by

$$\left.\begin{aligned} y_2^{(n)} &= x^{(n)} - \{F'(x^{(n)})\}^{-1} F(x^{(n)}), \\ x^{(n+1)} &= \varphi_5(x^{(n)}, y_2^{(n)}) = x^{(n)} - \frac{1}{2}\left[\{F'(x^{(n)})\}^{-1} + \{F'(y_2^{(n)})\}^{-1}\right] F(x^{(n)}). \end{aligned}\right\} \tag{24}$$

The WNM is given by

$$\left.\begin{aligned} y^{(n)} &= x^{(n)} - \frac{2}{3}\{F'(x^{(n)})\}^{-1} F(x^{(n)}), \\ x^{(n+1)} &= \varphi_6(x^{(n)}, y^{(n)}) \\ &= x^{(n)} - \frac{1}{2}\left[\frac{9}{4}\left(\{F'(x^{(n)})\}^{-1} F(y^{(n)})\right)^{-1} + \frac{3}{4}\{F'(x^{(n)})\}^{-1} F(y^{(n)}) - I\right]\{F'(x^{(n)})\}^{-1} F(x^{(n)}). \end{aligned}\right\} \tag{35}$$

The CM is given by

$$\left.\begin{aligned} y_2^{(n)} &= x^{(n)} - \{F'(x^{(n)})\}^{-1} F(x^{(n)}), \\ x^{(n+1)} &= \varphi_7(x^{(n)}, y_2^{(n)}) \\ &= y_2^{(n)} - \left[2\{F'(x^{(n)})\}^{-1} - \{F'(x^{(n)})\}^{-1} F'(y_2^{(n)})\{F'(x^{(n)})\}^{-1}\right] F(y_2^{(n)}). \end{aligned}\right\} \tag{25}$$

The DM is given by

$$\left.\begin{aligned} y_2^{(n)} &= x^{(n)} - \{F'(x^{(n)})\}^{-1} F(x^{(n)}), \\ y_4^{(n)} &= x^{(n)} - \{F'(x^{(n)})\}^{-1} \left[F(x^{(n)}) + F(y_2^{(n)})\right], \\ x^{(n+1)} &= \varphi_8(x^{(n)}, y_2^{(n)}, y_4^{(n)}) \\ &= x^{(n)} - \left[\frac{1}{6}F'(x^{(n)}) + \frac{2}{3}F'\left(\frac{x^{(n)} + y_4^{(n)}}{2}\right) + \frac{1}{6}F'(y_4^{(n)})\right]^{-1} F(x^{(n)}). \end{aligned}\right\} \tag{26}$$

In the iterative method ϕ_3, that is Newton's method, instead of computing the inverse operator we solve a linear system, where we have $k(k-1)(2k-1)/6$ products and $k(k-1)/2$ quotients in the LU decomposition and $k(k-1)$ products and k quotients in the resolution of two triangular linear systems. If we suppose that a quotient is equivalent to n products, then

$$P(k) = \frac{k(k-1)(2k+5)}{6} + n\frac{k(k+1)}{2} = \frac{k(2k^2 + 3(n+1)k + 3n - 5)}{6}. \tag{27}$$

In general, we denote by the number of scalar products per iteration by p^0 and the number of complete resolutions of a linear system (LU decomposition and resolution of two triangular systems) by p^1. We call p^2 the number of resolutions of two triangular systems when LU decomposition is computed in another step in the same iteration, then total number of products is (see Behl et al (2013))

$$P(k) = \frac{k\left(2p1k^2 + (3p1(n+1) + 6p2)k + 6p0 + p1(3n-5) + 6p2(n-1)\right)}{6}. \tag{28}$$

In Table 1, we present the values of $a0, a1, p0, p1, p2, \rho$ and $C(u_0, u_1, k)$ for each iterative method analyzed in this paper, $\phi_1 - \phi_8$.

Table 1. Coefficients used in (22) and (28), local order of convergence and computational cost of iterative methods $\phi_1 - \phi_8$,

Method	a^0	a^1	p^0	p^1	p^2	ρ	$C(u_0,u_1,k)$
ϕ_1	1	2	5	2	1	4	$k\left(2k^2 + 3(2u_1 + n + 2)k + 3u_0 + 3n + 10\right)/3$
ϕ_2	1	2	7	2	1	4	$k\left(2k^2 + 3(2u_1 + n + 2)k + 3u_0 + 3n + 16\right)/3$
ϕ_3	1	1	0	1	0	2	$k\left(2k^2 + 3(2u_1 + n + 1)k + 6u_0 + 3n - 5\right)/6$

ϕ_4	1	2	1	2	0	3	$k\left(2k^2+3(2u_1+n+1)k+3u_0+3n-2\right)/3$
ϕ_5	1	2	1	2	0	3	$k\left(2k^2+3(2u_1+n+1)k+3u_0+3n-2\right)/3$
ϕ_6	1	2	4	2	1	4	$k\left(2k^2+3(2u_1+n+2)k+3u_0+6n+4\right)/3$
ϕ_7	2	2	1	1	2	4	$k\left(2k^2+3(4u_1+n+5)k+12u_0+15n-11\right)/6$
ϕ_8	2	3	3	2	1	4	$k\left(2k^2+3(3u_1+n+2)k+6u_0+6n+1\right)/3$

5.1. Comparison Between the Efficiencies

Let us denote the efficiencies of $\phi_i, i=1\,\text{to}\,8,$ by $M_i(u_0,u_1,k)$. Consider the ratio

$$G_{i,j}=\frac{\log M_i(u_0,u_1,k)}{\log M_j(u_0,u_1,k)}=\frac{\log(\rho_i)C_j(u_0,u_1,k)}{\log(\rho_j)C_i(u_0,u_1,k)}, \tag{29}$$

where

$$C_s(u_0,u_1,k)=2p1_s k^2+(6u_1a1_s+3p1_s(n+1)+p2_s)k+6u_1a0_s+6p0_s+p1_s(3n-5)+6p2_s(n-1), s=i,j.$$

It is clear that if $G_{i,j}>1$, the iterative method ϕ_i is more efficient than ϕ_j. Taking into account that border between two computational efficiencies is given by $G_{i,j}=1$, this boundary (using (29)) can be expressed by an equation which is written as [see Grau-Sa'nchez et al. (2011)]

$$u_0=\delta_1 ku_1+\delta_2 k^2+\delta_3 k+\delta_4, \tag{30}$$

where

$$\delta_1=-\frac{\log(\rho_i)a1_j-\log(\rho_j)a1_i}{\log(\rho_i)a0_j-\log(\rho_j)a0_i},$$

$$\delta_2=-\frac{1}{3}\frac{\log(\rho_i)p1_j-\log(\rho_j)p1_i}{\log(\rho_i)a0_j-\log(\rho_j)a0_i},$$

$$\delta_3=-\frac{1}{2}\frac{\log(\rho_i)(2p2_j+p1_j(n+1))-\log(\rho_j)(2p2_i+p1_i(n+1))}{\log(\rho_i)a0_j-\log(\rho_j)a0_i},$$

$$\delta_4=-\frac{1}{6}\left[\frac{\log(\rho_i)(6p2_j(n-1)+p1_j(3n-5)+6p0_j)-\log(\rho_j)(6p2_i(n-1)+p1_i(3n-5)+6p0_i)}{\log(\rho_i)a0_j-\log(\rho_j)a0_i}\right],$$

and $\log(\rho_i)a0_j-\log(\rho_j)a0_i\neq 0$.

In order to compare the efficiency index of the iterative method ϕ_1, that is M_1, with the efficiency indices of the other methods in Fig. 1 in the $(u_1,u_0)-$plane for $k = 3,5,11$ respectively, for n=1, we present the boundary $G_{1,3}=1$ between M_1 and M_3 by dotted line S_1, the boundary $G_{1,4}=1$ between M_1 and M_4 by dashed line S_2, the boundary $G_{1,7}=1$ between M_1 and M_7 by dot-dashed line S_3, the boundary $G_{1,8}=1$ between M_1 and M_8 by solid line S_4. The point is that we can't compare the efficiency indices of the iterative methods ϕ_1 & ϕ_2 and ϕ_1 & ϕ_6 since the condition $\log(\rho_i)a0_j - \log(\rho_j)a0_i \neq 0$ is violated here. Line S_1 divides the maximum efficiency region between ϕ_1 & ϕ_3, being that $E_1 > E_3$ is above S_1. Similarly, line S_2 divides the maximum efficiency region between ϕ_1 & ϕ_4, being that $E_1 > E_4$ is above S_2, line S_3 divides maximum efficiency region between ϕ_1 & ϕ_7, being that $E_1 > E_7$ is above S_3, line S_4 divides maximum efficiency region between ϕ_1 & ϕ_8, being that $E_1 > E_8$ is above S_4.

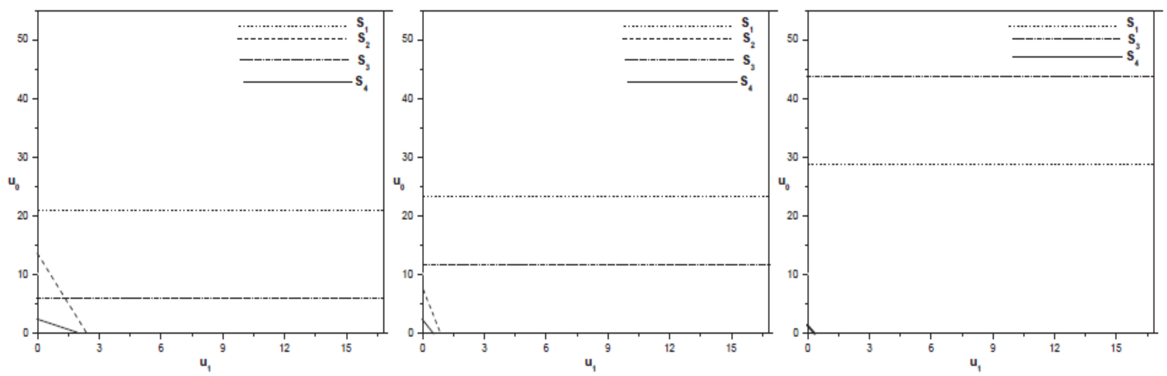

Figure. 1. (Boundary Lines in (u_1, u0)- plane for k=3,5,11 respectively, for *n*=1)

The above results concerning efficiency indices are summarized in the following theorem:

Theorem 5.1.

For all $k \geq 3$, we have

(a) $M_1 > M_3$ for $u_0 > k+5$,

(b) $M_1 > M_4$ for $u_0 > -\frac{2}{3}k^2 - 2k - \frac{5}{3} - 2ku_1 + \frac{(3k+8)\log(3)}{\log(\frac{4}{3})}$,

(c) $M_1 > M_7$ for $u_0 > \frac{k^2+11}{3}$,

(d) $M_1 > M_8$ for $u_0 > -ku_1 + 2$.

On similar lines, we can also compare the efficiency index of the proposed method **Error! Reference source not found.** (MJM_2) (ϕ_2) with that of Newton's method (ϕ_3) (NM), the third order method by Homeier (HM) (ϕ_4) Homeier (2004) and harmonic mean Newton's method $(HMNM)$ (ϕ_5) Grau-Sa'nchez et al. (2011), the fourth-order methods by Sharma et al. (WNM) (ϕ_6) Sharma et al. (2013), Cordero et al. (CM) (ϕ_7) Cordero et al. (2009) and Darvishi (DM) (ϕ_8) Darvishi and Barati (2007).

6. Numerical Experiments

Now, we present some numerical examples to illustrate the comparison of the performance of the newly developed methods namely, method **Error! Reference source not found.** (MJM_1), method **Error! Reference source not found.** (MJM_2) and method (21) (MJM_3) with that of classical Newton's method (NM), third order method by Homeier (HM) **Error! Reference source not found.** and the fourth-order methods by Cordero et al. (CM) Cordero et al. (2009), Darvishi (DM) Darvishi and Barati (2007), Sharma et al. (WNM) Sharma et al. (2013) and Jarratt's method (JM) Nedzhibov (2008) respectively for solving systems of nonlinear equations given in Table 2. Computations have been performed using $MATLAB^{®}$ version 7.5($R2007b$) in double precision arithmetic. We use $\varepsilon = 10^{-15}$ as a tolerance error. Following stopping criteria are taken for computer programs:

(i) $\| x^{(n+1)} - x^{(n)} \| < \varepsilon$, (ii) $\| F(x^{(n)}) \| < \varepsilon$.

We analyze the number of iterations needed to converge to the required solution.

Consider the following systems of nonlinear equations

Example 6.1.

$$\left.\begin{aligned} x_1^2 - 2x_1 - x_2 + 0.5 &= 0, \\ x_1^2 + 4x_2^2 - 4 &= 0. \end{aligned}\right\}$$

Solution is $(1.900676726367066, 0.311218565419294)^T$.

Example 6.2.

$$\left.\begin{aligned} x_1 + \exp(x_2) - \cos(x_2) &= 0, \\ 3x_1 - x_2 - \sin(x_2) &= 0. \end{aligned}\right\}$$

Solution is $(0,0)^T$.

Example 6.3.

$$\left.\begin{aligned} x_1^2 - x_2^2 + 3\log(x_1) &= 0,\\ 2x_1^2 - x_1x_2 - 5x_1 + 1 &= 0. \end{aligned}\right\}$$

Solution is $(1.319205803329892, -1.603556555187415)^T$.

Example 6.4.

$$\left.\begin{aligned} \exp(x_1) + x_1x_2 - 1 &= 0,\\ \sin(x_1x_2) + x_1 + x_2 - 1 &= 0. \end{aligned}\right\}$$

Solution is $(0,1)^T$.

Example 6.5.

$$\left.\begin{aligned} x_1^2 - x_2^2 - 1 &= 0,\\ x_1^3x_2^2 - 1 &= 0. \end{aligned}\right\}$$

Solution is $(1.236505703393025, 0.727286982232063)^T$.

Example 6.6.

$$\left.\begin{aligned} x_1^2 + x_2^2 + x_3^3 &= 9,\\ x_1x_2x_3 &= 1,\\ x_1 + x_2 - x_3^2 &= 0. \end{aligned}\right\}$$

Solution is $(2.224244828847784, 0.283884974072938, 1.583707612825272)^T$.

Example 6.7.

$$\left.\begin{aligned} \cos(x_2) - \sin(x_1) &= 0,\\ x_3^{x_1} - \frac{1}{x_2} &= 0,\\ \exp(x_1) - x_3^2 &= 0. \end{aligned}\right\}$$

Solution is $(0.909569494520044, 0.661226832274852, 1.575834143906999)^T$.

Table 2. (Total number of iterations)

Example no.	Initial guess	NM	HM	CM	DM	WNM	JM	MJM_1	MJM_2	MJM_3
6.1	(1.7, 0)	4	4	2	2	2	2	2	2	2
	(3.5, 2.5)	5	5	3	2	2	2	2	2	2
	(3, 2)	5	5	3	2	2	2	2	2	2
6.2	(0.3, 0.5)	4	4	2	2	2	2	2	1	2
	(1.7, 2.2)	5	4	3	2	2	2	2	2	2
6.3	(0.91, -2)	4	4	2	2	2	2	2	2	2
	(1.7, -2.2)	4	4	2	2	2	2	2	1	2
	(1.8, -2.1)	4	4	2	2	2	2	2	1	2
6.4	(0.7, 0.9)	4	4	2	2	2	2	2	2	2
	(-0.5, 0.5)	4	4	2	2	2	2	2	1	2
	(-0.1, 2)	3	3	1	1	1	1	1	1	2
6.5	(1.5, 1)	4	4	2	2	2	2	2	2	2
	(1.3, 0.4)	3	3	2	2	2	2	2	1	2
6.6	(2, 0.6, 1.5)	3	3	2	2	2	1	2	1	2
	(3, 0.05, 2)	4	4	2	2	2	2	2	2	2
6.7	(1, 0.5, 5)	4	4	2	2	2	2	2	2	2
	(1, 1, 2)	5	4	5	4	4	2	4	2	2

7. Conclusions

Evidently, we have proposed and analyzed a wide general class of Jarratt's method for solving nonlinear equations in the multivariate case. This class is a generalization over the family of Jarratt's method proposed by Behl et al. (2013) and depends on two disposable parameters. These methods have a fourth-order convergence and do not require the second-order Fréchet derivative. In terms of computational cost, all these methods require evaluations of one function and two first-order Fréchet derivatives. Finally, the computational results verify that the family of methods are efficient and exhibit equal or better performance, compared to other well-known methods available in literature.

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