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An Optimal Reinsurance Contract from Insurer's and Reinsurer's Viewpoints

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Abstract

This article constructs two classes of appropriate reinsurance contracts from both an insurer's and a reinsurer's viewpoints. The first class, say $\mathcal{C}$, has been constructed by minimizing the conditional tail expectation, say CTE, of an insurer's random risk. Then an optimal reinsurance contract has been obtained by estimating the reinsurance's random risk, using the Bayesian estimation method while the second class of reinsurance contracts, say $\mathcal{C}^*$, is obtained by minimizing a convex combination of the CTE of both the insurer's and reinsurer's random risks. These two approaches consider both the insurer's and reinsurer's viewpoints to establish an optimal reinsurance contract. A simulation study has been conducted to show practical implementation of our results.

Keywords: Optimal reinsurance contract; Conditional tail expectation (CTE); Value-at-Risk (VaR); Bayesian estimation.

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1. Introduction

Reinsurance contracts involve two parties, an insurance company and a reinsurance company. Suppose aggregate loss X is a nonnegative and continuous random variable, with cumulative distribution function F_X, defined on the measurable space $(\Omega, \mathcal{F}, P)$, where $\Omega = [0, \infty]$ and $\mathcal{F}$

is the Borel σ-field on Ω. Moreover, suppose that random claim X can be decomposed as the sum of insurance portion, say $I_f(X)$, and reinsurance portion, say $f(X)$, i.e $X = I_f(X) + f(X)$, where both $I_f(X)$ and $f(X)$ are continuous functions that satisfy $0 \leq I_f(x)$ and $f(x) \leq x$ for all $x \geq 0$. Under this decomposition, the total risk of insurance and reinsurance companies, respectively, can be restated as $T_f(X) = X - f(X) + \Pi(f(X))$ and $T_f^*(X) = f(X) - \Pi(f(X))$, where $\Pi(f(X))$ stands for the reinsurance premium.

An optimal reinsurance strategy has been obtained by determining the functional form of reinsurance portion $f(x)$ under an optimal criteria or estimating parameter(s) of $f(X)$ whenever it is constrained in a class of continuous functions. Finding such an optimal reinsurance strategy is an interesting actuarial problem from both theoretical and practical viewpoints. Designing an optimal reinsurance strategy was started in the 1960s by seminal results of Borch (1960, 1969), Kahn (1961), and Arrow (1963). The existing optimal reinsurance strategies can be classified into three categories. In the first category, the authors consider an optimal criterion. Then under such criterion they derive an optimal reinsurance strategy from the insurer's viewpoint. Arrow (1974), Beard et al. (1977), Gerber (1979), Bowers et al. (1997), Kass et al. (2001), Cai and Tan (2007), Cai et al. (2008), Chi and Tan (2011), Tan et al. (2011), Cheung et al. (2014), and Asimit et al. (2015), among others, are authors who designed such optimal reinsurance strategies. In the second category an optimal reinsurance has been achieved from the reinsurer's viewpoint under a certain optimal criterion. Borch (1960, 1969), Ignatov et al. (2004), Kaishev and Dimitrova (2006), Dimitrova and Kaishev (2010), Asimit et al. (2013), Cai et al. (2013) and Assa (2015), among others, are authors who designed such optimal reinsurance strategies. The third category was achieved by combining some well-known reinsurance strategies and determining new strategies with optimal properties. One author who designed this kind of reinsurance contract is Centeno (1985), who combined two well-known quota-share and excess of loss reinsurance strategies and defined a new reinsurance contract $f(X) = \min\{\alpha X, M\}$ where $0 \leq \alpha \leq 1$ and $M \geq 0$ are retentions that estimated by minimizing the coefficient of variation and the skewness of random risk X. Liang and Guo (2011) provided different estimators for α and M.

Hereafter, we assume the two first moments of random risk X are finite. Therefore, we seek an optimal reinsurance model in the space $L^2 := L^2(\Omega, \mathcal{F}, P)$.

This article considers the following two classes of reinsurance strategies. The first class of reinsurance models is obtained by decomposing reinsurance portion $f(x)$ as $f(x) = (1 - \beta)xI(x)_{[0,M_1)} + f^*(x)I(x)_{[M_1,\infty)}$, where $0 \leq \beta \leq 1$ and $M \geq 0$ are retentions and $I(\cdot)$ is the indicator function. In the first step, we assume β and M_1 are given, and then we determine an optimal $f^*(x)$ by minimizing the CTE of risk of the insurance company from retention M_1. Finally, we provide a Bayes estimator for two parameters β and M_1 using reinsurance's random risk. The second class of reinsurance models is obtained by considering the random risk of both insurance and reinsurance companies simultaneously. An optimal reinsurance contract in this class arrives by minimizing a convex combination of the CTE of risk of both insurance and reinsurance companies.

More precisely, the first class of reinsurance models can be restated as

$$\mathcal{C} := \left\{ \begin{array}{l} f(x) = (1-\beta)xI(x)_{[0,M_1)} + f^*(x)I(x)_{[M_1,\infty)} \;\; | \;\; 0 \le E(f^*(X)|X \ge M_1) \le \frac{\pi_2}{(1+\theta)}; \\ f(x) \;\; \text{is nonnegative, non-decreasing and continuous function such that } f(x) \le x \end{array} \right\}, \quad (1)$$

where reinsurance premium is evaluated under expectation principle, with safety factor $\theta \ge 0$, and restated as

$$\Pi = (1+\theta)E(f(X)) =: \Pi_1 + \Pi_2,$$

that satisfy

$$\Pi_1 := (1+\theta)(1-\beta)E(X|X \le M_1) \le \pi_1,$$

and

$$\Pi_2 := (1+\theta)E(f^*(X)|X \ge M_1) \le \pi_2 \le (1+\theta)E(X|X \ge M_1).$$

$\pi = \pi_1 + \pi_2$ is the maximum of the reinsurance premium that an insurance company accepted to pay that divided into two parts: π_1 is the maximum reinsurance premium for $X < M_1$, and π_2 is the maximum reinsurance premium for $X \ge M_1$.

In the first step, we assume β and M_1 are given then, we determine an optimal $f^*(x)$ by minimizing the following CTE:

$$\min_{f^*} \;\; CTE_\alpha(T_{f^*}) = \min_{f^*} \left\{ CTE_\alpha((X - f^*(X))I(X)_{[M_1,\infty)} + (1+\theta)E(f^*(X)|X \ge M_1)) \right\}. \quad (2)$$

Finally two parameters β and M_1 are estimated under a Bayesian approach (see below).

The second class of reinsurance models can be restated as

$$\mathcal{C}^* := \left\{ \begin{array}{l} f(x) \;\; | \;\; 0 \le E[f(X)] \le \frac{\pi}{(1+\theta)}; f(x) \;\; \text{is nonnegative, non-decreasing} \\ \text{and continuous function that } f(x) \le x \end{array} \right\}, \quad (3)$$

where the reinsurance premium is evaluated under expectation principle, with safety factor $\theta \ge 0$, $\Pi = (1+\theta)E(f(X)) \le \pi \le (1+\theta)E(X)$.

An optimal $f(x)$ is determined by minimizing the following convex combination of the CTE of risk of both insurance and reinsurance companies

$$\min_f \left\{ \omega CTE_\alpha(T_f) + (1-\omega)CTE_\alpha(T_f^*) \right\}, \quad (4)$$

where $0.5 \le \omega \le 1$.

The rest of this article is organized as following: Section 2 derives optimal solutions for the above two classes of appropriate reinsurance contracts. Practical applications of our results have been provided, via two simulation studies, in Section 3. Conclusions and suggestions have been given in Section 4.

2. CTE-based optimal reinsurance contract

The conditional tail expectation, say CTE, for random risk X at the confidence level of $100(1-\alpha)\%$ has been defined by

$$CTE_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_q(X)dq,$$

where $\text{VaR}_p(X) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}$. Many authors, such as Landsman and Valdez (2003) and Dhaene et al. (2006) defined $CTE_\alpha(Z) = E[Z|Z > VaR_\alpha(Z)]$. This definition is, however, not quite correct whenever $Z > VaR_\alpha(Z)$ with zero probability (see Wirch and Hardy (1999) and Tan et al. (2011)) for more details.

In the next two subsections, an optimal reinsurance contract has been determined within two classes of reinsurance contracts $\mathcal{C}$ and $\mathcal{C}^*$.

2.1. Optimal reinsurance contract within class $\mathcal{C}$

We begin by assuming that two parameters β and M_1 in the class of reinsurance contracts $\mathcal{C}$ are given and we are looking for an optimal f^*.

The following lemma summarizes results of Theorems 3.1, 3.2, and 3.3 of Tan et al. (2011) and provides a vital role in the rest of this section.

Lemma 1.

Suppose α, θ and π are positive constants such that $\alpha(1+\theta) \leq 1$ and $0 \leq \pi \leq (1+\theta)E[X]$. There exists positive constant M such that $f^*(x) = (x-M)_+$ minimizes $CTE_\alpha(T_f)$ for $0 \leq f(x) \leq x$ for all $x \geq 0$ and $E[f(X)] \in [0, \frac{\pi}{1+\theta}]$.

Using the result of Lemma 1, the following provides functional form of an optimal reinsurance contract within class $\mathcal{C}$, whenever β and M_1 are given.

Theorem 1.

Under conditions given by reinsurance contract class $\mathcal{C}$, the following reinsurance contract $f_{opt}(x)$ minimizes the CTE given by Equation (2).

$$f_{opt}(x) = \begin{cases} (1-\beta)x & x < M_1, \\ (1-\beta)M_1 & M_1 \leq x < M_2, \\ x + (1-\beta)M_1 - M_2 & x \geq M_2, \end{cases} \tag{5}$$

where

(i) $M_2 = M_1 + \hat{d}$ whenever $\pi_2 \in [0, (1+\theta)E[(Y-d_\alpha)_+]]$ and $\alpha(1+\theta) \leq 1$;
(ii) $M_2 = M_1 + d^*$ whenever $\pi_2 \in [(1+\theta)E[(Y-d_\alpha)_+], (1+\theta)E[(Y-d_\theta)_+]]$ and $\alpha(1+\theta) \leq 1$;
(iii) $M_2 = M_1 + d_\theta$ whenever $\pi_2 \in [(1+\theta)E[(Y-d_\theta)_+, \infty]]$ and $\alpha(1+\theta) \leq 1$;

where $\hat{d} > 0$ is determined by $(1+\theta)E[h_{opt}(Y)] = \pi_2$, d^* is determined by $(1+\theta)E[(Y-d^*)_+] = \pi_2$, $P\{Y \geq d^*\} \leq \frac{1}{1+\theta}$ and $P\{Y \geq d^*\} \geq \alpha$, and $d_\theta = \inf\{d : P[Y > d] \leq \frac{1}{1+\theta}\}$.

Proof:

Reinsurance contracts in class $\mathcal{C}$ have two parts: $(1-\beta)x$, for value of $x \in [0, M_1)$, and $f^*(x)$, for $x \geq M_1$. The second part $f^*(\cdot)$ should be found such that the optimal property of Equation (2) holds.

In the first step, we use an appropriate transformation and change the origin point of the Cartesian coordinate system from $(0,0)$ to $(M_1, (1-\beta)M_1)$ and take this point as the origin point of the shifted coordinate system. The new equivalent function that should be found according to optimal property is $h(y)$ for $y \geq 0$ such that $y = x - M_1$. Now the equivalent objective function is

$$\begin{cases} \min\limits_h CTE_\alpha[T_h] = \min\limits_h CTE_\alpha[y - h(y) + (1+\theta)E(h(Y))] \\ \quad \text{such that} \qquad 0 \leq h(y) \leq y \qquad \text{for all} \qquad y \geq 0 \\ \qquad\qquad\qquad E(h(Y)) \in \left[0, \frac{\pi_2}{(1+\theta)}\right] \end{cases}. \tag{6}$$

Now $h(y)$ is an optimal reinsurance contract that we are looking for. Using this transformation, one may employ Lemma () to determine the optimal reinsurance. It would be worthwhile mentioning that the set of all plausible functions, given by Lemma 1, is greater than the set of our plausible functions $\mathcal{C}$. Fortunately, since the optimal strategy obtained by Lemma 1, is a continuous function, it can be the optimal strategy for our model as well.

An application of Lemma () leads to the following results:

(i) If $\pi_2 \in [0, (1+\theta)E[(Y-d_\alpha)_+]]$ and $\alpha(1+\theta) \leq 1$, then $h_{opt}(y) = (y-\hat{d})_+$ is an optimal reinsurance loss function where the retention $\hat{d} > 0$ is determined by $(1+\theta)E[h_{opt}(Y)] = \pi_2$ and $d_\alpha = \inf\{d : P[Y > d] \leq \alpha\}$.

(ii) If $\pi_2 \in [(1+\theta)E[(Y-d_\alpha)_+], (1+\theta)E[(Y-d_\theta)_+]]$ and $\alpha(1+\theta) \leq 1$, then $h_{opt}(x) = (y-d^*)_+$ is an optimal reinsurance loss function if the retention d^* exists such that $(1+\theta)E[(Y-d^*)_+] = \pi_2$, $P\{Y \geq d^*\} \leq \frac{1}{1+\theta}$ and $P\{Y \geq d^*\} \geq \alpha$, and $d_\theta = \inf\{d : P[Y > d] \leq \frac{1}{1+\theta}\}$.

(iii) If $\pi_2 \in [(1+\theta)E[(Y-d_\theta)_+, \infty]]$ and $\alpha(1+\theta) \leq 1$, then $h_{opt}(y) = (y-d_\theta)_+$ is an optimal reinsurance loss function where d_θ is defined in the previous item.

Now, going back to the original coordinate system, the optimal reinsurance strategy will be

$$f_{opt}(x) = \begin{cases} (1-\beta)x & x < M_1, \\ (1-\beta)M_1 & M_1 \leq x < M_2, \\ x + (1-\beta)M_1 - M_2 & x \geq M_2, \end{cases}$$

This observation completes the desired results. □

The above theorem provides a functional form of an optimal reinsurance contract that falls in the class of reinsurance contracts $\mathcal{C}$. Now we employ the Bayesian approach to estimate unknown parameters β and M_1 and consequently determine optimal reinsurance contract in class $\mathcal{C}$.

The Bayesian approach has been used in various areas of actuarial science. The Bayesian estimation method combines some available information with the theoretical models to provide a more appropriate estimator for unknown parameters. The earliest application of Bayesian ideas in the actuarial science appears to be in Whitney (1918) for experience rating. A clear and strong argument in favor of using Bayesian methods in actuarial science is given in Bailey (1950). Makov et al. (1996), Hesselager and Witting (1998), Hossack et al. (1999), Makov (2001), England and Verral (2002), Payandeh (2010), and Payanadeh et al. (2012), among others, are authors who employed Bayesian methods in actuarial sciences.

Since the maximum of reinsurance premium π, for all reinsurance contracts in class $\mathcal{C}$, has to satisfy

$$(1+\theta)E[f_{opt}(X)] \leq \pi \tag{7}$$

we have, already, one equation to determine β and M_1.

Now we want to determine Bayesian estimator of β and M_1 using reinsurance's random risk. For convenience in presentation we will use $Z_i = f(X_i)$. Lemma 2 provides cumulative distribution function and density function of conditional random variable $Z|(\theta, \alpha, M_1)$.

Lemma 2.

Suppose $X|\theta$ has continuous distribution function $G_{X|\theta}(\cdot)$ and continuous density function $g_{X|\theta}(\cdot)$. Moreover, suppose that $Z_1, \cdots, Z_n|(\theta, \beta, M_1)$ is a sequence of i.i.d. random variables with common density function $g_{Z|(\theta,\beta,M_1)}(\cdot)$. Then the joint density function of $Z_1, \cdots, Z_n|(\theta, \beta, M_1)$ can be represented by

$$\begin{aligned} g(z_1, \cdots, z_n\,|\theta, \beta, M_1) = &\left(\frac{1}{1-\beta}\right)^{n_1} \prod_{i=1}^{n_1} g_{X|\theta}\left(\frac{z_i}{1-\beta}\right) \times [G_{X|\theta}(M_2) - G_{X|\theta}(M_1)]^{n_2} \\ &\times \prod_{i=n_1+n_2+1}^{n} g_{X|\theta}(z_i + M_2 - (1-\beta)M_1), \end{aligned}$$

where n_1 is the number of z_i's less than $(1-\beta)M_1$ and n_2 is the number of z_i's equal to $(1-\beta)M_1$ such that $n_1 + n_2 \leq n$.

Proof:

In the first step observe that for one sample $Z\,|(\theta, \alpha, M_1)$ the distribution function is

$$\begin{aligned} G_{Z|(\theta,\beta,M_1)}(z) = P(Z \leq z) = \quad & P((1-\beta)X \leq z, X < M_1) + P((1-\beta)M_1 \leq z, M_1 \leq X < M_2) \\ & + P(X + (1-\beta)M_1 - M_2 \leq z, X \geq M_2,) \\ = \quad & \begin{cases} G_{X|\theta}(\frac{z}{1-\beta}) & z < (1-\beta)M_1, \\ G_{X|\theta}(M_2) & z = (1-\beta)M_1, \\ G_{X|\theta}(z + M_2 - (1-\beta)M_1) & z > (1-\beta)M_1. \end{cases} \end{aligned}$$

Then the density function of $Z|(\theta, \beta, M_1)$ will be as follows:

$$g_{Z|\theta,\beta,M_1}(z) = \begin{cases} \frac{1}{1-\beta} g_{X|\theta}(\frac{z}{1-\beta}) & z < (1-\beta)M_1, \\ G_{X|\theta}(M_2) - G_{X|\theta}(M_1) & z = (1-\beta)M_1, \\ g_{X|\theta}(z + M_2 - (1-\beta)M_1) & z > (1-\beta)M_1, \end{cases} \tag{8}$$

Now suppose that n_1 is the number of z_i's less than $(1-\beta)M_1$ and n_2 is the number of z_i's equal to $(1-\beta)M_1$ such that $n_1 + n_2 \leq n$. The desired proof is obtained by the fact that joint density function for a sequence of independent random variables is the product of their respective marginal density functions. □

Lemma 3 develops the joint posterior distribution for (θ, β, M_1) given a random sample $z_1, \cdots, z_n$.

Lemma 3.

Suppose $Z_1, \cdots, Z_n|(\theta, \beta, M_1)$ is a sequence of i.i.d. random variables with common density function $f_{Z|\theta,\beta,M_1}(z)$. Moreover, suppose that $\pi_1(\Theta)$, $\pi_2(\mathcal{B})$, and $\pi_3(\mathcal{M})$ are prior distributions for θ, β, and M_1, respectively. Then, the joint posterior distribution for $(\theta, \beta, M_1\,|z_1, \cdots, z_n)$ is

$$\pi(\theta, \beta, M_1\,|z_1, \cdots, z_n) = \frac{\left(\frac{1}{1-\beta}\right)^{n_1} \prod_{i=1}^{n_1} g_{X|\theta}\left(\frac{z_i}{1-\beta}\right)\left[G_{X|\theta}(M_2) - G_{X|\theta}(M_1)\right]^{n_2}}{\int_{\mathrm{M}}\int_{\mathrm{A}}\int_{\Theta} \left(\frac{1}{1-\beta}\right)^{n_1} \prod_{i=1}^{n_1} g_{X|\theta}\left(\frac{z_i}{1-\beta}\right)\left[G_{X|\theta}(M_2) - G_{X|\theta}(M_1)\right]^{n_2}}$$
$$\times \frac{\prod_{i=n_1+n_2+1}^{n} g_{X|\theta}\left(z_i + M_2 - (1-\beta)M_1\right)\pi_1(\theta)\pi_2(\beta)\pi_3(M)}{\prod_{i=n_1+n_2+1}^{n} g_{X|\theta}\left(z_i + M_2 - (1-\beta)M_1\right)\pi_1(\theta)\pi_2(\beta)\pi_3(M)d\theta d\beta dM}$$

where n_1 is the number of z_i's less than $(1-\beta)M_1$ and n_2 is the number of z_i's equal to $(1-\beta)M_1$ such that $n_1 + n_2 \leq n$.

Proof:

The desired proof is obtained by using the joint density function of $Z_1, \cdots, Z_n|(\theta, \beta, M_1)$ along with prior distributions for θ, β, and M_1. □

The marginal posterior density functions for $(\beta\,|Z_1, \cdots, Z_n)$ and $(M_1\,|Z_1, \cdots, Z_n)$ are

$$\begin{aligned} \pi(\beta\,|Z_1, \cdots, Z_n) &= \int_{\Theta}\int_{\mathcal{M}} \pi(\theta, \beta, M_1\,|Z_1, \cdots, Z_n)\, dM_1 d\theta; \\ \pi(M_1\,|Z_1, \cdots, Z_n) &= \int_{\Theta}\int_{\mathcal{A}} \pi(\theta, \beta, M_1\,|Z_1, \cdots, Z_n)\, d\beta d\theta. \end{aligned} \tag{9}$$

The Bayes estimator for unknown parameters β and M_1 with respect to the squared error loss function can be obtained by evaluating the expectation of the above marginal posterior distributions.

2.2. Optimal reinsurance contract within class $\mathcal{C}^*$

This subsection provides optimal reinsurance strategy within class of reinsurance contracts $\mathcal{C}^*$ under the convex combination of two CTE, given by Equation (4).

Before representing the main result of this subsection, we collect some useful properties of the VaR and the CTE.

The following provides an appropriate property of the VaR which will be useful in the rest of this section.

Lemma 4.

Suppose X is a nonnegative random variable defined on the measurable space $(\Omega, \mathcal{F}, P)$, where $\Omega = [0, \infty]$ and $\mathcal{F}$ is the Borel σ-field on Ω. Moreover suppose that $g(\cdot)$ is a nondecreasing and continuous real-valued function. Then

$$VaR_p(g(X)) = g(VaR_p(X)), \tag{10}$$

where $p \in (0, 1)$.

Proof:

Using the fact that $VaR_p(X) \leq x \Leftrightarrow p \leq F_X(x)$ (Denuit et al., 2006, Lemma 1.5.15) one may conclude that $VaR_p(g(X)) \leq x \Leftrightarrow p \leq F_{g(X)}(x)$. Since $g(\cdot)$ is continuous function, we have

$$g(z) \leq x \quad \Leftrightarrow \quad z \leq \sup\{y \,|g(y) \leq x\},$$

for all z and x. Consequently, under the condition that $sup\{y|g(y) \leq x\}$ is finite, a simultaneous application of continuity of $g(\cdot)$ and $VaR_p(g(X)) \leq x \Leftrightarrow p \leq F_{g(X)}(x)$ leads to

$$\begin{aligned} p \leq F_{g(X)}(x) \quad &\Leftrightarrow \quad p \leq F_X\{\sup\{y \,|g(y) \leq x\}\} \\ &\Leftrightarrow \quad VaR_p(X) \leq \sup\{y \,|g(y) \leq x\} \\ &\Leftrightarrow \quad g(VaR_p(X)) \leq x. \end{aligned}$$

Therefore, $VaR_p(g(X)) \leq x \Leftrightarrow g(VaR_p(X)) \leq x$ holds for all values of x, which means the equality holds. In the case of $sup\{y|g(y) \leq x\}$ being infinite, the result is obvious. $\square$

The lemma entails several interesting properties, such as location invariance, of the VaR. This properties can be extended to the CTE.

Lemma 5.

Suppose $T_f(X) = X - f(X) + \Pi(f(X))$ and $T^*_f(X) = f(X) - \Pi(f(X))$, where $\Pi(f(X)) = (1 + \theta)E(f(X))$. Then,

(i) $\omega CTE_\alpha[T_f]+(1-\omega)CTE_\alpha[T^*_f] = CTE_\alpha \{\omega X - (2\omega - 1)f(X) + (1 + \theta)E[(2\omega - 1)f(X)]\}$

(ii) $\omega CTE_\alpha[T_f]+(1-\omega)CTE_\alpha[T^*_f] = \omega CTE_\alpha(X)+(2\omega-1)CTE_\alpha \{-f(X) + (1 + \theta)E[f(X)]\}.$

Proof:

Using Lemma 4, one may conclude that

$$\begin{aligned}\omega CTE_\alpha[T_f]+(1-\omega)CTE_\alpha[T_f^*] &= \frac{\omega}{\alpha}\int_0^\alpha \{VaR_s(X-f(X)+(1+\theta)E(f(X))\}\,ds\\ &\quad+\frac{(1-\omega)}{\alpha}\int_0^\alpha \{VaR_s(f(X)-(1+\theta)E(f(X))\}\,ds\\ &= \frac{1}{\alpha}\int_0^\alpha \{\omega VaR_s(X)-\omega f(VaR_s(X))+\omega(1+\theta)E(f(X))\}\,ds\\ &\quad+\frac{1}{\alpha}\int_0^\alpha \{(1-\omega)f(VaR_s(X))+(1-\omega)(1+\theta)E(f(X))\}\,ds.\end{aligned}$$

The last expression can be restated as the following two different ways.

$$\begin{aligned} &= \frac{1}{\alpha}\int_0^\alpha \{\omega VaR_s(X)-(2\omega-1)f(VaR_s(X))+(2\omega-1)(1+\theta)E(f(X))\}\,ds\\ \text{or} &= \frac{1}{\alpha}\int_0^\alpha \{VaR_s(\omega X-(2\omega-1)f(X)+(2\omega-1)(1+\theta)E(f(X))\}\,ds\\ &= \frac{\omega}{\alpha}\int_0^\alpha VaR_s(X)ds+\frac{(2\omega-1)}{\alpha}\int_0^\alpha \{VaR_s(-f(X)+(1+\theta)E(f(X))\}\,ds.\end{aligned}$$

The rest of proof is accomplished by definition of CTE. □

The following theorem provides an optimal reinsurance contract, within the class of reinsurance contracts $\mathcal{C}^*$, which minimizes the convex combination of insurer's and reinsurer's CTE, given by Equation (4).

Theorem 2.

Under conditions given by reinsurance contract class $\mathcal{C}^*$, the following reinsurance contact $f_{opt}(x)$ minimizes convex combination of insurer's and reinsurer's CTE, given by Equation (4),

$$f_{opt}(x)=(x-M)_+, \tag{11}$$

where

(i) $M=\hat{d}$, whenever $\pi\in[0,(1+\theta)E[(X-d_\alpha)_+]]$ and $\alpha(1+\theta)\le 1$;
(ii) $M=d^*$, whenever $\pi\in[(1+\theta)E[(X-d_\alpha)_+],(1+\theta)E[(X-d_\theta)_+]]$;
(iii) $M=d_\theta$, whenever $\pi\in[(1+\theta)E[(X-d_\theta)_+],\infty]$ and $\alpha(1+\theta)\le 1$,

and the retention $\hat{d}>0$ is determined by $(1+\theta)E(f_{opt}(X))=\pi$ and $d_\alpha=\inf\{d:P[X>d]\le\alpha\}$; d^* exist such that $(1+\theta)E[(X-d^*)_+]=\pi$, $P\{X\ge d^*\}\le\frac{1}{1+\theta}$ and $P\{X\ge d^*\}\ge\alpha$; and $d_\theta=\inf\{d:P[X>d]\le\frac{1}{1+\theta}\}$.

Proof:

Using Lemma 5, one may have

$$\omega CTE_\alpha[T_f]+(1-\omega)CTE_\alpha[T_f^*]=\omega CTE_\alpha(X)+(2\omega-1)CTE_\alpha\{-f(X)+(1+\theta)E[f(X)]\}.$$

Using the fact that $2\omega - 1 > 0$ along with application of Lemma 1, one may conclude that the optimal function falls in class $\mathcal{C}^*$ and is a stop-loss reinsurance. □

3. Practical applications

This section provides two numerical examples to show how the above results can be applied in practical situations. More precisely, it develops Bayes estimators for β and M_1 with respect to the squared error loss function. To achieve this goal all loss observations have to be classified into three different classes given by Equation (5).

The following example provides a practical example for this procedure to estimate unknown parameters for our optimal strategy with respect to class $\mathcal{C}$. The second example derives an optimal strategy with respect to class $\mathcal{C}^*$.

Example 1.

Suppose 100 random numbers are generated from one of distributions, given in the first column of Table I. Moreover, suppose that prior distributions $\pi_2(\beta)$ and $\pi_3(M_1)$ are given in the second and the third columns of Table 1, respectively. After estimating β and $M_1,$ using Theorem 1, one may estimate parameter M_2.

The fourth and the fifth columns of Table 1, respectively, represent mean and standard deviation of Bayes estimator of β and M_1 for 100 random numbers that generated 100 times from a given distribution. The last column of Table 1 shows estimator M_2. Such an estimator has been arrived at, using Theorem 1, whenever the mean of the Bayes estimator for β and M_1 in 100 iterations has been considered as an estimator for β and M_1.

Table I. Mean and standard deviation of bayes estimator for β and M_1 based upon 100 sample size and 100 iterations.

Risk Distribution	Prior distribution for β	Prior distribution for M_1	Mean (standard deviation) of estimated β	Mean (standard deviation) of estimated M_1	Mean (standard deviation) of estimated M_2
EXP(1)	Unif(0,1)	Exp(0.5)	0.7416 (0.0202)	0.0880 (2.490×10^{-5})	1.3408 (2.492×10^{-5})
EXP(8)	Unif(0,1)	Exp(2)	0.3848 (0.0646)	0.3738 (0.0396)	0.5373 (0.0396)
Weibull(2,1)	Beta(2,2)	Exp(2)	0.6116 (0.0164)	0.1204 (1.522×10^{-5})	1.1366 (1.520×10^{-5})
Weibull(2,3)	Beta(3,2)	Gamma(2,2)	0.8699 (0.0739)	0.3244 (0.0713)	3.3734 (0.0714)

Small standard deviation of our estimators shows that this estimation method is an appropriate method with respect to different distributions.

The following example employs the result of Theorem 2 to derive optimal reinsurance contract within class of reinsurance contracts $\mathcal{C}^*$.

Example 2.

Suppose random risk X has been distributed according to one of the distributions given in Table II. An optimal reinsurance contract within class $\mathcal{C}^*$ for each distribution has been given in Table II.

Table II. Optimal reinsurance strategy within class $\mathcal{C}^*$, for different distribution and $\theta = 0.4$.

distribution	optimal strategy	distribution	optimal strategy	distribution	optimal strategy
EXP(1)	$(x - 0.3365)_+$	Gamma(2,2)	$(x - 2.1168)_+$	Weibull(2,1)	$(x - 0.6729)_+$
EXP(2)	$(x - 0.1682)_+$	Gamma(2,3)	$(x - 3.7216)_+$	Weibull(3,2)	$(x - 1.7402)_+$
Exp(8)	$(x - 0.0420)_+$	Gamma(3,2)	$(x - 3.1752)_+$	Weibull(2,3)	$(x - 1.3911)_+$

It is worthwhile to mention that the above estimators have been evaluated from functional form of random risk X rather than a random sample observation, moreover, M is not a random variable. Therefore, we do not need to evaluate the standard deviation of the estimators to study the robustness of estimators with respect to different samples.

4. Conclusions & Suggestions

This article considers two classes of reinsurance contracts $\mathcal{C}$ and $\mathcal{C}^*$. Then, using the CTE, an optimal reinsurance strategy has been derived within $\mathcal{C}$ and $\mathcal{C}^*$. Via two simulation studies, practical applications of our findings have been given.

Our method can be extended under other evaluation criteria such as risk measurements, ruin probability, etc. It would be worthwhile to mention that Ohlin's lemma (1969) warrants that results of Theorem () can be extended to any optimal criteria $\rho(\cdot)$, whenever (**1**) $\rho(\cdot)$ can be restated as $\rho(\cdot) = E(\phi(\cdot))$, where $\phi(\cdot)$ is a convex function; (**2**) Cumulative distribution functions of random variable $(X - M)_+$ and $(1-\beta)xI(x)_{[0,M_1)} + f^*(x)I(x)_{[M_1,\infty)}$, for some certain function $f^*(\cdot)$, cross each other exactly one time; and (**3**) $E(f^*(X)|X > M_1) =$ constant.

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